E C O N O M I C S   B U L L E T I N

Inequality measurement and the leaky−bucket paradox

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Abstract

The transfer principle requires inequality measures to decrease for mean preserving contractions. How much leakage of transfers can preserve inequality? Conditions are shown for leaky transfers to preserve inequality. We find that positive remainders with positive or negative leakage as well as negative remainders with positive leakage may occur. This constitutes the leaky−bucket paradox.

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1. Introduction

The transfer principle requires an income inequality measure to decrease if a rank preserving transfer of a richer to a poorer income recipient occurs. It was already alluded to by Pareto (1897, p. 320), formulated by Pigou (1912, p. 24), and rekindled as a clear postulate on inequality measures by Dalton (1920, p. 351).

Most inequality measures satisfy the transfer principle, i.e., they decrease as a consequence of a transfer from a richer to a poorer income recipient. Suppose that, on its way from the transferer to the transferee, the transfer suffers from leakage (e.g., transaction cost, gift tax, etc.). Then the question arises how much leakage of the transfer is tolerated to maintain the degree of inequality. While, prima facie, one would expect that the transferee would experience some betterment, there exist inequality measures which require the transferee, too, to forfeit some income to restore the former degree of inequality. It is shown that the Schutz coefficient is of this type, whereas the Gini coefficient is a real joker among inequality measures. It does not even exclude negative leakage. Note that the analysis for leaky transfers is more complicated than dealing just with the transfer principle, as the mean of the income distribution changes, too, for leaky transfers.

2. Leaky–bucket operations and inequality measurement

Let $y = (y_1, y_2, ..., y_n)$ denote the incomes of a population and let them be arranged in a nondecreasing order. Let μ denote their mean. Furthermore, let $I : \mathbb{R}^n_+ \rightarrow [0, 1]$ denote an inequality measure, where $I(\mu, ..., \mu] = 0$ indicates a perfectly equal income distribution, and $I(0, 0, \ldots, 0, n\mu) \approx 1$ a perfectly unequal income distribution. Let $\tilde{y}$ denote a mean and rank preserving contraction of y and let $\tilde{y}_z$ denote a mean and rank preserving contraction of y such that income is transferred from $y_j > z$ to an income recipient i with income $y_i < z$.

Definition 1. An inequality measure I satisfies the strong [weak] principle of transfers if $I(\tilde{y}) < I(y)$ [$I(\tilde{y}) \leq I(y)$].

Definition 2 [Castagnoli and Muliere (1990, p. 177)]. An inequality measure I satisfies the strong [weak] z–modified principle of transfers if $I(\tilde{y}_z) < I(y)$ [$I(\tilde{y}_z) \leq I(y)$].

Notice that the principle of transfers implies the z–modified principle of transfers, but not vice versa. Henceforth we will restrict our attention only to inequality measures which satisfy at least one of the two transfer principles.

Of course, μ is at the same time also the mean of $\tilde{y}$ and $\tilde{y}_z$. It is tempting to entertain a leaky–bucket mental image and look for the tolerance of inequality measures for leaks in making transfers between two individuals.[1] As inequality decreases [does not increase for the weak forms] as a consequence of the principle of transfers, we can ask how much of the money taken away from the tranferer may be siphoned off such that the remainder, given to the transferee, keeps the degree of the income inequality intact.[2] We shall examine whether this remainder is necessarily positive to maintain the same degree of income inequality.

We shall exemplify our analysis by considering two principal rank dependent inequality measures satisfying the principle of transfers and the z–modified principle of transfers, respectively, viz. the Gini coefficient and the Schutz coefficient.[3]

For the Gini coefficient, originally proposed by Gini (1912), we will use the form put forward by Sen (1973, p. 31):

$$G = 1 + \frac{1}{n} - \frac{2}{n^2\mu}\sum_{k=1}^{n}(n+1-k)y_k. \tag{1}$$

This form reveals an important value judgement behind the Gini coefficient: Higher–ranked income recipients are assigned lower (rank dependent) weights, and lower–ranked income recipients are assigned higher weights. Income inequality is the more decreased the lower the rank of an income recipient is, who gets an additional amount of money.

An ancestor of the Schutz coefficient was originally proposed by von Bortkiewicz (1898, p. 1209) [cf. also von Bortkiewicz (1931, pp. 204-19)] as the mean absolute deviation about the mean. Bresciani–Turroni (1910) modified it as the relative mean deviation. Some years later it was rediscovered

[1] Creedy and Hurn (1999), p. 249.

[2] If the level of social welfare should be maintained instead, we get into the realm of ethical inequality measures [cf. Chakravarty (1990)]. Their crucial concept is the equally distributed equivalent income, η, which is defined as that equally distributed income which confers the same social welfare as does y. As a rule $\eta < \mu$, which gave rise to the Dalton–Atkinson index of inequality $I_{DA} = 1 - \eta/\mu$ [Atkinson (1970)]. I_{DA} represents the maximum aggregate leakage of transfers from richer to poorer income recipients that is tolerable to maintain social welfare, expressed as a fraction of aggregate pre–transfer income. Thus, I_{DA} is a generalized leaky–bucket measure. Ethical inequality measures are not dealt with in this paper.

[3] Both measures belong to the family of rank–dependent inequality measures. These capture the notion of *relative deprivation* for inequality measurement. Relative deprivation contends that an individual feels more deprived the more individuals are ahead of him or her on the income ladder, where higher ranks in the income ladder imply more deprivation. Relative deprivation was introduced into the social sciences by Stouffer et al. (1949), and further elaborated by Runciman (1966). It was first applied by Sen (1976) to an area such as poverty measurement.

by Schutz (1951):

$$S = \frac{1}{2n\mu}\sum_{k=1}^{n}|\mu - y_k| = \frac{1}{n\mu}\sum_{y_k \leq \mu}(\mu - y_k). \tag{2}$$

The Schutz coefficient measures the proportion of total income which would have to be transferred from above–mean to below–mean incomes in order to attain perfect income inequality.

Direct inspection shows that the Gini coefficient satisfies the principle of transfers, while the Schutz coefficient satisfies only the μ–modified principle of tranfers. For the Gini coefficient this is accomplished because of the lower weight assigned to the transferer and the higher weight assigned to the transferee, for the Schutz coefficient by the lower income gap of the transferee. Note that μ remains intact in both cases.

However, leaky–bucket operations impinge also upon the mean of the income distribution after the truncated transfer has taken place. This makes things somewhat more complicated.

Proposition 1. The Schutz coefficient satisfies leaky–bucket operations for μ–modified transfers only with negative remainders.

Proof. Subtract $\Delta_j > 0$ from $y_j > \mu$ and look for Δ_i to be added to $y_i < \mu$ such that the Schutz coefficient does not change. In particular:

$$\frac{\sum_{y_k \leq \mu}(\mu - y_k)}{n\mu} \doteq \frac{\sum_{y_k \leq \mu}(\mu - \frac{\Delta_j}{n} + \frac{\Delta_i}{n} - y_k) - \Delta_i}{n\mu - \Delta_j + \Delta_i} \tag{3}$$

Denote $\# \{y_k \mid y_k \leq \mu\} := p$, we have, after a rearrangement of (3):

$$\Delta_i = \frac{\sum_{y_k \leq \mu}(\mu - y_k) - \mu p}{\sum_{y_k \leq \mu}(\mu - y_k) + (n-p)\mu}\Delta_j \tag{4}$$

Yet $\sum_{y_k \leq \mu}(\mu - y_k) - \mu p = -\sum_{y_k \leq \mu} y_k < 0$ and $\sum_{y_k \leq \mu}(\mu - y_k) + (n-p)\mu > 0$, because $n > p$.

Therefore, the fraction in (4) has a negative sign, which means

$$\Delta_i < 0 < \Delta_j. \tag{5}$$

Thus, the Schutz coefficient satisfies leaky–bucket operations only with negative remainders. The degree of inequality as measured by the Schutz coefficient is maintained only if the would–be transferee, too, is placed in a worse income position.

■

What explains the leaky–bucket paradox for the Schutz coefficient? When an above–mean income recipient loses income, the Schutz coefficient decreases because mean income decreases, which affects the numerator in (2) more than the denominator. When a below–mean income recipient gains income, the Schutz coefficient decreases again because aggregate penury has decreased. Thus, but a smaller share of total income is required to wipe out income inequality. Both effects of a μ–modified transfer reinforce such that no $\Delta_i > 0$ exists which can restore the degree of inequality as measured by the Schutz coefficient. Leaky–bucket operations which maintain the degree of inequality in the Schutz sense invariably lead to Pareto–inferior states.

For the next proposition define

$$C := \frac{\sum_{k=1}^{n} k y_k}{n\mu} = \frac{1}{2}[n(G+1)+1]. \tag{6}$$

Negative leakage obtains if transfers require additional subsidies to preserve the degree of inequality, which means that the transferee's extra income has to exceed the very transfer itself.

Proposition 2. The Gini coefficient satisfies leaky–bucket operations

(i) *with positive remainders and positive leakage iff $i < j < C$;*

(ii) *with positive remainders and negative leakage iff $j > i > C$;*

(iii) *with negative remainders and positive leakage iff $j > C > i$.*

(iv) *For $i = C$, no leaky–bucket operations exist.*

Proof. Substract $\Delta_j > 0$ from y_j and look for Δ_i to be added to y_i, $i < j$, such that the Gini coefficient does not change. In particular:

$$\frac{2}{n^2\mu}A \doteq \frac{2}{n^2\mu - n\Delta_j + n\Delta_i}[A + (n+1-i)\Delta_i - (n+1-j)\Delta_j], \tag{7}$$

where $A := \sum_{k=1}^{n}(n+1-k)y_k$. Rearrangement of (7) gives

$$\Delta_i = \frac{A - n\mu(n+1-j)}{A - n\mu(n+1-i)}\Delta_j. \tag{8}$$

Note that

$$A - n\mu(n+1-j) = jn\mu - \sum_{k=1}^{n} ky_k; \tag{9}$$

$$A - n\mu(n+1-i) = in\mu - \sum_{k=1}^{n} ky_k. \tag{10}$$

Furthermore, note that $A - n\mu(n+1-j) > 0$ for $j = n$, and $A - n\mu(n+1-i) < 0$ for $i = 1$, and that

$$A - n\mu(n+1-j) > A - n\mu(n+1-i). \tag{11}$$

(i) Let $A - n\mu(n+1-j) < 0$ and $A - n\mu(n+1-i) < 0$.

Then $i < j < C$ and

$$0 < \frac{A - n\mu(n+1-j)}{A - n\mu(n+1-i)} < 1. \tag{12}$$

Thus, (8) shows that $0 < \Delta_i < \Delta_j$, which proves (i).

(ii) Let $A - n\mu(n+1-j) > 0$ and $A - n\mu(n+1-i) > 0$.

Then $j > i > C$ and, by (11),

$$\frac{A - n\mu(n+1-j)}{A - n\mu(n+1-i)} > 1. \tag{13}$$

Thus, (8) shows that $\Delta_i > \Delta_j > 0$, which proves (ii).

(iii) Let $A - n\mu(n+1-j) > 0$ and $A - n\mu(n+1-i) < 0$.

Then $j > C > i$ and

$$\frac{A - n\mu(n+1-j)}{A - n\mu(n+1-i)} < 0. \tag{14}$$

Thus, (8) shows that $\Delta_j > 0 > \Delta_i$, which proves (iii).

(iv) Finally, suppose $A - n\mu(n+1-i) = 0$. Then (8) shows that no feasible Δ_i exists for $\Delta_j > 0$, which concludes the proof.

■

Proposition 2 shows that the Gini coefficient is a joker with respect to leaky–bucket operations. The expected properties of leaky–bucket operations obtain only when transfers occur between two members of the lower income strata. There, the maintenance of the degree of inequality involves some betterment for the transferee and a positive leakage which may be siphoned off for some other purposes.

However, if transfers occur between two members of the upper income strata, the transferee has to be given more than the transfer itself in order to restore the former degree of income inequality. Thus, the leakage of the transfer has to become negative to preserve the same income inequality. The remainder is positive and exceeds the very transfer.

If the transferer is a member of the upper income strata and if the transferee belongs to the lower income strata, then remainders have to become negative to preserve the same degree of income inequality: The poor income recipient, too, has to be made worse off in order to restore the degree of income inequality. Leaky–bucket operations imply Pareto inferiority in this case. This part of the result is akin to the effect of μ–modified transfers on the Schutz coefficient.

Finally, we cannot exclude that leaky–bucket operations become infeasible at all if i happens to be equal to C.

3. Conclusion

This paper has shown that leaky–bucket operations perform as expected for the Gini coefficient, i.e., involving positive remainders and positive leakage to preserve the degree of inequality, only if transfers are confined to the lower strata of the income ladder. Transfers among high–ranked income recipients require positive remainders for the transferee which exceed the transfer itself, thus causing the leakage to become negative. For this lucky transferee the familiar biblical quotation applies "to him that hath shall be given". The most disadvantaged persons are low–ranked transferees whose partner to the transfer happens to be a high–ranked income recipient. Both the Schutz coefficient and the Gini coefficient require that this transferee even has to forfeit some of his or her previous income, if the degree of income inequality is to be preserved. In these cases the leakage exceeds the transfer, thus causing the remainder for the transferee to become negative. Ignoring possible welfare–enhancing effects of the proceeds of the transfer for at least one of the partners to the transfer, transfers of this kind are Pareto inferior. Finally, instances exist where leaky–bucket transfers are infeasible to preserve the degree of inequality as measured by the Gini coefficient.

Experimental research on the perception of leaky–bucket operations is still in its infancy and no systematic results are available. Sporadical results of experimental research evidence as yet positive remainders for lower levels of total income. For higher levels of total income there are instances of the occurrance of negative remainders.[4] The structure of leaky–bucket operations which preserve the perceived degree of income inequality as expounded in Propositions 1 and 2 still awaits experimental tests. Their results should be able to evaluate the descriptive goodness of the Gini coefficient and the Schutz coefficient to capture the perception of income inequality.

[4]Cf. Creedy and Hurn (1999), p. 251. If the results of Glejser, Gevers, Lambot, and Morales (1977) are interpreted in terms of remainders, their positive remainder for the (7, 10) case is 0.765, whereas their (14, 18) case shows a negative remainder of 0.2838.

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