E C O N O M I C S   B U L L E T I N

# On the existence of self−enforcing equilibria

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## *Abstract*

It is argued that if an out−of−equilibrium player observing a deviation from a presumed strategically stable path of play believes that a player also observing the deviation is more likely to deviate than a player who does not observe the deviation then it is possible to justify, in some extensive form game, the non−existence of a self−enforcing equilibrium.

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# 1. Introduction

According to van Damme (1991, p. 3), "the *core problem of noncooperative game theory* can be formulated as: given a game with more than one Nash equilibrium, which one of these should be chosen as the solution of the game?". Many requirements have been suggested to help in the task of equilibrium selection. One of them, applied to extensive form games, is sequential rationality; see Kreps and Wilson (1982, pp. 870-872). The idea motivating sequential rationality is that strategy profiles should also prescribe payoff maximizing choices at unreached information sets. Specifically, a *sequentially rational equilibrium* is an equilibrium such that, for every information set *h* reached with probability zero when the equilibrium is played, there is a probability distribution on *h* making the strategy prescribed at *h* a best reply to the strategies prescribed at information sets going after *h*. In particular, if *h* is a singleton, a sequentially rational equilibrium must dictate at *h* a strategy which is a best reply at *h* to the strategies prescribed at information sets going after *h*.

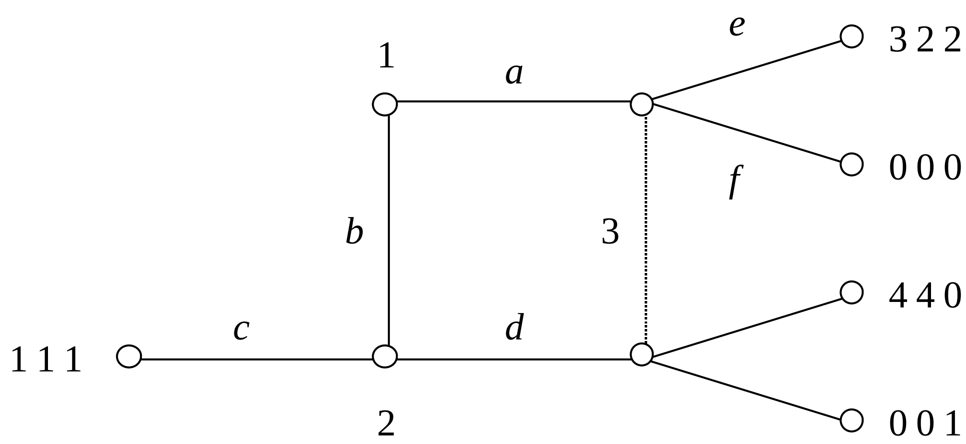


Fig. 1

Consider, for instance, game $G_1$ of Figure 1, taken from Selten (1975, p. 33) and also analysed in Kreps and Wilson (1982, pp. 870-871) and Kreps (1989, pp. 22-24). Profile (*a*, *c*, *e*) is an equilibrium but it is not a sequentially rational equilibrium: if 2's information set is reached, it is *d* not *c* the best reply to *e*. Thus, if 1 expects 3 to choose *e* and 2 to choose *d* when given the move, it would be better for him to select *b* instead of *a*. The conclusion is then that (*a*, *c*, *e*) is not self-enforcing. In fact, since self-enforcingness is an informal (intuitive) notion, requirements like sequential rationality contribute to make its meaning more specific. In this respect, consistency with sequential rationality appears to be necessary for an equilibrium to be self-enforcing. Notice finally that being a sequentially rational equilibrium is strictly less demanding that being a sequential equilibrium (Kreps and Wilson (1982, p. 872)), which in his turn is strictly less demanding that being a perfect equilibrium (Selten (1975, p. 38)).

It is plain that rejection of $(a, c, e)$ relies on the presumption that 3 will not substantially alter the probability with which he is expected to choose $e$. In other words, it is presumed that the occurrence of an event (1's deviation) only (significantly) affects the player who observes the deviation and not the player who ignores that it has occurred. At a general level, this seems to be a sound and reasonable principle: is it not a player who observes the unexpected occurrence of an event relevant for his strategy choice more likely to reconsider his choice than a player who ignores its occurrence?

This note shows that this principle is capable of making all sequentially rational equilibria of some game strategically unstable, in the same sense as sequential rationality makes $(a, c, e)$ strategically unstable in $G_1$.

## 2. The game

Consider game $G_2$ of Figure 2 and let $s \in \{a, c, e, g, h, j, k\}$ simultaneously denote action $s$ and the probability with which $s$ is taken.

**Claim 1.** In every equilibrium of $G_2$, $c = j = 1$, $0 \leq a \leq 1/2$, $0 \leq e \leq 1/3$ and $0 \leq g \leq 1$.

*Proof*. If $a = 1$ then 5's best reply (BR) is $j = 0$, in which case $a = 0$ is 1's BR. This means that 2 is given the move with positive probability in every equilibrium. Suppose $c \neq 1$. When given the move, 2's necessary condition for randomization is $ej = 1/3$ whereas, with $c \neq 1$, 4's is $ej = 1/2$. Thus, it cannot be that both 2 and 4 randomize. Case 1: $g > k$. Hence, $e = 1$ is 3's BR. Case 1a: 2 randomizes but 4 does not. Since $e = 1$, $ej = 1/3$ yields $j = 1/3$, so $g = 0$ is 4's BR and 3 does better by playing $e = 0$: contradiction. Case 1b: 4 randomizes but 2 does not. Now, $j = 1/2$, $c = 0$ is 2's BR and $j = 0$ becomes 5's BR: contradiction. Case 1c: neither 2 nor 4 randomize. As $g > k$, $g = j = 1$, $c = 0$ is 2's BR and $j = 0$ is 5's BR: contradiction. Case 2: $g < k$. Then $e = 0$ is 3's BR and 2's BR is $c = 1$: contradiction. Case 3: $g = k$. Since 2 and 4 cannot simultaneously randomize, either $g = k = 1$ (so $c = 1$ is 2's BR: contradiction) or $g = k = 0$ (so $a = 1$ is 1's BR and 5 does better by playing $k = 1$: contradiction). In sum, given that no equilibrium of $G_2$ has $c \neq 1$, the existence of some equilibrium in $G_2$ implies $c = 1$. If $j \neq 1$, $a = 0$ is 1's BR. This makes $j = 1$ the only BR for 5. To sustain $j = 1$ as a BR, $0 \leq a \leq 1/2$ and to sustain $c = 1$ as a BR, $0 \leq e \leq 1/3$. As regards 4, he may ascribe any probability to $g$. QED

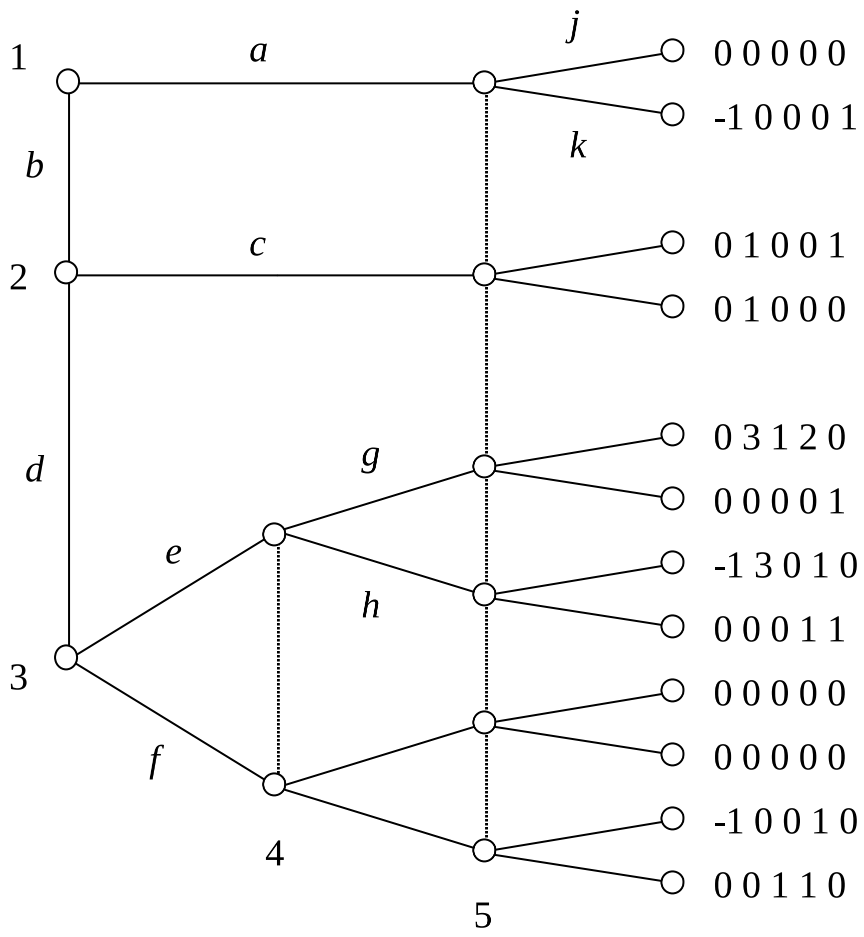


Fig. 2

**Claim 2.** In every sequentially rational equilibrium of $G_2$, $c = h = j = 1$, $0 \leq a \leq 1/2$ and $0 \leq e \leq 1/3$.

*Proof*. By Claim 1, in every equilibrium $c = j = 1$, $0 \leq a \leq 1/2$ and $0 \leq e \leq 1/3$, strategies that are sequentially rational for players on the equilibrium path of play. To sustain $0 \leq e \leq 1/3$ as a best reply at 3's information set given $j = 1$ it must be that $g = 0$, which is a best reply for 4 given $j = 1$ and any probability distribution over 4's information set assigning probability not greater than 1/2 to the upper node in 4's information set. QED

A noteworthy implication of Claim 2 is that the probability with which player 4 chooses *g* can be interpreted as the probability that he deviates from sequentially rational equilibrium play, whereas the probability with which player 5 chooses *k* can be interpreted as the probability that he deviates from sequentially rational equilibrium play. If it is accepted that being sequentially rational is necessary for an equilibrium to be self-enforcing, then players should in principle expect $h = j = 1$.

Suppose now that player 3 is given the move. He then faces a situation similar to player 1's in $G_1$ when he plans to deviate from (*a*, *c*, *e*): to which extent will the player unacquainted with the deviation play as expected? It could be reasonably assumed that players unacquainted with a deviation will stick to their expected choices. A weaker principle will nonetheless be postulated, namely, that a player who does not observe a deviation is *less likely* to reconsider his presumed choice than another player who does observe it. In the case at hand, there is little trouble with defining likelihood that 4 and 5 reconsider their expected choices: since 4's expected choice is $h = 1$, the probability that 3 attributes to $g$ represents his assessment of the likelihood that 4 changes the expected choice and, similarly, since 5's expected choice is $j = 1$, the probability that 3 attributes to $k$ represents his assessment of the likelihood that 5 changes the expected choice.

Summing up, adoption of the above belief formation principle amounts for player 3 of $G_2$ to believe that $g > k$. If this is the case, the only best reply at his information set is $e = 1$, which is part of no sequentially rational equilibrium. What is more: if 2 believes that 3 adopts such principle to form beliefs then he will expect $e = 1$ and $j$ very close to 1, in which case $c = 0$ is the only best reply at his information set, a strategy that is part of no sequentially rational equilibrium. As a result, no sequentially rational equilibrium of $G_2$ is self-enforcing. What if the length of the hierarchy of beliefs is increased? If 5 believes 2 believes that 3 follows the above principle, $j = 0$ becomes his only best reply. If 2 believes 5 believes 2 believes so, it would be better for him to return to $c = 1$. If 5 believes 2 believes 5 believes 2 believes so, he would be induced to change his choice from $j = 0$ to $j = 1$... As a consequence, with common belief in the suggested belief formation principle no sequentially rational equilibrium of $G_2$ is strategically stable[1].

## 3. Discussion

There is a simple way to test the preceding conclusion: insofar as the event that 3 is called upon to move has probability zero under the assumption that some equilibrium is strategically stable, it is enough to observe that player 3 is at least once called upon to move when players are put to play $G_2$ several times.

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[1] Another way to justify $e = 1$ if 3 is called upon to move extends the argument used in the game of Figure 1 to sentence (*a*, *c*, *e*). Why not allowing a simultaneous reconsideration of strategy choices to all those players observing the deviation, keeping the expected choices of the players that may play after the deviation but who do not observe it fixed? This approach implies, for the game of Figure 2, that 3 and 4 must take $b = d = j = 1$ as given and then decide what to choose in the resulting induced game. In that game, $e$ is a weakly dominant choice for player 3.

The crux of the matter is clearly how reasonable is to consider 4's deviation from sequentially rational equilibrium play in $G_2$ more likely than 5's. If the reader is convinced by the sequentially rational argument that ($a$, $c$, $e$) in $G_1$ is not self-enforcing, (s)he should not find unreasonable to hold that 4's deviation could be considered more likely than 5's. After all, player 4 in $G_2$ (like player 2 in $G_1$) receives factual evidence during the play that a deviation has occurred (so he can motivate a continuation of the deviation on these grounds), while player 5 (like player 3 in $G_1$) does not.

Moreover, if players are assumed to choose sequentially rational equilibrium strategies unless they are furnished with factual evidence in the opposite or can *prove* (from the information they possess) that others will not choose sequentially rational equilibrium strategies, then the first player between 4 and 5 having a reason to reject his only sequentially rational equilibrium strategy is 4, with 5 only justified in rejecting his after having inferred that the situation faced by 2, 3 and 4 leads player 2 to defect. Contending otherwise would lead to the possibility of sustaining ($a$, $c$, $e$) in $G_1$ on the grounds that 1 believes that 3 will recognize (having less factual evidence than 2) player 1's deviation when it occurs.

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