E C O N O M I C S B U L L E T I N

A simple proof for the existence of Walrasian equilibrium under monotone preferences

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Abstract

When individual preferences are strictly monotone, the continuity of the excess demand functions that is usually assumed to show the existence of a Walrasian equilibrium does not hold for price vectors in which at least one component is equal to zero. In this paper we provide a simple proof of existence for precisely this important case. It is based upon the economic intuition that equilibrium prices will never be equal to zero if preferences are strictly monotone, and it runs surprisingly similar to the one usually adopted for the case of non−monotone preferences.

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1. Introduction

In basically all graduate textbooks on microeconomics it is standard to include a chapter on general equilibrium theory and to discuss the existence of a Walrasian equilibrium in detail, at least within the pure exchange version of the underlying model. In this context, the existence of an equilibrium price vector is usually shown by application of Brouwer's fixed point theorem to excess demand functions that are assumed to be continuous on the unit simplex and to satisfy Walras' law. The main disadvantage of this approach, however, is that the resulting theorem does not hold for the important case of (strictly) monotone preferences because the excess demand functions are then no longer continuous or even not defined for price vectors in which at least one component is equal to zero.[1] For this reason, the existence of an equilibrium under strictly monotone preferences is mostly not discussed in further detail. Instead, this problem is "overcome" by stating that it is mainly technical and that it can be dealt with by applying more sophisticated mathematical methods.[2] In fact, there are of course textbooks that do establish the existence of an equilibrium under monotone preferences rigorously. However, they apply rather complex proofs that differ substantially from the standard one that makes use of Brouwer's fixed point theorem.[3] In our opinion, both approaches are somewhat unsatisfactory. Leaving out the proof is problematical because the monotonicity assumption is subsequently imposed to prove the welfare theorems and hence plays a crucial role for results that are *based upon* the existence of a Walrasian equilibrium. Presenting a complex proof is not entirely satisfactory, either. While the standard proof via Brouwer's fixed point theorem is quite straightforward and intuitively obvious, the proof using Kakutani is much more involved and requires several intermediate steps to deal with technical details. Moreover, it is always hard to explain why a theorem on correspondences is necessary even though we deal with excess demand *functions*, where Brouwer should be expected.

In this paper we intend to close this gap in the (textbook) literature by providing a simple proof for the existence of a Walrasian equilibrium under strictly monotone preferences that closely parallels the one usually adopted for the case of non-monotone preferences. It is based upon the economic intuition (as well as the mathematical result) that, under strictly monotone preferences, the demand for a

[1] For an example, see Arrow, Hahn (1971, pp. 29-30).

[2] See, e.g., Luenberger (1995, p. 215) and Varian (1992, p. 322).

[3] See Aliprantis, Brown, Burkinshaw (1990, pp. 32-34), Arrow, Hahn (1971, pp. 31-33), and Mas-Colell, Whinston, Green (1995, pp. 585-587).

commodity approaches infinity as its price tends to zero.[4] If, on the other hand, all goods only exist in finite amounts, there should obviously be no free goods in equilibrium, i.e., all equilibrium prices ought to be strictly positive. Thus, the difficulties associated with the definition and the continuity of the excess demand functions for price vectors with zero components can be expected to cause no essential problems. This conjecture turns out to be true, and it is in fact possible to establish the existence of an equilibrium price vector along these lines.

Our approach should be contrasted with related work by Geanakoplos (2001) who also provides a proof for the existence of Walrasian equilibria via Brouwer's fixed point theorem. While his analysis is more general as it is not confined to the case of monotone preferences in pure exchange economies it is also conceptually different because the proofs are based on minimum expenditure functions and the application of the "Satisficing Principle". In contrast, our objective is to extend the *classical* proof to the case of monotone preferences without recurrence to additional concepts. Furthermore, a central feature of our approach is that we maintain the premise that individual households derive their optimal behaviour from price signals only, such that the individual budget sets are unbounded in case of free goods.

2. Notations and assumptions

We consider the general equilibrium model of a pure exchange economy that is standard in most graduate textbooks, so we do not develop the model setup at full length but only introduce the basic notations and the main assumptions that are subsequently used. Furthermore, to keep the exposition as short and simple as possible, we have refrained from presenting the model in terms of assumptions on individual preferences but use utility functions throughout. In fact, all we need is what Aliprantis, Brown, Burkinshaw (1990, p. 29) call a neoclassical exchange economy, in which the properties denoted in our assumptions 1-4 are all valid.[5] The economy consists of $i = 1, ..., m$ individuals and $j = 1, ..., n$ commodities. Each individual i is equipped with a nonnegative and finite endowment ω_{ij} of good j. Let $\boldsymbol{\omega}_i \in \mathbb{R}^n_+$ denote the vector of endowments of agent i, where $\mathbb{R}^n_+$ is the set of nonnegative vectors in $\mathbb{R}^n$. In order to exclude trivial cases, suppose that each good is available in strictly positive amount, i.e., we impose

Assumption 1. $\sum_{i=1}^{m} \boldsymbol{\omega}_i > \mathbf{0}$, i.e., $\sum_{i=1}^{m} \omega_{ij} > 0$ for all $j = 1, ..., n$.

[4] Of course, this assertion is not literally correct if more than one price approaches zero, see section 3 on this point.

[5] For details on the model setup the reader is referred to Aliprantis, Brown, Burkinshaw (1990) or one of the other textbooks listed in the references.

Regarding individual preferences we make the following simplifying assumption.

Assumption 2. Let the preferences of individual i be represented by the continuous utility function $u_i : \mathbb{R}^n_+ \to \mathbb{R}$, $i = 1, ..., m$.

Denote by $p_j \geq 0$ the price for good j, by $\mathbf{p} \in \mathbb{R}^n_+$ the price vector, and by $\mathbf{p}'$ its transpose. All individuals intend to choose their consumption quantities x_{ij}, $i = 1, ..., m$, $j = 1, ..., n$, so as to maximize utility subject to the budget constraint $\mathbf{p}'\mathbf{x}_i \leq \mathbf{p}'\boldsymbol{\omega}_i$, where $\mathbf{x}_i = (x_{i1}, ..., x_{in})' \in \mathbb{R}^n_+$. For every strictly positive price vector $\mathbf{p} \in \mathbb{R}^n_{++} = \{\mathbf{p} \in \mathbb{R}^n : p_j > 0,\ j = 1, ..., m\}$ the budget sets $\mathcal{B}(\mathbf{p}, \boldsymbol{\omega}_i) = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{p}'\mathbf{x} \leq \mathbf{p}'\boldsymbol{\omega}_i\}$ are compact and hence the existence of a utility maximizing consumption bundle $\mathbf{x}_i$ for all $i = 1, ..., m$ follows directly from assumption 2. For simplicity, we assume that this bundle is unique.[6]

Assumption 3. For each $\mathbf{p} \in \mathbb{R}^n_{++}$ and each $\boldsymbol{\omega}_i \in \mathbb{R}^n_+$ there is exactly one utility maximizing consumption bundle $\mathbf{x}_i(\mathbf{p}, \boldsymbol{\omega}_i) \in \mathcal{B}(\mathbf{p}, \boldsymbol{\omega}_i)$, $i = 1, ..., m$.

Under assumption 3 we obtain the single-valued demand functions $\mathbf{p} \mapsto \mathbf{x}_i(\mathbf{p}, \boldsymbol{\omega}_i)$, $i = 1, ..., m$, $\mathbf{p} \in \mathbb{R}^n_{++}$. They yield the individual excess demand functions $\mathbf{p} \mapsto \mathbf{z}_i(\mathbf{p}, \boldsymbol{\omega}_i) = \mathbf{x}_i(\mathbf{p}, \boldsymbol{\omega}_i) - \boldsymbol{\omega}_i$ as well as the aggregate excess demand function $\mathbf{p} \mapsto \mathbf{z}(\mathbf{p}) := \mathbf{z}(\mathbf{p}, \boldsymbol{\omega}_1, ..., \boldsymbol{\omega}_m) = \sum_{i=1}^m \mathbf{z}_i(\mathbf{p}, \boldsymbol{\omega}_i)$, where the initial endowment of goods is held fixed. An equilibrium price vector $\overline{\mathbf{p}}$ is then characterized by the condition $\mathbf{z}(\overline{\mathbf{p}}) \leq \mathbf{0}$.

An important consequence of assumption 3 is that the aggregate excess demand function $\mathbf{z} : \mathbb{R}^n_{++} \to \mathbb{R}$ is continuous[7] on its domain. We thus obtain a natural modification of the standard assumption that $\mathbf{z}$ is defined and continuous on $\mathbb{R}^n_+$ which may be imposed if preferences are *not* monotone. Here, the possibility that agents will be satiated for sufficiently large consumption bundles is excluded by the following central assumption.

Assumption 4. All individual preferences are strictly monotone, i.e., $\mathbf{x}_i \geq \mathbf{y}_i$, $\mathbf{x}_i \neq \mathbf{y}_i$, implies $u_i(\mathbf{x}_i) > u_i(\mathbf{y}_i)$ for $\mathbf{x}_i, \mathbf{y}_i \in \mathbb{R}^n_+$ and for all $i = 1, ..., m$.

3. Existence of an equilibrium price vector

Before proving the existence of an equilibrium price vector under assumptions 1-4, we state the following well-known and useful results:[8]

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[6] This can, for example, be ensured by the assumption of strictly convex preferences.

[7] This follows directly from the maximum theorem because the budget sets $\mathcal{B}(\mathbf{p}, \boldsymbol{\omega}_i)$ are compact.

[8] The main result needed for the proof of existence is contained in lemma 2. Similar properties to those stated in lemmas 1 and 2 can, e.g., be found in Theorems 1.3.3, 1.3.6, and 1.3.9 in Aliprantis, Brown, Burkinshaw (1990, pp. 21-26), see also proposition 17.B.2 in Mas-Colell, Whinston, Green (1995, p. 581). We establish both lemmas formally so as to keep our analysis self-contained.

LEMMA 1. *Let assumptions 1-4 be valid and let a price vector* $\overline{\mathbf{p}} \in \mathbb{R}^n_+$ *with* $\overline{\mathbf{p}} \notin \mathbb{R}^n_{++}$ *be given. Then there exists no utility maximizing consumption bundle* $\overline{\mathbf{x}}_i \in \mathcal{B}(\overline{\mathbf{p}}, \boldsymbol{\omega}_i)$ *for any* i.

Proof. Suppose that $\overline{\mathbf{x}}_i \in \mathcal{B}(\overline{\mathbf{p}}, \boldsymbol{\omega}_i)$ is utility maximizing, i.e., $u_i(\overline{\mathbf{x}}_i) \geq u_i(\mathbf{x}_i)$ for all $\mathbf{x}_i \in \mathcal{B}(\overline{\mathbf{p}}, \boldsymbol{\omega}_i)$. Since $\overline{\mathbf{p}} \notin \mathbb{R}^n_{++}$ there is at least one $r \in \{1, ..., n\}$ with $\overline{p}_r = 0$ such that $\widetilde{\mathbf{x}}_i = \overline{\mathbf{x}}_i + \lambda \mathbf{e}_r \in \mathcal{B}(\overline{\mathbf{p}}, \boldsymbol{\omega}_i)$, where $\lambda > 0$ and where $\mathbf{e}_r$ denotes the rth canonical vector in $\mathbb{R}^n$. Since preferences are strictly monotone, it follows that $u(\widetilde{\mathbf{x}}_i) > u(\overline{\mathbf{x}}_i)$, which is a contradiction. ■

LEMMA 2. *Let assumptions 1-4 be valid and let a sequence* $\{\mathbf{p}_k\}$ *of price vectors* $\mathbf{p}_k \in \mathbb{R}^n_{++}$, $k = 1, 2, ...$, *with* $\lim_{k \to \infty} \mathbf{p}_k = \overline{\mathbf{p}}$ *and* $\overline{\mathbf{p}} \notin \mathbb{R}^n_{++}$ *be given. Furthermore, suppose that* $\overline{\mathbf{p}} \neq \mathbf{0} \in \mathbb{R}^n$. *Then*

$$\limsup_{k \to \infty} \|\mathbf{z}(\mathbf{p}_k)\| = +\infty, \tag{1}$$

where $\|\mathbf{x}\| = \sqrt{\mathbf{x}'\mathbf{x}}$ *denotes the Euclidean norm of any vector* $\mathbf{x} \in \mathbb{R}^n$.

Proof. Suppose that (1) does not hold. Then there exists some $d \in \mathbb{R}$, $d > 0$, with $\mathbf{z}(\mathbf{p}_k) \in U_d(\mathbf{0}) = \{\mathbf{z} : \|\mathbf{z}\| \leq d\}$ for all $k = 1, 2, ...$, and $\mathbf{x}_i(\mathbf{p}_k, \boldsymbol{\omega}_i) \in U_d(\boldsymbol{\omega}_i) = \{\mathbf{x}_i \colon \|\mathbf{x}_i - \boldsymbol{\omega}_i\| \leq d\}$ for all $k = 1, 2, ...$ and all $i = 1, ..., m$. Consequently, the sequence $\mathbf{x}_i^k := \mathbf{x}_i(\mathbf{p}_k, \boldsymbol{\omega}_i)$ has a convergent subsequence $\mathbf{x}_i^{k_s}$ with $\lim_{s \to \infty} \mathbf{x}_i^{k_s} = \overline{\mathbf{x}}_i$. Since $\mathbf{p}'_{k_s} \mathbf{x}_i^{k_s} \leq \mathbf{p}'_{k_s} \boldsymbol{\omega}_i$ for all s, continuity of the inner product implies $\overline{\mathbf{p}}' \overline{\mathbf{x}}_i \leq \overline{\mathbf{p}}' \boldsymbol{\omega}_i$, hence $\overline{\mathbf{x}}_i \in \mathcal{B}(\overline{\mathbf{p}}, \boldsymbol{\omega}_i)$. Now let any $\mathbf{x} \in \mathcal{B}(\overline{\mathbf{p}}, \boldsymbol{\omega}_i)$ be given. Since $\overline{\mathbf{p}} \neq \mathbf{0}$, we may assume without loss of generality that $\overline{\mathbf{p}}' \boldsymbol{\omega}_i > 0$ (cf. assumption 1). Hence there exists a convergent sequence $\mathbf{y}^s$, $\mathbf{y}^s \in \mathcal{B}(\overline{\mathbf{p}}, \boldsymbol{\omega}_i)$, with limit $\mathbf{x}$, such that $\overline{\mathbf{p}}' \mathbf{y}^s < \overline{\mathbf{p}}' \boldsymbol{\omega}_i$ for all s, implying $\mathbf{p}'_{k_s} \mathbf{y}^s \leq \mathbf{p}'_{k_s} \boldsymbol{\omega}_i$ for sufficiently large s. As all $\mathbf{x}_i^{k_s}$ are utility maximizing we have $u_i(\mathbf{x}_i^{k_s}) \geq u_i(\mathbf{y}^s)$ and, therefore, by continuity of u_i we obtain $u_i(\overline{\mathbf{x}}_i) \geq u_i(\mathbf{x})$. Since $\mathbf{x} \in \mathcal{B}(\overline{\mathbf{p}}, \boldsymbol{\omega}_i)$ was chosen arbitrarily, this implies that $\overline{\mathbf{x}}_i$ is utility maximizing for $\overline{\mathbf{p}}$. This, however, contradicts lemma 1. ■

Since all excess demand functions are bounded from below, lemma 2 implies that the demand for at least one commodity grows unboundedly as one or more price(s) approach(es) zero. This has a straightforward economic interpretation. As all individuals are non-satiated under strictly monotone preferences they will demand "infinitely much" of at least one good as some prices tend to zero, provided that they earn a positive income. The latter is guaranteed by assumption 1 together with $\overline{\mathbf{p}} \neq \mathbf{0}$ for at least one individual.

We are now in position to prove the existence of an equilibrium. Here, we slightly modify the usual fixed point argument[9] so as to cope with price vectors belonging to the boundary of the unit simplex.

PROPOSITION 1. *Under assumptions 1-4 there exists a price vector* $\overline{\mathbf{p}} \in \mathbb{R}^n_{++}$ *satisfying* $\mathbf{z}(\overline{\mathbf{p}}) \leq \mathbf{0}$.

Proof. Let z_j, $j = 1, ..., n$, denote the jth component of the aggregate demand function $\mathbf{z}$. It is well-known that all z_j are homogeneous of degree zero, so without loss of generality we may consider an equilibrium price vector as an element of the unit simplex $S^{n-1} = \{\mathbf{p} \in \mathbb{R}^n_+ : \sum_{j=1}^n p_j = 1\}$. Let an arbitrary vector $\mathbf{c} = (c_1, ..., c_n)' \in \mathbb{R}^n_{++}$ be given and define the functions $g_j^c : S^{n-1} \to \mathbb{R}$ for all $j = 1, ..., n$ according to

$$g_j^c(\mathbf{p}) = \frac{p_j + z_j(\mathbf{p}+\mathbf{c})^+}{1 + \sum_{k=1}^n z_k(\mathbf{p}+\mathbf{c})^+} \quad \text{for all } \mathbf{p} \in S^{n-1}, \tag{2}$$

where $z_j(\mathbf{p}+\mathbf{c})^+ = \max[0, z_j(\mathbf{p}+\mathbf{c})]$ denotes the positive part of z_j. Obviously, g_j^c is well-defined and continuous on S^{n-1} because of $\mathbf{p}+\mathbf{c} \in \mathbb{R}^n_{++}$ and assumptions 2 and 3. Moreover, since $\sum_{j=1}^n g_j^c(\mathbf{p}) = 1$, the n-dimensional function $\mathbf{g}^c = (g_1^c, ..., g_n^c)'$ maps the unit simplex S^{n-1} into itself. Thus, there exists a fixed point $\mathbf{p}^c = (p_1^c, ..., p_n^c)' \in S^{n-1}$ of $\mathbf{g}^c$ by Brouwer's fixed point theorem, i.e.,

$$\mathbf{p}^c = \mathbf{g}^c(\mathbf{p}^c). \tag{3}$$

Combining (2) and (3) we obtain for all $j = 1, ..., n$

$$p_j^c \cdot \sum_{k=1}^n z_k(\mathbf{p}^c+\mathbf{c})^+ = z_j(\mathbf{p}^c+\mathbf{c})^+ \tag{4}$$

which is equivalent to

$$\left(p_j^c + c_j\right) \cdot \sum_{k=1}^n z_k(\mathbf{p}^c+\mathbf{c})^+ = c_j \cdot \sum_{k=1}^n z_k(\mathbf{p}^c+\mathbf{c})^+ + z_j(\mathbf{p}^c+\mathbf{c})^+. \tag{5}$$

Multiplying equation (5) by $z_j(\mathbf{p}^c+\mathbf{c})$ and summing up the resulting equation over all $j = 1, ..., n$ then gives

$$\begin{aligned} &\left(\sum_{j=1}^n z_j(\mathbf{p}^c+\mathbf{c})\left(p_j^c + c_j\right)\right)\left(\sum_{k=1}^n z_k(\mathbf{p}^c+\mathbf{c})^+\right) \\ &\quad = \sum_{j=1}^n z_j(\mathbf{p}^c+\mathbf{c})c_j \cdot \sum_{k=1}^n z_k(\mathbf{p}^c+\mathbf{c})^+ + \sum_{j=1}^n z_j(\mathbf{p}^c+\mathbf{c})z_j(\mathbf{p}^c+\mathbf{c})^+. \end{aligned} \tag{6}$$

[9] See, e.g., Mas-Colell, Whinston, Green (1995, pp. 588-589) or Varian (1992, pp. 319-322).

This implies

$$0=\sum_{j=1}^{n} z_j(\mathbf{p}^c+\mathbf{c})c_j\cdot\sum_{k=1}^{n} z_k(\mathbf{p}^c+\mathbf{c})^+ +\sum_{j=1}^{n} z_j(\mathbf{p}^c+\mathbf{c})z_j(\mathbf{p}^c+\mathbf{c})^+ \tag{7}$$

because the strong version of Walras' law holds under strictly monotone preferences. Since all z_j are bounded from below, equation (7) gives

$$0\geq\sum_{k=1}^{n}\left(-\sum_{j=1}^{n}\omega_j c_j+z_k(\mathbf{p}^c+\mathbf{c})\right)z_k(\mathbf{p}^c+\mathbf{c})^+. \tag{8}$$

Now consider a sequence of vectors $\mathbf{c}_s\in\mathbb{R}^n_{++}$, $s=1,2,...$, with $\lim_{s\to\infty}\mathbf{c}_s=\mathbf{0}$. By replicating the above arguments we obtain a sequence of functions $\mathbf{g}^{c_s}$ and a sequence of associated fixed points $\mathbf{p}^{c_s}\in S^{n-1}$ with $\mathbf{p}^{c_s}=\mathbf{g}^{c_s}(\mathbf{p}^{c_s})$. Since S^{n-1} is compact, there exists a convergent subsequence $\mathbf{p}^{c_{s_t}}$ of $\mathbf{p}^{c_s}$ with $\lim_{t\to\infty}\mathbf{p}^{c_{s_t}}=\overline{\mathbf{p}}\in S^{n-1}$. Obviously, we have $\overline{\mathbf{p}}\neq\mathbf{0}\in\mathbb{R}^n$, but we may even conclude that *no* component of $\overline{\mathbf{p}}$ is equal to zero. If we had $\overline{\mathbf{p}}\in S^{n-1}$ but $\overline{\mathbf{p}}\notin\mathbb{R}^n_{++}$, then lemma 2 (in connection with the immediate consequence stated right below it) would imply $\limsup_{t\to\infty} z_j(\mathbf{p}^{c_{s_t}}+\mathbf{c}_{s_t})=\infty$ for at least one j, thus contradicting (the suitable version of) equation (8). Consequently, we have $\overline{\mathbf{p}}\in S^{n-1}\cap R^n_{++}$. Taking the limits in (the analogue of) equation (8) then gives

$$0\geq\lim_{t\to\infty}\sum_{k=1}^{n}\left(-\sum_{j=1}^{n}\omega_j(c_{s_t})_j+z_k(\mathbf{p}^{c_{s_t}}+\mathbf{c}_{s_t})\right)z_k(\mathbf{p}^{c_{s_t}}+\mathbf{c}_{s_t})^+=\sum_{k=1}^{n}z_k(\overline{\mathbf{p}})z_k(\overline{\mathbf{p}})^+, \tag{9}$$

where the equality follows from the continuity of z_j and z_j^+. Equation (9) directly implies $z_k(\overline{\mathbf{p}})\leq 0$ for all $k=1,...,n$. ■

Comparing the above proof to the one usually adopted when $\mathbf{z}$ is continuous on $\mathbb{R}^n_+$, we see that only two additional arguments are necessary. The first one concerns the introduction of the sequence of parameters $\mathbf{c}_s$ and the limiting behaviour of the associated (sub-)sequence of fixed points $\mathbf{p}^{c_s}$ when the parameters approach zero. The second modification is the application of lemma 2 which ensures that a zero price can never prevail in equilibrium.

Remark. As a trivial corollary to proposition 1 we obtain a formal justification for the intuitively obvious result that all commodities are "scarce" in equilibrium if all agents are non-satiated and if initial endowments are bounded, i.e., there are no free goods and all markets clear. This follows immediately from the existence of an equilibrium price vector $\overline{\mathbf{p}}\in\mathbb{R}^n_{++}$ and the fact that Walras' law implies $\mathbf{z}(\overline{\mathbf{p}})=\mathbf{0}$ in this case.

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