ECONOMICS BULLETIN

A generalized envelope theorem with an application to congestion−prone facilities

Marvin Kraus
Boston College

Abstract

A generalized envelope theorem is presented which has the Envelope Theorem as a special case. Relative to the Envelope Theorem, it provides greater flexibility in determining the rate of change of a value function with respect to one of its arguments. We revisit a classic result on economies of scale for a congestion−prone facility, using the flexibility of the Generalized Envelope Theorem to provide a simpler, more intuitive proof.

I am grateful to an anonymous referee for valuable suggestions.
Citation: Kraus, Marvin, (2002) "A generalized envelope theorem with an application to congestion−prone facilities." *Economics Bulletin,* Vol. 3, No. 28 pp. 1−4
Submitted: August 5, 2002. **Accepted:** December 4, 2002.
URL: http://www.economicsbulletin.com/2002/volume3/EB−02C60005A.pdf

1. Introduction

Applied to cost curves, the Envelope Theorem (see, e.g., Simon and Blume (1994)) states that a firm's long run total cost (LRTC) curve is the envelope of its family of short run total cost curves, where for the sake of illustration, we characterize the short run by some fixed employment of capital. LRTC also envelopes families of (short run) total cost curves for which capital does vary with output, but nonoptimally. But because the statement of the Envelope Theorem does not cover this, it is not as widely appreciated. In this note, we provide a more general statement of the Envelope Theorem which in the case of the cost curve application is more inclusive with respect to short run cost curves, and demonstrate its usefulness in practical applications. Specifically, we show that it permits a simpler, more intuitive approach than would otherwise be possible to the problem of analyzing economies of scale for the output of a congestible facility.

2. The Generalized Envelope Theorem

We begin with a statement of the standard Envelope Theorem.

Envelope Theorem. Let $\phi(y,k)$ be a C^1 function of $k \in R^n$ and the scalar y, and consider the problem $\min_k \phi(y,k)$. Let $k^*(y)$ be a minimizer, and assume that $k^*(y)$ is a C^1 function. Define $v(y) = \phi(y, k^*(y))$, and let y_o be some particular value of y. Then

$$\frac{dv(y_o)}{dy} = \frac{\partial\phi(y_o, k^*(y_o))}{\partial y}.$$

In the cost curve application, y_o is some particular level of output, k is employment of capital, $\phi(\cdot)$ is a short run total cost function, and the value function $v(\cdot)$ is the long run total cost function. The Envelope Theorem asserts that long run marginal cost at y_o can be evaluated as short run marginal cost at y_o, provided the latter is determined from a short run total cost relationship for which capital is held fixed at the level $k^*(y_o)$ which minimizes cost for y_o.

Continuing with the cost curve application, now consider the cost relationship $w(y) \equiv \phi(y, k(y))$, where $k(y)$ is any C^1 function such that $k(y_o) = k^*(y_o)$. The fact that $k(y)$ is smooth and satisfies $k(y_o) = k^*(y_o)$ means that $v(y)$ envelopes $w(y)$, and that $w'(y_o)$ provides an alternative way of evaluating long run marginal cost at y_o. In general,

Generalized Envelope Theorem. Let $\phi(y,k)$ be a C^1 function of $k \in R^n$ and the scalar y, and consider the problem $\min_k \phi(y,k)$. Let $k^*(y)$ be a minimizer, and assume that $k^*(y)$ is a C^1 function. Let y_o be some particular value of y, and let $k(y)$ be any C^1 function such that $k(y_o) = k^*(y_o)$. Define $v(y)$ and $w(y)$ by $v(y) = \phi(y, k^*(y))$ and $w(y) = \phi(y, k(y))$. Then

$$\frac{dv(y_o)}{dy} = \frac{dw(y_o)}{dy}.$$

Proof. $\frac{dv(y_o)}{dy} = \frac{\partial\phi(y_o, k^*(y_o))}{\partial y} + \sum_{i=1}^{n} \frac{\partial\phi(y_o, k^*(y_o))}{\partial k_i} \cdot \frac{dk_i^*(y_o)}{dy} = \frac{\partial\phi(y_o, k^*(y_o))}{\partial y}$, since $\frac{\partial\phi(y_o, k^*(y_o))}{\partial k_i} = 0$ for $i = 1, ..., n$ by the first order conditions. At the same time,

$$\frac{dw(y_o)}{dy} = \frac{\partial\phi(y_o, k(y_o))}{\partial y} + \sum_{i=1}^{n} \frac{\partial\phi(y_o, k(y_o))}{\partial k_i} \cdot \frac{dk_i(y_o)}{dy}$$

$$= \frac{\partial\phi(y_o, k^*(y_o))}{\partial y} + \sum_{i=1}^{n} \frac{\partial\phi(y_o, k^*(y_o))}{\partial k_i} \cdot \frac{dk_i(y_o)}{dy}, \qquad (1)$$

since $k(y_o) = k^*(y_o)$. Again using the first order conditions $\frac{\partial\phi(y_o, k^*(y_o))}{\partial k_i} = 0$ for $i = 1, ..., n$, (1) reduces to

$$\frac{dw(y_o)}{dy} = \frac{\partial\phi(y_o, k^*(y_o))}{\partial y}$$

$$= \frac{dv(y_o)}{dy}. \qquad \text{Q.E.D.}$$

Remarks. The standard Envelope Theorem is the special case of the preceding theorem in which the function $k(y)$ used to define $w(y)$ is the constant function for which $k(y) = k^*(y_o)$ for all y. The theorem is a generalization of the Envelope Theorem for unconstrained static optimization problems. A corresponding generalization is possible for envelope theorems for constrained static optimization problems, including those with inequality constraints, as well as dynamic optimization problems.

3. Application of the Theorem

In this section, we will demonstrate the usefulness of the theorem for the problem of identifying what type of local returns to scale characterize the output of a congestible facility. The reason why this problem is important is that the type of local returns to scale determines whether the facility will incur a deficit or a surplus (or break even) under the socially optimal price and capacity (Mohring and Harwitz (1962), Strotz (1965)). Since the optimum is achieved with long run marginal cost pricing, the result is essentially the standard result on the profitability of long run marginal cost pricing extended to commodities for which consumers play a producing role by providing some inputs directly. The best-known example is consumer-supplied travel time in the case of highway trips.

The most common cost specification for congestible facilities is to assume a short run total cost relationship of the form

$$\phi(y, k) = yf(y/k) + g(k), \qquad (2)$$

where y is output and k is a scalar measure of capacity. $g(k)$ is the cost of providing a capacity of k, and $f(y/k)$ which is often called user cost, is short run average variable cost. Equation (2) is used in the standard highway model, in which y is trip output and $f(\cdot)$ includes the time cost of a trip. Equation (2) applies to any congestible facility for which user cost is scale-invariant in the sense that if twice the output is produced with twice the capacity, there is no effect on user cost.

It is well known in the highway literature that, under the structure in (2), the elasticity of long run total cost with respect to output is alternatively less than, equal to, or greater than one according to whether there are increasing, constant, or decreasing returns to providing capacity. The intuition for this result is that short run total variable cost (i.e., total user cost) is homogeneous of degree one in y and k, working neither towards economies nor diseconomies of scale, leaving the outcome to the nature of returns to scale in capacity provision. The actual proof of the result runs as follows, where $k^*(y)$ and $v(y)$ are as previously defined, $E_{v:y}$ denotes the elasticity of long run total cost with respect to y, and *sgn* before an expression indicates its sign:

$$sgn(E_{v:y} - 1) = sgn(v'(y)y/v(y) - 1)$$
$$= sgn(v'(y)y - v(y)). \quad (3)$$

Using $v'(y) = \phi_y(y, k^*(y))$ as well as equation (2), (3) can be rewritten

$$sgn(E_{v:y} - 1) = sgn(y^2 f'(y/k^*(y))/k^*(y) - g(k^*(y)))$$
$$= sgn(k^*(y)\, g'(k^*(y)) - g(k^*(y))),$$

since the first order condition for (2) is $-(y/k)^2 f'(y/k) + g'(k) = 0$. Thus,

$$sgn(E_{v:y} - 1) = sgn(E_{g:k} - 1),$$

where $E_{g:k}$ is the elasticity of capacity costs with respect to capacity evaluated at $k^*(y)$.

The Generalized Envelope Theorem permits a simpler, more intuitive derivation of this result. As a preliminary to the derivation, we point out that, like the Envelope Theorem, the Generalized Envelope Theorem holds in terms of elasticities, so that in the notation of the theorem, $E_{v(y_o):y} = E_{w(y_o):y}$. Thus, in the present application, we can consider $sgn(E_{w(y_o):y} - 1)$, which is equivalent to comparing $w(ty_o)$ to $tw(y_o)$ for t infinitesimally different from 1. Turning to the actual proof, the function $k(y)$ we use to define $w(y)$ is $k(y) = k^*(y_o)y/y_o$. This exploits the special structure of (2) by having k vary proportionately to y. We have

$$w(ty_o) - tw(y_o) = ty_o f(ty_o/tk^*(y_o)) + g(tk^*(y_o))$$
$$- t[y_o f(y_o/k^*(y_o)) + g(k^*(y_o))]$$
$$= g(tk^*(y_o)) - tg(k^*(y_o))$$

which has the same sign as $E_{g:k} - 1$, where $E_{g:k}$ is evaluated at $k = k^*(y_o)$.

Not only is this simpler, but it has the same structure as the intuitive argument given above.

4. Conclusion

A generalized envelope theorem has been presented which has the Envelope Theorem as a special case. Relative to the Envelope Theorem, it provides greater flexibility in determining the rate of change of a value function with respect to one of its arguments. We revisited a classic result on economies of scale for a congestion-prone facility, using the flexibility of the Generalized Envelope Theorem to provide a simpler, more intuitive proof.

References

Mohring, H., and M. Harwitz (1962) *Highway Benefits: An Analytical Framework*, Northwestern University Press: Evanston, IL.

Simon, C.P., and L. Blume (1994) *Mathematics for Economists,* Norton: New York.

Strotz, R.H. (1965) "Urban Transportation Parables" in *The Public Economy of Urban Communities* by J. Margolis, Ed., Resources for the Future: Washington, DC., 127-169.