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A Dual Egalitarian Solution

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Abstract

In this note we introduce an egalitarian solution, called the dual egalitarian solution, that is the natural counterpart of the egalitarian solution of Dutta and Ray (1989). We prove, among others, that for a convex game the egalitarian solution coincides with the dual egalitarian solution for its dual concave game.

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1 Introduction

Often a situation involving several persons who can obtain benefits by cooperating can be formulated in terms of a cooperative savings game with transferable utility. Examples of such classes of games are bankruptcy games (O'Neill (1982) and Aumann and Maschler (1985)), sequencing games (Curiel *et al.* (1989)), market games (Shapley and Shubik (1969)), and linear production games (Owen (1975)). Moreover, some interesting classes of these games mentioned above are convex, i.e., an agent contributes more to the benefits of a coalition when the coalition becomes larger.

A decision making situation involving several persons can sometimes also be formulated in terms of a cooperative cost game. Among the cost games that can be found in the literature are airport games (Littlechild and Owen (1973)) and minimum cost spanning tree games (Bird (1976)). These games satisfy the property of concavity, i.e., the marginal burden of an agent to the costs of a coalition decreases when the coalition becomes larger.

In this note we start with recalling the definition of the egalitarian solution, a solution concept for cooperative games introduced by Dutta and Ray (1989). This solution unifies the two conflicting concepts of individualistic utility maximization and the social goal of equality. The egalitarian solution is a singleton or the empty set. For convex games Dutta and Ray (1989) describe an algorithm to locate the unique egalitarian solution, and they show, in addition, that it is in the core of the game. Finally, they prove that for convex games the egalitarian solution Lorenz dominates all other allocations in the core. For an extensive discussion on the egalitarian solution we refer to Dutta and Ray (1989).

If a game is concave but not additive, then the egalitarian solution of Dutta and Ray (1989) is empty. Therefore, we define, along the lines of Dutta and Ray (1989), a dual of Lorenz domination and construct a dual egalitarian solution. We show that Lorenz domination and dual Lorenz domination are equivalent. Furthermore, the dual egalitarian solution is either a singleton or the empty set. After that, we study the relations with the egalitarian solution of Dutta and Ray (1989) and the core of the game. Next, we introduce an algorithm similar to that of Dutta and Ray (1989) to calculate the dual egalitarian solution for concave games. We show that the dual egalitarian solution is in the dual core of the concave game and that the dual egalitarian solution Lorenz dominates all other allocations in the dual core.

Our main result is that for a convex game the egalitarian solution coincides with the dual egalitarian solution for its dual (concave) game. Similar duality results are provided by Funaki (1994) and concern, among others, the core, the Shapley value (Shapley (1953)), and the prenucleolus (Schmeidler (1969)). To this series we can also add the modified nucleolus (Sudhölter (1997)) and the τ-value (Tijs (1986)).

The work is organized as follows. Section 2 deals with notation and definitions regarding cooperative games with transferable utility. Moreover, we recall the definition of the egalitarian solution of Dutta and Ray (1989). In Section 3 we introduce the dual egalitarian solution and present the relations with the egalitarian solution. Finally, we prove that for a convex game the egalitarian solution coincides with the dual egalitarian solution for its dual (concave) game.

2 Preliminaries

A cooperative game with transferable utility (game, for short) is a pair (N, v), where $N = \{1, \dots, n\}$ is the player set and v the characteristic function, which assigns to every subset[1] $S \subseteq N$ a value $v(S)$, with $v(\emptyset) = 0$.

A game (N, v) is called *convex* if

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T) \quad \text{for all } S \subseteq T \subseteq N \text{ and } i \in N \backslash T,$$

and *concave* if the reverse inequality holds.

The *core* of a game (N, v) is defined by

$$C(N, v) := \{x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subseteq N\},$$

and its *dual core* is defined by

$$C^*(N, v) := \{x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \leq v(S) \text{ for all } S \subseteq N\}.$$

A game is called (dual) balanced if and only if its (dual) core is non-empty. Let S be a non-empty subset of N. The game $(S, v_{|S})$ (or (S, v), for short) is called a subgame of (N, v). A game is called totally (dual) balanced if and only if the (dual) cores of all subgames are non-empty.

The *dual game* of a game (N, v) is the game (N, v^*), where v^* is defined by

$$v^*(S) := v(N) - v(N \backslash S) \text{ for all } S \subseteq N.$$

It is easily shown that $C(N, v) = C^*(N, v^*)$.

Next, we recall the definition of the egalitarian solution of Dutta and Ray (1989). For this we need some more notation, most of which is due to Dutta and Ray (1989). Let S be a non-empty subset of $\{1, \dots, n\}$. We denote the cardinality of S by $|S|$. To avoid unnecessary notational complications we denote any $x \in \mathbb{R}^S$ by $x = (x_1, \dots, x_{|S|})$. For two vectors x and y in $\mathbb{R}^S$, we write $x = y$ if all their components are equal, and $x > y$ if $x_i \geq y_i$ for all $i = 1, \dots, |S|$, with strict inequality for some i. For any $x \in \mathbb{R}^S$, we denote by $\hat{x}$ the vector obtained by permuting the indices of x such that $\hat{x}_1 \geq \hat{x}_2 \geq \cdots \geq \hat{x}_{|S|}$. For $x \in \mathbb{R}^S$ and $T \subseteq S$, we denote the projection of x on $\mathbb{R}^T$ by $x(T) = (x_i)_{i \in T}$.

Let (N, v) be a game. An allocation $x \in \mathbb{R}^S$ is *feasible* for S if $\sum_{i \in S} x_i = v(S)$. The *Lorenz map* E is defined on the domain

$$\mathcal{A} := \left\{ A : A \subseteq \mathbb{R}^k \text{ for some } k, \text{ and there exists } \lambda \in \mathbb{R} \text{ such that } \sum_{i=1}^{k} x_i = \lambda \text{ for all } x \in A \right\}.$$

[1] $S \subseteq N$ denotes that S is a subset of N and $S \subset N$ denotes that S is a strict subset of N.

For each set $A \in \mathcal{A},\ EA$ is the set of all allocations in A that are *Lorenz undominated* within A. Formally,

$$EA := \left\{x \in A : \text{ there is no } y \in A \text{ such that } \textstyle\sum_{i=1}^{j} \hat{y}_i \leq \sum_{i=1}^{j} \hat{x}_i \text{ for all } j = 1, \ldots, k, \text{ with strict inequality for some } j\right\}.$$

The *Lorenz core* is defined inductively as follows. The Lorenz core of a singleton coalition is defined by $L(\{i\}, v) := \{v(\{i\})\}$. Now suppose that the Lorenz cores for all coalitions of cardinality k or less have been defined, where $1 \leq k < n$. The Lorenz core of a coalition of size $k+1$ is defined by

$$L(S, v) := \left\{x \in \mathbb{R}^S : x \text{ is feasible for } S, \text{ and there is no } T \subset S \text{ and } y \in EL(T, v) \text{ such that } y > x(T)\right\}.$$

If $x \in \mathbb{R}^S$ and there is $T \subset S$ and $y \in EL(T, v)$ such that $y > x(T)$, then we say that y *Lorenz-blocks* (L-blocks) x. We shall also say in this case that T L-blocks x. Furthermore, note that $C(S, v) \subseteq L(S, v)$.

Dutta and Ray (1989) called the set of Lorenz undominated allocations in the Lorenz core of (N, v), i.e., $EL(N, v)$, the *egalitarian solution*. They proved that $EL(N, v)$ is either empty or a singleton. If it is non-empty, then its unique element is called the *egalitarian allocation* of the game (N, v). For convex games Dutta and Ray (1989) describe a finite algorithm to locate the unique egalitarian allocation. Moreover, they showed that the egalitarian allocation is an element of the core of (N, v) and that it Lorenz dominates all other core allocations.

3 The dual egalitarian solution

In this section, we will introduce a dual of the egalitarian solution of Dutta and Ray (1989), thereby following closely the ideas of Dutta and Ray (1989) concerning egalitarianism. Theorem 3.2 shows that the dual egalitarian solution is either a singleton or the empty set. After that, we study the relations with the egalitarian solution of Dutta and Ray (1989) and the core of the game. Next, we introduce an algorithm similar to that of Dutta and Ray (1989) to calculate the dual egalitarian solution for concave games. We show that the dual egalitarian solution is in the dual core of the concave game and that the dual egalitarian solution Lorenz dominates all other allocations in the dual core. Finally, we present our main result: for a convex game the egalitarian allocation is equal to the dual egalitarian allocation for its dual (concave) game.

For starters, we need some extra notation. Let $S \subseteq \{1, \ldots, n\}$. For any $x \in \mathbb{R}^S$, we denote by $\bar{x}$ the vector obtained by permuting the indices of x such that $\bar{x}_1 \leq \bar{x}_2 \leq \cdots \leq \bar{x}_{|S|}$. Let (N, v) be a game. The *dual Lorenz map* D is defined on the same domain as the Lorenz map E, i.e.,

$$\mathcal{A} = \left\{A : A \subseteq \mathbb{R}^k \text{ for some } k, \text{ and there exists } \lambda \in \mathbb{R} \text{ such that } \sum_{i=1}^{k} x_i = \lambda \text{ for all } x \in A\right\}.$$

For each such set $A \in \mathcal{A},\ DA$ is the set of all allocations in A that are *dual Lorenz undominated* within A. Formally,

$$DA := \Big\{x \in A : \text{ there is no } y \in A \text{ such that } \textstyle\sum_{i=1}^{j} \bar{y}_i \geq \sum_{i=1}^{j} \bar{x}_i \text{ for all } j = 1, \ldots, k, \text{ with strict inequality for some } j\Big\}.$$

The *dual Lorenz core* of a singleton coalition is defined by $L^*(\{i\}, v) := \{v(\{i\})\}$. Now suppose that the dual Lorenz cores for all coalitions of cardinality k or less have been defined, where $1 \leq k < n$. The dual Lorenz core of a coalition of size $k+1$ is defined by

$$L^*(S, v) := \Big\{x \in \mathbb{R}^S : x \text{ is feasible for } S, \text{ and there is no } T \subset S \text{ and } y \in DL^*(T, v) \text{ such that } y < x(T)\Big\}.$$

If $x \in \mathbb{R}^S$ and there is $T \subset S$ and $y \in DL^*(T, v)$ such that $y < x(T)$, then we say that y *dual Lorenz-blocks* (L^*-blocks) x. We shall also say in this case that T L^*-blocks x. Furthermore, note that $C^*(S, v) \subseteq L^*(S, v)$.

We call the set of dual Lorenz undominated allocations in the dual Lorenz core of (N, v), i.e., $DL^*(N, v)$, the *dual egalitarian solution*. Next, we will show that $DL^*(N, v)$ is either empty or a singleton. Hence, if $DL^*(N, v)$ is non-empty, its unique element will be called the *dual egalitarian allocation* of the game (N, v). But first we show that the operators D and E coincide.

Lemma 3.1 *The operators D and E coincide.*

Proof. Let $A \in \mathcal{A}$. Then there is a coalition $S = \{1, \ldots, k\}$ and a number $\lambda \in \mathbb{R}$ such that $\sum_{i=1}^{k} z_i = \lambda$ for all $z \in A$. Let $z \in A$. Consider the vectors $\hat{z}$ and $\bar{z}$. Recall that $\hat{z}_1 \geq \hat{z}_2 \geq \cdots \geq \hat{z}_k$ and $\bar{z}_1 \leq \bar{z}_2 \leq \cdots \leq \bar{z}_k$. It is easily verified that for all $i = 1, \ldots, k$ we have $\bar{z}_i = \hat{z}_{k-i+1}$.

Let $x, y \in A$. From the observation above and $\sum_{i=1}^{k} x_i = \lambda = \sum_{i=1}^{k} y_i$ it follows that the following conditions are equivalent:

$$\sum_{i=1}^{j} \bar{y}_i \geq \sum_{i=1}^{j} \bar{x}_i \text{ for all } j = 1, \ldots, k \text{ with a strict inequality for some } j \quad \Leftrightarrow$$

$$\sum_{i=1}^{j} \bar{y}_i \geq \sum_{i=1}^{j} \bar{x}_i \text{ for all } j = 1, \ldots, k-1 \text{ with a strict inequality for some } j \quad \Leftrightarrow$$

$$\sum_{i=j+1}^{k} \bar{y}_i \leq \sum_{i=j+1}^{k} \bar{x}_i \text{ for all } j = 1, \ldots, k-1 \text{ with a strict inequality for some } j \quad \Leftrightarrow$$

$$\sum_{i=j+1}^{k} \hat{y}_{k-i+1} \leq \sum_{i=j+1}^{k} \hat{x}_{k-i+1} \text{ for all } j = 1, \ldots, k-1 \text{ with a strict inequality for some } j \quad \Leftrightarrow$$

$$\sum_{i=1}^{j} \hat{y}_i \leq \sum_{i=1}^{j} \hat{x}_i \text{ for all } j = 1, \ldots, k-1 \text{ with a strict inequality for some } j \quad \Leftrightarrow$$

$$\sum_{i=1}^{j} \hat{y}_i \leq \sum_{i=1}^{j} \hat{x}_i \text{ for all } j = 1, \ldots, k \text{ with a strict inequality for some } j$$

From this it immediately follows that $DA = EA$. □

From Lemma 3.1 it follows that dual Lorenz domination coincides with Lorenz domination. Henceforth, we will use the notation E instead of D. The proof of the next Theorem is omitted since it runs completely analogously to the proof of Theorem 1 of Dutta and Ray (1989).

Theorem 3.2 *There is at most one dual egalitarian allocation.*

The following examples from Dutta and Ray (1989) give an idea about the relations between the egalitarian solution, the dual egalitarian solution, and the core of the game.

The game (N, v) in the first example (Example 1 from Dutta and Ray (1989)) is totally balanced game with $EL(N, v) = \emptyset$, but $EL^*(N, v^*) \neq \emptyset$.

Example 3.3 Let $N = \{1, 2, 3\}$, $v(\{i\}) = 0$ for all $i \in N$, $v(\{1, 2\}) = v(\{1, 3\}) = v(N) = 1$, and $v(\{2, 3\}) = 0$. Dutta and Ray (1989) showed that $EL(N, v) = \emptyset$. Using that $L^*(\{1, 2\}, v^*)$ $= (1, 0)$, $L^*(\{1, 3\}, v^*) = (1, 0)$, and $L^*(\{2, 3\}, v^*) = \emptyset$ one easily verifies that $L^*(N, v^*) =$ $(1, 0, 0)$ and, hence, $EL^*(N, v^*) = \{(1, 0, 0)\}$. Note that the game (N, v) is not convex, since $v(\{1, 2\}) - v(\{1\}) = 1 > 0 = v(\{1, 2, 3\}) - v(\{1, 3\})$. $\diamond$

The game (N, v) in the second example (Example 2 from Dutta and Ray (1989)) is not convex. It holds that $EL(N, v) \neq \emptyset$, $EL^*(N, v^*) = \emptyset$, and $C(N, v) = \emptyset$.

Example 3.4 Let $N = \{1, 2, 3\}, v(\{1\}) = 0, v(\{2\}) = v(\{3\}) = 1$, $v(\{1, 2\}) = v(\{1, 3\}) =$ 1.4, and $v(\{2, 3\}) = v(N) = 2.2$. Dutta and Ray (1989) observed that $EL(N, v) = \{(0, 1.1,$ $1.1)\}$ and $C(N, v) = \emptyset$. It is easy to verify that $EL^*(\{1\}, v^*) = \{(0)\}$ and $EL^*(\{2\}, v^*) =$ $EL^*(\{3\}, v^*) = \{(0.8)\}$. Since $v^*(N) = 2.2$, it follows that $EL^*(N, v^*) = \emptyset$. $\diamond$

Finally, we consider Example 3 from Dutta and Ray (1989). In this example we have a non-convex game (N, v) with $EL(N, v) = EL^*(N, v^*) \neq \emptyset$, $C(N, v) \neq \emptyset$, and $EL(N, v) \cap$ $C(N, v) = \emptyset$,

Example 3.5 Let $N = \{1, 2, 3, 4\}, v(\{i\}) = 0$ for all $i \in N$, $v(N) = 2, v(\{2, 3\}) = 1.05$, $v(\{3, 4\}) = 1.9$, and for all other S, $v(S)$ is the minimal superadditive function compatible with these values. Dutta and Ray (1989) observed that $EL(N, v) = \{(0.05, 0.05, 0.95, 0.95)\} \notin$ $C(N, v) \neq \emptyset$. One can verify that $EL^*(N, v^*) = EL(N, v)$. $\diamond$

Next, we describe an algorithm for locating the (unique) dual egalitarian allocation in a concave game. The algorithm is analogous to the algorithm of Dutta and Ray (1989) for locating the egalitarian allocation in a convex game. Denote the *average worth* of coalition S with respect to a characteristic function v' by

$$a(S, v') := \frac{v'(S)}{|S|}.$$

Let (N, w) be a concave game. Define $N_1 := N$ and $w_1 := w$.

STEP 1: Let T_1 be the largest coalition with the *lowest* average worth in the game (N, w_1). Define

$$x_i^*(N, w) := a(T_1, w_1) \qquad \text{for all } i \in T_1. \tag{1}$$

STEP k: Suppose that $T_1, \ldots, T_{k-1}$ have been defined recursively and $T_1 \cup \cdots \cup T_{k-1} \neq N$. Define a new game with player set $N_k := N \backslash (T_1 \cup \cdots \cup T_{k-1})$. For all subcoalitions $T \subseteq N_k$, define $w_k(T) := w_{k-1}(T_{k-1} \cup T) - w_{k-1}(T_{k-1})$. The game (N_k, w_k) is concave since (N_{k-1}, w_{k-1}) is concave. Define T_k to be the largest coalition with the lowest average worth in this game. Define

$$x_i^*(N, w) := a(T_k, w_k) \qquad \text{for all } i \in T_k. \tag{2}$$

We remark that the concavity of the game (N_k, w_k) ensures that there is a largest coalition with the lowest average worth in (N_k, w_k). Let x^* be the allocation defined by equations (1) and (2).

Theorem 3.6 *In a concave game (N, w), x^* as constructed by the algorithm above is the unique dual egalitarian allocation. Moreover, x^* is in the dual core $C^*(N, w)$ and Lorenz dominates every allocation in the dual core.*

The proof of Theorem 3.6 is omitted since it runs completely analogously to the proofs of Theorem 2 and 3 of Dutta and Ray (1989).

From Examples 3.3, 3.4, and 3.5 it follows that $EL(N, v)$ does not need to coincide with $EL^*(N, v^*)$. For convex games, however, they do coincide. This is our main result:

Theorem 3.7 *For a convex game (N, v) it holds that $EL(N, v) = EL^*(N, v^*)$.*

Proof. Let (N, v) be a convex game. Dutta and Ray (1989) showed that $EL(N, v) \subseteq C(N, v)$ and that the unique egalitarian solution Lorenz dominates all other core allocations in $C(N, v)$. Note also that (N, v^*) is a concave game, and hence, according to Theorem 3.6, we have that $EL^*(N, v) \subseteq C^*(N, v^*)$. Moreover, the unique dual egalitarian solution Lorenz dominates all other dual core allocations in $C^*(N, v)$. Now the Theorem follows from the observation that $C(N, v) = C^*(N, v^*)$. □

Given Theorem 3.7 it is not difficult to axiomatically characterize the dual egalitarian solution on the class of concave games. The procedure runs along the lines of Funaki (1994) and consists of dualizing the axioms that characterize the egalitarian solution. We refer to Funaki (1994) for the details. Thus, dualizing the axioms in Dutta (1990) and Klijn *et al.* (2000) yields characterizations of the dual egalitarian solution. Since this procedure is straightforward we have omitted it.

We conclude with the following open question: if for a game (N, v) it holds that $EL(N, v) \neq \emptyset$ and $EL^*(N, v^*) \neq \emptyset$, then $EL(N, v) = EL^*(N, v^*)$? We know by Theorem 3.7 that if (N, v) is convex this is true. Note also that by Example 3.5 $EL(N, v)$ and $EL^*(N, v^*)$ can be both non-empty for non-convex games (N, v).

References

Aumann, R. and M. Maschler (1985) “Game Theoretic Analysis of a Bankruptcy Problem from the Talmud” *Journal of Economic Theory* **36**, 195-213.

Bird, C. (1976) “On Cost Allocation for a Spanning Tree: A Game Theoretic Approach” *Networks* **6**, 335-350.

Curiel, I., Pederzoli, G., and S. Tijs (1989) “Sequencing Games” *European Journal of Operational Research* **40**, 344-351

Dutta, B. (1990) “The Egalitarian Solution and Reduced Game Properties in Convex Games” *International Journal of Game Theory* **19**, 153-169.

Dutta, B. and D. Ray (1989) “A Concept of Egalitarianism under Participation Constraints” *Econometrica* **57**, 615-635.

Funaki, Y. (1994) “Dual Axiomatizations of Solutions of Cooperative Games” Working Paper 13, Faculty of Economics, Toyo University, Tokyo, Japan.

Klijn, F., Slikker, M., Tijs, S., and J. Zarzuelo (2000) “The Egalitarian Solution for Convex Games: Some Characterizations” it Mathematical Social Sciences **40**, 111-121.

Littlechild, S. and G. Owen (1973) “A Simple Expression for the Shapley-Value in a Special Case” *Management Science* **20**, 370-372.

O’Neill, B. (1982) “A Problem of Rights Arbitration from the Talmud” *Mathematical Social Sciences* **2**, 345-371.

Owen, G. (1975) “On the Core of Linear Production Games” *Mathematical Programming* **9**, 358-370.

Schmeidler, D. (1969) “The Nucleolus of a Characteristic Function Game” *SIAM Journal on Applied Mathematics* **17**, 1163-1170.

Shapley, L. (1953) “A Value for n-Person Games” in *Contributions to the Theory of Games II*, *Annals of Mathematics Studies* **28** by H. Kuhn and A. Tucker, Eds., Princeton University Press: Princeton, 307-317.

Shapley, L. and M. Shubik (1969) “On Market Games” *Journal of Economic Theory* **1**, 9-25.

Sudhölter, P. (1997) “The Modified Nucleolus: Properties and Axiomatizations” *International Journal of Game Theory* **26**, 147-182.

Tijs, S. (1986) Coursebook Game Theory, University of Nijmegen, Nijmegen, The Netherlands.