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Technological efficiency and the welfare of the capital−poor in a simple model of endogenous growth

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Abstract

The note demonstrates the existence of a trade−off between technological efficiency (and economic growth) and the welfare of infinitely−lived capital−poor agents in a simple model of endogenous growth with a convex technology. The trade−off can exist even when the wage rate for unskilled labour and its rate of growth are higher along growth paths associated with more efficient technologies.

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1 Introduction

Linear or AK models of growth form an important part of the 'new growth theory.'[1] In these models growth is sustained endogenously through the accumulation of capital K, broadly defined to include both physical and human capital, and the incentive to accumulate is supported by the presence, over time, of a lower bound on the rate of return to capital. Alesina and Rodrik (1994) and Bertola (1993) have extended linear models of growth due respectively to Barro (1990) and Romer (1986) to consider the relationship between an economy's steady-state rate of growth and the initial distribution of capital endowments amongst infinitely-lived individuals in society. In their papers an economy's (constant) rate of growth is determined by majority voting over rates of taxation while individual preferences over the set of alternative tax rates (and growth paths) depend on initial individual endowments of capital and unskilled labour. Thus, paths with higher long-run rates of growth may imply higher or lower levels of individual welfare depending on an individual's initial endowments.

This note looks at a simple one-sector version of the AK model with transitional dynamics due to Jones and Manuelli (1990) where the class of possible technologies is indexed by an efficiency parameter, and the long-run rate of growth increases one-for-one with the value of this efficiency parameter. The note shows that for economies with sufficiently large intial stocks of capital, the lifetime welfare of individuals with sufficiently small initial wealth holdings may be smaller along growth paths associated with more efficient technologies and characterised by higher long-run rates of growth.

That the introduction of more efficient growth-enhancing technologies, biased towards the use of physical or human capital[2], can adversely affect the relative earnings of the capital-poor, is a possibility that has long engaged the attention of economists. In recent years too, a fairly large literature has developed explaining how the advent of new skill-biased technologies can account for the substantial increases in income and wage inequality observed in several OECD countries during the past twenty five years[3]. However, higher rates of return to capital not only imply a larger incentive for initially capital-poor individuals to acquire capital over time, but for technologies with constant returns to scale and diminishing marginal returns to all factors, higher economy-wide rates of capital accumulation imply higher wages for unskilled labour in the long run. Therefore, if technologies are convex and the welfare of the capital-poor is evaluated over a sufficiently long period of time, it is not clear that lower relative earnings imply lower levels of absolute welfare for the capital-poor.

This note provides an example of an economy with a convex technology and infinitely-long individual lifetimes where the lifetime welfare of capital-poor individuals maybe smaller along growth paths associated with more capital-biased technologies. This is true even if

[1] See Barro and Sala-i-Martin (1995) and Aghion and Howitt (1998) for a review and McGrattan (1998) for an empirical defense of these models.

[2] We define a technology to be more capital-biased than another when it is associated with a higher income share for capital for any given set of ratios of factor inputs. The rental price of capital is taken to equal its marginal product.

[3] See, for example, Berman, Bound and Machin (1998). Aghion, Caroli and García-Peñalosa (1999) provide a useful survey of the literature.

these growth paths are characterised throughout by higher real wage rates for unskilled labour as well as by higher rates of growth of this wage rate.

This note is arranged as follows. Section 2 presents the model under consideration. Section 3 characterises the unique perfect foresight competitive equilibrium growth path for the model economy. Section 4 establishes the proposition that individuals with sufficiently small initial wealth holdings may be worse off along equilibrium growth paths associated with more efficient technologies even if the real wage rate and the rate of growth of the real wage rate is higher along these growth paths. Section 5 presents some comments in conclusion.

2 The model

We consider a single-good perfectly competitive economy with a constant population of infinitely-lived individuals differentiated only by their initial holdings of wealth. Wealth equals ownership of capital stock less net debts outstanding[4]. Each individual is endowed with a unit of raw labour at each point in time. Individuals are classified into a finite number of groups q. Let $c_i(t)$ and $a_i(t)$ denote respectively the rate of consumption and the amount of wealth owned by an individual in group i at time t and let l_i denote the constant proportion of population in group i $(i = 1, 2, ..., q)$.

For any time t, we denote the per capita values of the rate of consumption, the capital stock and output by $c(t)$, $k(t)$ and $y(t)$ respectively. The wage rate is denoted by $w(t)$ and the rate of return to capital by $r(t)$. Note that $c(t) = \sum_{i=1}^{q} l_i c_i(t)$ and $k(t) = \sum_{i=1}^{q} l_i a_i(t)$. The capitalised value of an individual's lifetime stream of wage earnings time t onwards is denoted by $W(t)$. That is, $W(t) = \int_t^{\infty} w(\tau) e^{-\int_t^{\tau} r(v)dv} d\tau$.

We consider a family of technologies $F(K, L; G) = AK^{\alpha}L^{1-\alpha} + GK$, $\alpha \in (0,1)$, $G > 0$, ordered by the efficiency parameter G.[5] Technologies with higher values of G are also more capital-biased. Each member of this family satisfies the standard properties of an aggregate neoclassical production function.[6] The marginal product of capital increases without bound as capital per unit labour approaches zero and it approaches a positive lower bound G as capital per unit labour increases without bound. As Jones and Manuelli (1990) demonstrate, given a sufficiently large value for G, the latter property is sufficient to generate endogenous growth, the long-run rate of growth being positively related to the value of G. Let,

$$f(k) = Ak^{\alpha} + Gk; A > 0, \alpha \in (0,1), G > \theta > 0$$

[4]Net debts outstanding is defined as equal to the amount of loans taken (and awaiting repayment) less the amount of loans given (and awaiting repayment).

[5]This is part of the general class of technologies considered by Jones and Manuelli (1990).

[6]Note, however, that production is possible without the use of labour. Therefore, it is more appropriate to think of capital as including both human and nonhuman capital. The factor 'labour' would then represent raw or unskilled labour.

where k denotes capital per unit labour and $f(k)$ the average product of labour. θ, as defined later, is the rate of time preference of all individuals. At any time t, we have

$$w(t) = f(k(t)) - k(t) f'(k(t)) = (1-\alpha) Ak(t)^{\alpha} \text{ and } r(t) = f'(k(t)) = \alpha Ak(t)^{\alpha-1} + G \tag{1}$$

Note that the rate of return to capital is positively related to the value of the efficiency parameter G while the wage rate is independent of the value of G.

The specification of the individual's optimisation problem is standard. At each instant $\tau \geq 0$, every individual in group $i \in \{1, 2, ..., q\}$ chooses a pair (c_i, a_i), $c_i : [0, \infty) \to R$, $a_i : [0, \infty) \to R$, which solves the following problem:

$\max \int_{\tau}^{\infty} [\ln c_i(t)] \, e^{-\theta(t-\tau)} dt$
subject to: $\dot{a}_i(t) = w(t) + r(t) a_i(t) - c_i(t)$, *for all* $t \geq \tau$;
$\lim_{t\to\infty} a_i(t) \, e^{-\int_{\tau}^{t} r(v)dv} \geq 0; c_i(t) \geq 0$, *for all* $t \geq \tau$; $a_i(\tau)$ *given.*

For a greater part of this note we will be considering an arbitrary pair of distinct technologies belonging to an arbitrary family of technologies $\{Ak^{\alpha} + Gk | A > 0, \alpha \in (0,1); A, \alpha \; fixed; G > \theta\}$ ordered by the efficiency parameter G. The more efficient technology in the pair will be represented by $f^h(k) = Ak^{\alpha} + G^h k$ and the less efficient by $f^l(k) = Ak^{\alpha} + G^l k$ $(G^h > G^l)$. Similarly, the variables along the growth path associated with the more efficient technology will be marked by the superscript h and the corresponding variables for the less efficient technology by the superscript l. For example, capital stocks per capita at time t along the two associated growth paths will be distinguished as $k^h(t)$ and $k^l(t)$.

3 Equilibrium growth paths

It is well known that along any equilibrium growth path for the above model

$$\forall i \in \{1, 2, ...q\} \wedge \forall t \geq 0 : c_i(t) = \theta \{a_i(0) + W(0)\} \, e^{\int_0^t \{r(v) - \theta\} dv} \tag{2}$$

$$\forall i \in \{1, 2, ...q\} \wedge \forall t \geq 0 : a_i(t) = \frac{a_i(0) + \int_0^t \{w(\tau) - c_i(\tau)\} \, e^{-\int_0^{\tau} r(v)dv} d\tau}{e^{-\int_0^t r(v)dv}} \tag{3}$$

Moreover, the paths of per capita consumption and capital stock must satisfy the set of conditions

$$\textbf{(S)} \; \forall t \geq 0 : \dot{c}(t) = c(t) \{f'(k(t)) - \theta\}, \; \dot{k}(t) = f(k(t)) - c(t), \; c(t) > 0;$$
$$\lim_{t\to\infty} k(t) \, e^{-\int_0^t f'(k(v))dv} = 0; \; k(0) \; (> 0) \; \textit{given.}$$

Barro and Sala-i-Martin (1995, pp.162 – 164) show that there exists a unique pair (c, k) satisfying the set of conditions (S) with $k(t) \geq 0$ for all t. The pair characterises a growth path for the economy along which per capita consumption, capital stock and output increase without bound and the rates of growth[7] of these variables asymptotically approach $G-\theta$. That is,

$$\forall t \geq 0 : \dot{k}(t) > 0 \; and \; \lim_{t\to\infty} k(t) = \infty \tag{4}$$

$$\lim_{t\to\infty} \hat{c}(t) = \lim_{t\to\infty} \hat{k}(t) = \lim_{t\to\infty} \hat{y}(t) = G-\theta \tag{5}$$

It is then easily proved that if the initial distribution of wealth is such that

$$\forall i \in \{1,2,...,q\} : a_i(0) > -W(0) \tag{6}$$

where $W(0)$ is determined by (1) and the set of conditions (S), then there exists a unique perfect foresight competitive equilibrium growth path for the economy.[8] Along this growth path, the time paths of individual consumption and wealth holding satisfy equations (2) and (3) while the time paths of per capita consumption and capital stock satisfy the set of conditions (S) and equations (4) and (5).

Given the characteristics of this growth path we can derive specific lower and upper bounds for the rate of growth of the wage rate and for the present value of an individual's lifetime stream of wage earnings. These are given in the following result.

Lemma 1 *Given an initial amount $k(0) > 0$ of capital per capita let the pair (c,k) satisfy the set of conditions (S) and the conditions (4) and (5). Then it must be true that*

$$\forall t \geq 0 : \alpha\left[\alpha A k(t)^{\alpha-1} + G - \theta\right] < \hat{w}(t) < \alpha\left[Ak(t)^{\alpha-1} + G - \theta\right] \tag{7}$$

$$and \; \frac{w(0)}{(1-\alpha)r(0)+\alpha\theta} < W(0) < \frac{w(0)}{(1-\alpha)G+\alpha\theta} \tag{8}$$

Proof. See Appendix. ■

4 Technological efficiency and the welfare of the capital-poor

Let us consider a family of technologies $\{Ak^\alpha + Gk | A > 0, \alpha \in (0,1); A, \alpha \; fixed; G > \theta\}$ ordered by the efficiency parameter G. Take any two distinct members of this family. Conforming to the notation introduced in setion 2, the more efficient technology has the index value G^h and the less efficient technology has the index value G^l. Let us assume we

[7]We denote the growth rate of any variable x at an instant t by $\hat{x}(t)$.

[8]Existence of an equilibrium growth path therefore requires that the initial level of net indebtedness of any individual in the economy be not too large. The proof of existence is basically premised on showing that along the path satisfying conditions (S) the value of the lifetime stream of individual wage earnings is finite; for which, see Lemma 1 below.

are given an initial distribution of wealth and an initial per capita capital stock $k(0) > 0$ in the economy such that condition (6) is satisfied for both technologies. That is,

$$\forall i \in \{1, 2, ..., q\} : a_i(0) > \max\left[-W^h(0), -W^l(0)\right] \tag{9}$$

where $W^h(0)$ and $W^l(0)$ are determined, given the corresponding technologies, from (1) and the conditions (S).

Given the initial capital stock and the initial distribution of wealth there is thus a unique equilibrium growth path associated with each technology. Now, suppose that U_i^h and U_i^l denote repectively the present values of the lifetime stream of utility attained by an individual in group i along the growth paths associated with the more efficient and the less efficient technologies. Using (2) we can show that

$$U_i^h - U_i^l = \frac{1}{\theta} \ln\left\{\frac{a_i(0) + W^h(0)}{a_i(0) + W^l(0)}\right\} + \int_0^\infty \int_0^t \left\{r^h(v) - r^l(v)\right\} dv e^{-\theta t} dt$$

From (1) and (4), it follows that, $\forall t \geq 0 : r^h(t) - r^l(t) < \alpha A k(0)^{\alpha-1} + G^h - G^l$.

Therefore, $U_i^h - U_i^l \;<\; \frac{1}{\theta}\left\{\ln\left\{\frac{a_i(0) + W^h(0)}{a_i(0) + W^l(0)}\right\} + \frac{1}{\theta}\left\{\alpha A k(0)^{\alpha-1} + G^h - G^l\right\}\right\}$

Therefore, if $a_i(0) \;<\; \dfrac{W^l(0) - e^{\frac{1}{\theta}\left\{\alpha A k(0)^{\alpha-1} + G^h - G^l\right\}} W^h(0)}{e^{\frac{1}{\theta}\left\{\alpha A k(0)^{\alpha-1} + G^h - G^l\right\}} - 1}$ then $U_i^h - U_i^l < 0$

Note that, if $W^h(0) \geq W^l(0)$ then the L.H.S. in the inequality can be satisfied only by violating (9). However, this is not true if $W^h(0) < W^l(0)$. We have therefore proved the following proposition.

Proposition 1 *If the present value of an individual's lifetime stream of wage earnings is greater for the growth path associated with the more efficient technology, then all individuals with sufficiently small initial holdings of wealth will be worse off under the more efficient technology.*

From the inequalities in (8) (Lemma 1) it follows that if $\alpha A k(0)^{\alpha-1} < G^h - G^l$ then $W^h(0) < W^l(0)$. Also, (4) and (7) (Lemma 1) imply that for all t, $\hat{w}^h(t) > \hat{w}^l(t)$ if $Ak(0)^{\alpha-1} < G^h - G^l$. Note from (1) that $w^h(0) = w^l(0)$. Therefore, given Proposition 1, the following holds.

Proposition 2 *If the initial amount of capital per capita in the economy is sufficiently large then the rate of growth of the real wage is always higher along the growth path associated with the more efficient technology (the initial wage rate being the same for both technologies) but individuals with sufficiently small initial holdings of wealth are worse off along the same growth path.*

The possible trade-off between technological efficiency and the distribution of welfare can be more starkly represented if we assume that instantaneous utility is a function of the *relative* rate of consumption. It is possible to prove that for the family of instantaneous utility functions given by

$$u(c_i(t)) = \frac{1}{1-\sigma}\left(\frac{c_i(t)}{c(t)}\right)^{1-\sigma}, \sigma > 0,$$

the equilibrium growth path, for a given technology and initial conditions, is exactly the same as in the case of the logarithmic felicity function. The following counterpart of Proposition 2 is then easily obtained.

Proposition 3 *If the initial amount of capital per capita in the economy is sufficiently large then the rate of growth of the real wage is always higher along the growth path associated with the more efficient technology (the initial wage rate being the same for both technologies) but the welfare of every individual with initial wealth holding less than the average is smaller along the same path.*

5 Conclusion

This note shows that trade-offs between technological efficiency (and growth) and the lifetime welfare of the capital-poor are possible in the dynamic context of a simple AK model of endogenous growth with a convex technology and infinitely-lived individuals. Such trade-offs follow from the possible capital bias of more efficient technologies which may adversely affect the value of human wealth associated with unskilled labour. Moreover, these trade-offs may arise even if more efficient technologies are associated with a higher time-profile of real wage rates for unskilled labour.

Although the results presented relate to a specific class of technologies, the intuition behind the results can be presented in the context of a general aggregate production function $F(K,L)$ with diminishing marginal factor products F_K, F_L, and constant returns to scale in K and L. Note that if a technology is both more efficient (higher values of F for every combination of K and L) and more capital-biased (higher values of $(F_K.K)/(F_L.L)$ for every combination of K and L) than another, then the associated marginal product of capital F_K must be greater for any combination of K and L. For any given time profile of individual wage earnings, higher values of F_K would imply a smaller capitalised value for the lifetime stream of wage earnings of individuals $W(0)$ because the rate of discount on future wage earnings (the interest rate payable on borrowing against future wage income) would be higher.

Higher rates of return to capital however tend to induce higher rates of growth of the capital stock which, in turn, promote higher rates of growth of wages. Thus, a more efficient technology may be associated with a higher time-profile of wage earnings even if the marginal product of labour F_L for any combination of K and L is the same for all technologies considered. If the negative impact of higher interest rates on $W(0)$ dominates the positive impact of higher wage rates as demonstrated here for the case $F(K,L) = AK^{\alpha}L^{1-\alpha} + GK$

, a more efficient technology would be associated with a smaller value of $W(0)$ even if the associated time profile of wage earnings is higher.

Higher values of F_K associated with more efficient but more capital-biased technologies induce individuals to save more, leading to higher rates of growth of individual consumption (see conditions (S)). A smaller value for $W(0)$ however implies a reduction in human wealth associated with unskilled labour and therefore has a negative effect on the entire time profile of individual consumption. This negative effect is proportionately greater in magnitude the smaller is an individual's initial capital or wealth holding as a share of the sum of that wealth holding and $W(0)$. If an individual's initial net worth is sufficiently small, the negative effect of lower human wealth associated with unskilled labour may outweigh the positive impact of higher growth rates of consumption on an individual's lifetime welfare. Hence, as demonstrated in this note, capital-poor individuals may actually end up worse off along growth paths associated with more efficient technologies even though these growth paths may be characterised by a higher time profile of wage earnings.

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6 Appendix

Proof. [Lemma 1] We first prove the inequalities in (7).

Given that (c,k) satisfies the set of conditions (S), it follows that for all $t \geq 0$,

$$\frac{dk(t)}{dt} e^{-\int_0^t \frac{f(k(v))}{k(v)} dv} - k(t) \frac{f(k(t))}{k(t)} e^{-\int_0^t \frac{f(k(v))}{k(v)} dv} = -c(t) e^{-\int_0^t \frac{f(k(v))}{k(v)} dv}$$

i.e. $$\int_t^\infty \frac{d}{d\tau}\left\{k(\tau) e^{-\int_0^\tau \frac{f(k(v))}{k(v)} dv}\right\} d\tau = -\int_t^\infty c(\tau) e^{-\int_0^\tau \frac{f(k(v))}{k(v)} dv} d\tau$$

Using conditions (S) it follows that

$$\lim_{\tau\to\infty}\left\{k(\tau) e^{-\int_0^\tau \frac{f(k(v))}{k(v)} dv}\right\} - k(t) e^{-\int_0^t \frac{f(k(v))}{k(v)} dv} = -\int_t^\infty c(t) e^{\int_t^\tau \{f'(k(v))-\theta\} dv} e^{-\int_0^\tau \frac{f(k(v))}{k(v)} dv} d\tau$$

Therefore, from (1), $$\lim_{\tau\to\infty}\left\{k(\tau) e^{-\int_0^\tau f'(k(v)) dv}\right\} \lim_{\tau\to\infty} e^{-\int_0^\tau \frac{w(v)}{k(v)} dv} - k(t) e^{-\int_0^t \frac{f(k(v))}{k(v)} dv}$$
$$= -c(t) e^{-\int_0^t \frac{f(k(v))}{k(v)} dv} \int_t^\infty e^{\int_t^\tau \{f'(k(v))-\theta\} dv} e^{-\int_t^\tau \left\{f'(k(v))+\frac{w(v)}{k(v)}\right\} dv} d\tau$$

Using the transversality condition in (S), $$c(t) = \frac{k(t)}{\int_t^\infty e^{-\int_t^\tau \left\{\frac{w(v)}{k(v)}+\theta\right\} dv} d\tau} \tag{10}$$

From (1) we get, $\frac{d}{dk(t)}\left\{\frac{w(t)}{k(t)}\right\} < 0$. Then, from (4), $\forall t \geq 0 : \frac{d}{dt}\left\{\frac{w(t)}{k(t)}\right\} < 0$.

Also, from (4), by L'Hopital's Rule, $\lim_{t\to\infty} \frac{w(t)}{k(t)} = \lim_{t\to\infty} \frac{dw(t)}{dk(t)} = \lim_{t\to\infty} \alpha(1-\alpha) k(t)^{\alpha-1} = 0$

Therefore, from (10) we can show that for all $t \geq 0$, $\theta < \frac{c(t)}{k(t)} < \theta + \frac{w(t)}{k(t)}$

Since $w(t) = (1-\alpha) Ak(t)^\alpha$ and $c(t) = Ak(t)^\alpha + Gk(t) - \dot{k}(t)$, it follows that

$$\forall t \geq 0 : \alpha Ak(t)^{\alpha-1} + G - \theta < \hat{k}(t) < Ak(t)^{\alpha-1} + G - \theta \tag{11}$$

From (1) we know that $\hat{w}(t) = \alpha\hat{k}(t)$, for all $t \geq 0$. Thus, (7) is proved.

We next go on to prove the inequalities in (8). Let us define for all $k \geq 0$, $g(k) = Ak^\alpha$.

Then, $f(k) = g(k) + Gk$.

Integrating by parts, $W(0) = \int_0^\infty w(t)\, e^{-\int_0^t r(v)dv} dt = \int_0^\infty w(t)\, e^{-\int_0^t \{g'(k(v))+G\}dv} dt$

$$= \left\{ w(t)\, e^{-\int_0^t g'(k(v))dv} \int e^{-Gt} dt \right\}_0^\infty - \int_0^\infty \frac{d}{dt} \left\{ w(t)\, e^{-\int_0^t g'(k(v))dv} \right\} \left(\int e^{-Gt} dt \right) dt$$

We have already proved that $\lim_{t\to\infty} \frac{w(t)}{k(t)} = 0$ and we know that $\lim_{t\to\infty} k(t)\, e^{-\int_0^t r(v)dv} = 0.$

Thus, simplifying we get, $W(0) = \frac{w(0)}{G} + \frac{1}{G} \int_0^\infty \left\{ \frac{dw(t)}{dt} - g'(k(t))\, w(t) \right\} e^{-\int_0^t r(v)dv} dt.$

Since $\hat{w}(t) = \alpha \hat{k}(t),\ W(0) = \frac{w(0)}{G} + \frac{1}{G} \int_0^\infty \left\{ \alpha \hat{k}(t) - g'(k(t)) \right\} w(t)\, e^{-\int_0^t r(v)dv} dt.$

Note that $g'(k(t)) = \alpha A k(t)^{\alpha-1}$. Also, from (4), $k(t)^{\alpha-1} < k(0)^{\alpha-1},$ for all $t > 0$.

Therefore, using (11) we get,

$$\frac{w(0)}{G} + \left\{ \alpha (G-\theta) - (1-\alpha)\, \alpha A k(0)^{\alpha-1} \right\} \frac{W(0)}{G} < W(0) < \frac{w(0)}{G} + \alpha (G-\theta) \frac{W(0)}{G}$$

Simplifying, $\frac{w(0)}{(1-\alpha)\, r(0) + \alpha\theta} < W(0) < \frac{w(0)}{(1-\alpha)\, G + \alpha\theta}$.

Hence Lemma 1 is proved. ■