E C O N O M I C S   B U L L E T I N

Threshold levels and the realization of a group benefit

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Abstract

This paper considers a voluntary contribution threshold game in which a group benefit is realized only if the number of contributors at least reaches a threshold level, and analyzes the effect of the threshold level on the likelihood that the group benefit is realized. Changes in the threshold level in interior symmetric equilibrium have two effects on the likelihood, the direct, threshold effect and the indirect, strategic effect. While the direct effect is always negative, the indirect effect can be either positive or negative. And the net effect is not necessarily negative.

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1. Introduction

Many researchers (e.g., Bliss and Nalebuff, 1984; Palfrey and Rosenthal, 1991; Bilodeau and Slivinski, 1996; Xu, 2001) have studied voluntary contribution threshold games. In these games, a group benefit or goal is realized only if the number of contributors in the group at least passes over a threshold level. These authors have focused on the free rider problem for a *given* threshold level. In this paper I focus on the effect of different threshold levels on the likelihood that the group benefit is realized.

It seems plausible, as conventional wisdom would suggest, that the group benefit, which is a pure public good, is less likely to be provided, when the threshold level increases. Suppose that a group project is produced with 50 percent chance if it takes at least two contributors to produce. Will increases in the threshold level (such that the project takes, say, at least three or more contributors to produce) lead to a greater likelihood that the group project will be completed? The immediate response of an average person would probably be No. In this paper I challenge the conventional wisdom and show that the answer to the above question is not so straightforward.

Real-world examples abound where a threshold level must be passed over before a benefit is received by everyone in a group. Many elections, from the presidential to local ones, involve a simple majority voting; the candidate getting more of the votes is elected to the office. There are also many cases of a majority of 2/3 voters to pass an agenda. Articles 73 and 79 of the German Fundamental Law (Constitution) require a majority of 2/3 voters to privatize some essential public services such as the postal service, telecommunications, and railways. And the Charter of the U. N. requires a majority of 2/3 to pass the budget. In some cases, unanimity of all players is called. For example, all the five members in the U. N. security council must vote unanimously (with an "abstain" vote counted as an invalid vote) to pass a resolution. The Portuguese Constitution requires the absolute majority of all the potential voters for privatization. And, in the Italian law on jointly owned block of flats, investment improving expenditures require the unanimity of the owners (Marrelli and Stroffolini, 2001).

In each of the above examples, the threshold level required for the realization of the group benefit is fixed. Because the group benefit is a public good, each player has an incentive to be a free rider. Once the threshold level is passed over, additional contributors, while personally incurring costs of contribution, add nothing to the provision of the public good. One interesting and important question is: If the threshold level is lowered for a given group of fixed size, will there exist equilibria in which the group benefit be more likely to be privately provided?[1]

The answer to the above question has great implications on issues of significant importance. For instance, the Alameda County Congestion Management Agency (1999) endorses the concept of a state constitutional amendment that would enable voters of the county and other counties in California to approve transportation sales tax measures by a simple majority, rather

[1] If players have conflicting interests, then a higher threshold level may render it harder to come up with the required number of contributors. This can be most clearly seen by looking at a two-player zero-sum game where a project benefits one player but hurts the other. If the threshold level is one, then the player who benefits from the project will contribute if his cost of contribution is less than the benefit. If the project needs contributions from both players, then it will never be completed, as the player who is to be hurt by the project will always refuse to contribute.

than the current 2/3 majority. Clearly, the Agency believes that lowering the level of threshold to pass a measure is more likely to get the measure passed. The critical problem here is whether voters act strategically, i.e., if their voting behavior will change with the constitutional amendment. If their voting behavior is invariant to the amendment, it does make sense that a lower threshold level is more conducive to the passage of the measure. But there is no reason to believe that voters do not behave strategically (Ledyard, 1981; Palfrey and Rosenthal, 1983). If voters do behave strategically, and furthermore, if each voter becomes less likely to vote for the transportation sales tax measure after the amendment, then there is no presumption that the measure will be more likely to be passed.[2]

In the paper I consider a situation in which all players benefit equally from a group project or good. But it takes at least as many players to make voluntary contributions as required by the threshold level to produce the group benefit. Contribution is costly, and the cost of contribution is solely borne by the contributor himself; no side payments are allowed. A player obtains the highest payoff (as long as the threshold level is less than the group size) as a successful free rider; the group benefit, as a pure public good, is produced without the player's contribution.

As argued by Palfrey and Rosenthal (1991), for models of this kind of public good problem to be applicable to natural settings, they should incorporate some element of private information. In reality, each player will generally be incompletely informed about certain characteristics of other players. In the paper I, following Palfrey and Resenthal (1991), model this uncertainty by assuming that each player's cost of contribution, relative to the group benefit, is the player's private information. Each player is uncertain about other players' costs of contribution which are independently drawn from a common distribution. The focus of the paper is on the effect of the threshold level on the probability that the group benefit is realized.

Clearly, if the threshold level is two or above, there is always an equilibrium in which all players do not contribute. If each player thinks that none of the other players will contribute, it is thus in the player's best interest not to contribute, either. In the paper I focus on analyzing interior symmetric equilibrium in which each player ex ante contributes with the same positive probability.

Changes in the threshold level in interior symmetric equilibrium have two effects on the likelihood that the group benefit is realized. One is the direct, threshold effect. Holding fixed each player's probability to contribute, when the threshold level increases, the group good is less likely to be provided, as the likelihood that the group good is provided is the probability that the number of contributors in the group at least reaches the threshold level. In other words, the direct effect is always negative.

The other is the indirect, strategic effect. Players' contribution decisions are dependent. In equilibrium, each player contributes if and only if his cost of contribution is no greater than a cutoff level. The cutoff level is equal to the benefit the player derives from the additional availability of the group good, which is the benefit of the group good times the probability that the player, if contributing, will put the group over the threshold level. The equilibrium probability that each player contributes is a function of the threshold level. When the threshold level increases, each player may be either more or less likely to contribute. As a result, the indirect, strategic effect can be either positive or negative.

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[2] As the referee points out, not all referenda will fit into the public good setup. This would be the case when there is an opposing party to a referendum. One such example is the referendum of separating Quebec from Canada; there were citizens for it, but also citizens against it. My analysis does not apply to these cases, as I illustrate in footnote 1.

The (net) effect, being the sum of the direct effect and the indirect effect, of the threshold level in interior symmetric equilibrium is thus not necessarily negative on the likelihood that the group benefit is realized. Indeed, it is possible that the indirect, strategic effect is positive and dominates the negative direct effect. As a result, as the threshold level increases, the group benefit is more likely to be privately provided.

The remainder of the paper is structured as follows. Section 2 sets up and analyzes the model, and Section 3 contains the conclusion.

2. The model

A group consists of $N > 1$ players. A group benefit or goal is realized if at least $m \in \{1, \ldots, N\}$ players contribute. Call m the threshold level.

The group benefit is equally enjoyed by all players in the group and is normalized to be 1. Each player's cost of contribution, however, is his private information. Let c_i be player i's cost of contribution. Each player knows his own cost of contribution $c_i \in [\underline{c}, \bar{c}]$, but knows only about the distributions of other players' costs of contribution. For simplicity, assume that players' costs of contribution are independently and identically distributed according to a cumulative distribution function F(c), with the corresponding density function $f(c) = F'(c) > 0$, for $c \in [\underline{c}, \bar{c}]$. Assume that $\underline{c} < 1 < \bar{c}$. This assumption implies that each player's ex ante probability to contribute is strictly less than 1.

Each player is risk neutral, and a player's payoff, when his cost of contribution is c, is as follows:

1 - c	if he contributes and at least m - 1 others contribute;
1	if he does not contribute and at least m others contribute;
- c	if he contributes and less than m - 1 others contribute;
0	if he does not contribute and less than m others contribute.

A pure strategy for each player is a mapping from his cost of contribution $[\underline{c}, \bar{c}]$ to his action set, {"contribute", "don't contribute"}. Denote s_i the strategy of player i, and $s_{-i} = \{s_1, s_2, \ldots, s_{N-1}, s_N\}$ the strategies of all players except player i.

Players' behavior is consistent with conditions for a Bayesian Nash equilibrium. A Bayesian Nash equilibrium is a profiles of strategies $\{s_1^*(.), \ldots, s_N^*(.)\}$ such that given other players strategies, $s_{-i}^*(.)$, strategy $s_i^*(c_i)$ maximizes player i's expected payoff, for all possible value of c_i and for all i.

Since players are symmetric ex ante, it is natural to focus on symmetric equilibrium. Let p be the subjective probability that player i thinks each of the other players contributes. Let k be the number of contributors in the residual group of $N - 1$ players. If player i contributes, his payoff is $\text{Prob}(k \geq m - 1) - c_i$. If he does not contribute, his expected payoff is $\text{Prob}(k \geq m)$. So, player i contributes if and only if $c_i \leq c^*(m, p)$, where $c^*(m, p) = \text{Prob}(k = m - 1) = C_{N-1}^{m-1} p^{m-1}(1-p)^{N-m}$.

In other words, player i's equilibrium strategy abides by a cutoff rule: He contributes if and only if his cost of contribution is no greater than the expected benefit he derives from the additional availability of the public good, which is the probability that m – 1 others contribute (see also Palfrey and Rosenthal, 1991).

In symmetric equilibrium,

$p = F(C_{N-1}^{m-1} p^{m-1}(1 - p)^{N-m})$. (1)

From (1), we can make the following observations. First, p is a function of m. Second, for m = 1, there is a unique equilibrium p > 0. Note that, even though the public good can be provided by a single player, there are chances that more than one player contributes at the same time. This happens because each player is uncertain about other players' costs of contribution (which are assumed to be identically and independently distributed), and each contributes if and only if his cost of contribution does not exceed the cutoff level. Third, for m > 1, there is always an equilibrium p = 0. If the group good takes at least two contributors to produce and each player believes that no one else will contribute, then it is in the player's best interest not to contribute. In the analysis below, I focus on interior symmetric equilibrium in which each player ex ante contributes with the same positive probability. Further, there may be multiple solutions to (1). Some of the equilibria may be stable while others may be not.

A Bayesian Nash equilibrium is expectationally stable if the following tâtonnement process converges to equilibrium p (Palfrey and Rosenthal, 1991). Suppose that player i expects that all other players deviate their contributing probability from p to p'. The cutoff level for player i to contribute then is $c^*(m, p') = C_{N-1}^{m-1} p'^{m-1}(1- p')^{N-m}$. Hence, the ex ante probability that player i contributes is $G(p') = F(C_{N-1}^{m-1} p'^{m-1}(1- p')^{N-m})$. p is an expectationally stable equilibrium (ESE) if there exists an interval $P^E(p) \subset [0, 1]$ containing p such that, for all $p' \in P^E(p)$, $[p' - G(p')](p' - p) > 0$ if $p' \neq p$. Dividing both sides of the above inequality by $(p' - p)^2$, we have $[p'- G(p')]/(p' – p) > 0$, which can be rewritten as $[G(p') – p + p - p']/(p' – p) > 0$. Note that $p = G(p)$. Plugging this into the last inequality and letting $p' \rightarrow p$, it follows immediately that for p to be an ESE, we must have $G'(p) < 1$.

ESE has a nice property. If the group benefit is not 1 but B, then one can easily see that in symmetric equilibrium, $p = F(C_{N-1}^{m-1} p^{m-1}(1 - p)^{N-m}B)$. It can be easily shown that $dp/dB > 0$ in ESE. In other words, for a given threshold level, each player is more likely to contribute as the group benefit increases.

When the threshold level is m, the probability that the group benefit is realized is

$P(m) = p^N + C_N^{N-1} p^{N-1}(1 - p) + \ldots + C_N^m p^m(1- p)^{N-m}$. (2)

Recall that in the above equation, p is a function of m. Clearly,

$dP/dm = \partial P/\partial m + \partial P/\partial p \; dp/dm$. (3)

The first term captures the direct effect of the threshold level on the likelihood that the group benefit is realized, and the second term captures the indirect effect.

Note that, for a given p, $P(m + 1) – P(m) = - C_N^m p^m(1- p)^{N-m} < 0$. In other words, $\partial P/\partial m < 0$. This is the direct, threshold effect of the threshold level on the realization of the group benefit. The direct effect is always negative in that a higher threshold level leads to a lower likelihood that the group benefit is realized.

The indirect, strategic effect describes the effect on P of m through p. Intuitively, for a given threshold level, if each player is more likely to contribute, then the group benefit is more likely to be realized, that is, $\partial P/\partial p > 0$. This can be formally proved as follows. Differentiating P with respect to p, we have $\partial P/\partial p = Np^{N-1} - C_N^{N-1} p^{N-1} + (N - 1) C_N^{N-1} p^{N-2}(1- p) + \ldots - (N - m - 1) C_N^{m+1} p^{m+1}(1- p)^{N-m-2} + (m + 1) C_N^{m+1} p^m(1- p)^{N-m-1} - (N - m) C_N^m p^m(1- p)^{N-m-1} + m C_N^m p^{m-1}(1- p)^{N-m} = m C_N^m p^{m-1}(1- p)^{N-m}$, where use is made of the fact that $(N - i) C_N^i = (i + 1) C_N^{i+1}$.

The sign of dp/dm can be obtained from (1). There are two ways of looking at it. The first is totally differentiating both sides of (1) with respect to p and m. It is easy to see that the sign of dp/dm is generally indeterminate. But a more instructive way is to compare c*(m + 1, p) with c*(m, p). Note that c*(m, p) > c*(m + 1, p) if and only if $C_{N-1}^{m-1} p^{m-1}(1-p)^{N-m} > C_{N-1}^{m} p^{m}(1-p)^{N-m-1}$, or equivalently, if p < m/N. But there is no guarantee that this will be the case (see Example 1 below). It implies that dp/dm can be either positive or negative.

The result that dp/dm is ambiguously signed can be most clearly illustrated by looking at two-player contribution threshold games. Let p(m) be the solution to (1) when the threshold level is m. Then, for N = 2, p(1) satisfies that p(1) = F(1 – p(1)), and p(2) satisfies that p(2) = F(p(2)). Clearly, p(1) > p(2) if and only if 1 - p(1) > p(2). Let p(1) = α < 1/2, and p(2) = β < α = p(1). Note that for β < α < 1/2, β < 1 - α. It is easy to construct cumulative distribution functions such that F(1 - α) = α, and F(β) = β.

On the other hand, let p(1) = α > 1/2, and p(2) = β > α = p(1). Then, β > 1 - α. It is easy to construct distributions such that F(1 - α) = α, and F(β) = β.

If dp/dm < 0, then ∂P/∂p dp/dm < 0. Thus, the indirect effect and the direct effect work in the same direction. Consequently, as the threshold level increases, the group benefit is less likely to be realized.

On the other hand, if dp/dm > 0, then ∂P/∂p dp/dm > 0. Thus, the indirect effect and the direct effect work in opposite directions. Moreover, if the direct effect dominates the indirect effect, there is an inverse relationship between the threshold level and the likelihood that the group benefit is realized. If the direct effect is dominated by the indirect effect, there is a positive relationship between the threshold level and the likelihood that the group benefit is realized.

I demonstrate now by an example that, depending on cost distributions, the group benefit can be either more or less likely to be privately provided, as the threshold level increases.

Example 1. Let N = 3. If m = 1, then p(1) satisfies that $p(1) = F([1 - p(1)]^2)$, and $P(1) = p(1)^3 + 3p(1)^2[1- p(1)] + 3p(1)[1- p(1)]^2$. If m = 2, then p(2) satisfies that p(2) = F(2p(2)[1 – p(2)]), and $P(2) = p(2)^3 + 3p(2)^2[1- p(2)]$. And if m = 3, then p(3) satisfies that $p(3) = F(p(3)^2)$, and $P(3) = p(3)^3$.

If each player's cost of contribution is uniformly distributed over the interval [0, 3/2], then F(c) = 2c/3. Simple algebra shows that p(1) ≈ 0.314, p(2) = 1/4, and p(3) = 0. It can also easily be verified that all p(i), i = 1, 2, 3, are ESE. Given p(1) > p(2) > p(3), it follows immediately from (3) that P(1) > P(2) > P(3). Indeed, it can be easily verified that P(1) ≈ 0.677, P(2) = 0.15625, and P(3) = 0.

On the other hand, for cost distribution such that $p_1 = 2/5 = F(9/25)$, $p_2 = 3/4 = F(3/8)$, and $p_3 = 19/20 = F(361/400)$, P(1) = 0.784, P(2) = 0.84374, and P(3) = 0.857375. It is easy to show that p(2) is an ESE. For p(3) to be an ESE, it is required that $2p(3)f(p(3)^2) = 19/10\, f(361/400) < 1$, or f(361/400) < 10/19.

3. Conclusion

This paper considers a voluntary contribution threshold game in which a group benefit or good is realized only if the number of contributors at least reaches a threshold level, and analyzes the effect of the threshold level on the likelihood that the group benefit is realized. Changes in the threshold level in interior symmetric equilibrium have two effects on the likelihood, namely, the direct, threshold effect and the indirect, strategic effect. The direct effect is always negative. The indirect, strategic effect can be either negative or positive, depending on whether each player is less or more likely to contribute as the threshold level increases. There is generally no inverse relationship between the threshold level and the likelihood that the group benefit is realized. Laboratory experiment can be conducted to test the relationship.

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