E C O N O M I C S   B U L L E T I N

A note on two notions of arbitrage

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Abstract

In the framework of economics models with unbounded short sales a number of different conditions limiting arbitrage opportunities have been introduced. Dana, Le Van and Magnien [JET.87(1999)169] appeal to the condition of compactness of the individually rational utility set and show that all prior conditions in the literature limiting arbitrage opportunities imply this compactness. More recently,Page, Wooders and Monteiro [JME.34(2000)439] introduced the condition of inconsequential arbitrage. In this note, we add to the conclusion of Dana, Le Van and Magnien [JET.87(1999)169] that inconsequential arbitrage implies the compactness of the individually rational utility set and also demonstrate that the converse does not hold.

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1 Introduction

The seminal papers of Hart [8], Grandmont [5] and Green [6] on temporary equilibrium models and competitive securities market models have motivated a large literature studying economies with unbounded short sales; see for example Hammond [7], Werner [14], Page ([11], [12]), Nielsen[10], Dana, Le Van and Magnien [3], Monteiro, Page and Wooders [9] and Allouch [2]. The common feature of these models, unlike the classic model of Arrow-Debreu-MacKenzie, is that there is no exogenous lower bound for agents' consumption sets; arbitrage opportunities may be unbounded and thus, equilibrium may not exist. To solve this problem several conditions limiting arbitrage opportunities have been introduced. Roughly, these conditions, based on recession cones, ensure that all utility-increasing and mutually compatible net trades can be exhausted. An extensive discussion is provided in Dana, Le Van and Magnien [4]. Indeed, Dana, Le Van and Magnien [4] show that *compactness of the individually rational utility* set $\mathcal{U}$ is implied by all prior conditions in the literature limiting arbitrage opportunities. When the individually rational utility set is compact arbitrage opportunities are exhausted in utility space and this suffices, along with standard assumptions of nonsatiation, for existence of equilibrium.

More recently, Page, Wooders and Monteiro [13] introduced the condition of *inconsequential arbitrage* and demonstrated existence of equilibrium using an adaptation of Hart's [8] 'back-up' argument. Rather than using recession cones, inconsequential arbitrage is defined in terms of individual utility-increasing and mutually compatible net trade directions. As shown in Page, Wooders and Monteiro [13] the prior conditions in the literature based on recession cones all imply that inconsequential arbitrage is satisfied. However, the question of the relationship between compactness of the individually rational utility set and inconsequential arbitrage is left unresolved.

In this present note we show that inconsequential arbitrage implies that $\mathcal{U}$ is compact and that the converse does not hold[1]. The idea of the proof is to set up a sequence of n-bounded economies. We then show that, provided that inconsequential arbitrage is satisfied, when n is sufficiently large, the individually rational utility set of the original economy coincides with the individually rational utility set of the sequence of bounded economies[2]. Our result follows from the fact that for a bounded economy the utility set is bounded as well. An example is provided to show that compactness

[1] This result was originally reported in my Ph.D. dissertation [1], and then quoted in Page et al. [13].

[2] This is in line with the Compactly Dominated Feasibility [CDF] condition in Allouch [2] which requires that there exists a compact subset $\overline{\mathcal{A}}$ of the set of *individually rational* and *attainable allocations* $\mathcal{A}$, such that the allocations of $\mathcal{A}$ are weakly Pareto dominated by the allocations of $\overline{\mathcal{A}}$.

of $\mathcal{U}$ does not imply inconsequential arbitrage. The example illustrates a situation where there is a mutually compatible net trade direction in which utility is strictly increasing. Thus the back-up argument does not hold, but $\mathcal{U}$ is compact.

The note is organised as follows. In Section 2, we introduce the basic model and the two conditions: compactness of the individually rational utility set $\mathcal{U}$ and inconsequential arbitrage. Section 3 is devoted to our main result. We will show that inconsequential arbitrage implies that the utility set is compact. In the last section, we provide an example where $\mathcal{U}$ is compact and inconsequential arbitrage does not hold.

2 The basic model

We consider an asset exchange economy $\mathcal{E} = ((X_i, u_i, e_i)_{i=1\ldots,m})$, with ℓ assets and m investors. For every $i = 1.\ldots, m$, $X_i \subset R^\ell$ is the choice set of the i-th investor, $e_i \in X_i$ her/his initial endowment vector and $u_i : X_i \to R$ her/his utility function.

We denote by $\mathcal{A}$ the set of *individually rational* and *attainable allocations* of the economy $\mathcal{E}$, that is:

$$\mathcal{A} = \{(x_i) \in \prod_{i=1}^{m} X_i \mid \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} e_i \text{ and } u_i(x_i) \geq u_i(e_i), \forall i\}.$$

We also denote by $\mathcal{U}$ the *individually rational utility set* of the economy $\mathcal{E}$, that is:

$$\mathcal{U} = \{(v_i) \in R^m_+ \mid \exists x \in \mathcal{A}, \ s.t. \ u_i(e_i) \leq v_i \leq u_i(x_i), \forall i\}.$$

We make the following assumptions. *For all* $i = 1, \ldots, m$*,*

[A.1] *X_i is a closed, convex subset of R^ℓ;*

[A.2] *u_i is strictly quasi-concave;*

[A.3] *u_i is upper semi-continuous.*

We say that $y \in R^{\ell m}$ is an *arbitrage direction* for the economy $\mathcal{E}$ if y is the limit of some sequence $\{(\lambda^n x^n)\}_n$ with $\lambda^n \downarrow 0$ and $\{(x^n)\}_n \in \mathcal{A}$. We shall denote by:

$$\text{arb}(\mathcal{E}) = \{y \in R^{\ell m} \mid \exists \{(x^n)\}_n \in \mathcal{A}, \lambda^n \downarrow 0, y = \lim_{n \to +\infty} \lambda^n x^n\}$$

the set of all arbitrage directions for $\mathcal{E}$. We shall denote also by

$$\mathrm{arbseq}(y) = \{\{(x^n)\}_n \in \mathcal{A} \mid \exists \lambda^n \downarrow 0, y = \lim_{n\to+\infty} \lambda^n x^n\},$$

the set of all *arbitrage sequences* corresponding to $y \in \mathrm{arb}(\mathcal{E})$.

Definition 2.1 *The economy $\mathcal{E}$ satisfies the inconsequential arbitrage condition if for all $y \in \mathrm{arb}(\mathcal{E})$ and $\{(x^n)\}_n \in \mathrm{arbseq}(y)$, there exists an $\epsilon > 0$ such that for all n sufficiently large*

$$x_i^n - \epsilon y_i \in X_i \text{ and } u_i(x_i^n - \epsilon y_i) \geq u_i(x_i^n), \forall i.$$

3 Inconsequential arbitrage implies $\mathcal{U}$ is compact

In order to prove that inconsequential arbitrage implies that the individually rational utility set $\mathcal{U}$ is compact, we need a notion of bounded economies. Given a positive integer n, an *n-bounded economy* is denoted by $\mathcal{E}^n = ((X_i^n, u_i, e_i)_{i=1\dots,m})$ where $X_i^n = X_i \cap \overline{B}(0,n)$. We choose n large enough so that $e_i \in B(0,n)$, for all $i = 1.\dots,m$. For each positive integer n, the set of individually rational and attainable allocations $\mathcal{A}^n$ and the set of individually rational utility allocations $\mathcal{U}^n$ for the bounded economy $\mathcal{E}^n$ are defined in a similar way to the definition of $\mathcal{A}$ and $\mathcal{U}$. That is:

$$\mathcal{A}^n = \{(x_i) \in \prod_{i=1}^m X_i^n \mid \sum_{i=1}^m x_i = \sum_{i=1}^m e_i \text{ and } u_i(x_i) \geq u_i(e_i), \forall i\}, \text{ and}$$

$$\mathcal{U}^n = \{(v_i) \in R_+^m \mid \exists x \in \mathcal{A}^n, \ s.t.\ u_i(e_i) \leq v_i \leq u_i(x_i), \ \forall i\}.$$

It is obvious that under [A.1]-[A.3], $\mathcal{U}^n$ is compact, since $\mathcal{A}^n$ is compact.

We now state the main result of this section.

Proposition 3.1 *Under* [A.1]-[A.3]*, if the economy $\mathcal{E}$ satisfies the inconsequential arbitrage condition, then there exists an integer n_0 such that for all $n \geq n_0$, $\mathcal{U}^n = \mathcal{U}$ and therefore $\mathcal{U}$ is compact.*

Proof of Proposition 3.1. Suppose the contrary. Since $\mathcal{U}^n \subset \mathcal{U}^{n+1} \subset \mathcal{U}$, it follows that for all n, $\mathcal{U} \not\subset \mathcal{U}^n$. Then, we can take a sequence of attainable allocations $\{(x^n)\}_n \in \mathcal{A}$ such that

$$\forall x \in \mathcal{A}^n, \exists i \text{ such that } u_i(x_i) < u_i(x_i^n).$$

We define the set

$$\mathcal{B}^n = \{x \in \mathcal{A} \mid u_i(x_i) \geq u_i(x_i^n), \forall i\}.$$

Let us consider the optimisation problem

$$\mathcal{P}^n = \begin{cases} \inf \sum_{i=1}^m \|x_i\| \\ x \in \mathcal{B}^n \end{cases}$$

Claim 3.1 *$\mathcal{P}^n$ has a solution $z^n \in \mathcal{B}^n$.*

Proof of Claim 3.1. It is clear that $\mathcal{B}^n$ is a nonempty closed subset of $R^{\ell m}$. Moreover, the function $f^n : x \longmapsto \sum_{i=1}^m ||x_i||$ defined on $\mathcal{B}^n$ is continuous and coercive. Then the problem $\mathcal{P}^n$ has a solution $z^n \in \mathcal{B}^n$.□

Claim 3.2 $\lim_{n\to+\infty} \sum_{i=1}^m \|z_i^n\| = +\infty$.

Proof of Claim 3.2. Since $z^n \in \mathcal{B}^n$, it follows that $z^n \notin \mathcal{A}^n$. But z^n is an attainable allocation, and therefore we must have $z^n \notin \prod_{i=1}^m (X_i \cap \overline{B}(0, n))$. Hence $\sum_{i=1}^m \|z_i^n\| > n$.□

Let y denote any cluster point of the sequence $\frac{z^n}{\sum_{i=1}^m \|z_i^n\|}$.

Claim 3.3 *For n sufficiently large and for all $0 < \epsilon \leq 1$ one have:*

$$\sum_{i=1}^m \|z_i^n - \epsilon y_i\| < \sum_{i=1}^m \|z_i^n\|.$$

Proof of Claim 3.3. We first remark that, if $y_i = 0$, then $\|z_i^n - \epsilon y_i\| = \|z_i^n\|$. Moreover, $I_0 := \{i \mid y_i \neq 0\} \neq \emptyset$, since $\sum_{i=1}^m \|y_i\| = 1$. Hence for all $i \in I_0$, we have

$$\begin{aligned} \|z_i^n - \epsilon y_i\| &\leq \|z_i^n - \frac{\epsilon z_i^n}{\sum_{i=1}^m \|z_i^n\|}\| + \|\frac{\epsilon z_i^n}{\sum_{i=1}^m \|z_i^n\|} - \epsilon y_i\| \\ &= \|z_i^n\| + \epsilon(\|\frac{z_i^n}{\sum_{i=1}^m \|z_i^n\|} - y_i\| - \|\frac{z_i^n}{\sum_{i=1}^m \|z_i^n\|}\|). \end{aligned}$$

Since

$$\lim_{n\to+\infty}(\|\frac{z_i^n}{\sum_{i=1}^m \|z_i^n\|} - y_i\| - \|\frac{z_i^n}{\sum_{i=1}^m \|z_i^n\|}\|) = -\|y_i\| < 0,$$

we obtain for n sufficiently large

$$(\|\frac{z_i^n}{\sum_{i=1}^m \|z_i^n\|} - y_i\| - \|\frac{z_i^n}{\sum_{i=1}^m \|z_i^n\|}\|) < 0.$$

We can conclude $\|z_i^n - \epsilon y_i\| < \|z_i^n\|$, for n sufficiently large .□

To end the proof we notice that from the inconsequential arbitrage condition, it follows, that for some $\epsilon > 0$ and for n sufficiently large, $z^n - \epsilon y \in \mathcal{B}^n$, which contradicts Claim 3.3 since z^n is a solution of $\mathcal{P}^n$.□

4 Example

In this section we provide an example in which we have compact $\mathcal{U}$ while inconsequential arbitrage is not satisfied. Consider the economy with two consumers and three commodities.

Consumer 1 has the following characteristics:

$X_1 = R^3_+$

$u_1(x,y,z) = \begin{cases} \frac{x+y}{y+1} & \text{if } x \in [0,1] \\ x & \text{if } x \geq 1 \end{cases}$

$e_1 = (0,0,0).$

Consumer 2 has the following characteristics:

$X_2 = [-1, +\infty[\times R_- \times R_+$

$u_2(x,y,z) = z$

$e_2 = (0,0,0).$

The set of individually rational and attainable allocations is:

$$\mathcal{A} = \{(x,y,0), (-x,-y,0) \in R^6 \mid x \in [0,1], y \in R_+\}.$$

Hence

$$\begin{aligned} u(\mathcal{A}) &= \{(\frac{x+y}{y+1}, 0) \mid x \in [0,1], y \in R_+\} \\ &= [0,1] \times \{0\}. \end{aligned}$$

Since $\mathcal{U} = (u(\mathcal{A}) + R^2_-) \cap R^2_+$, we have also $\mathcal{U} = [0,1] \times \{0\}$, and therefore $\mathcal{U}$ is compact. Moreover, one can easily show that $\mathcal{U} = \mathcal{U}^2$.

In order to prove that inconsequential arbitrage is not satisfied, we define the sequence $\{(x^n)\}_n \in \mathcal{A}$, where

$$x_1^n = (0,n,0) \text{ and } x_2^n = (0,-n,0).$$

Then

$$u_1(x_1^n) = 1 - \frac{1}{n+1} \text{ and } u_2(x_2^n) = 0.$$

Let

$$y_1 = (0,1,0) \text{ and } y_2 = (0,-1,0).$$

It is clear that $y \in \text{arb}(\mathcal{E})$ and $\{(x^n)\}_n \in \text{arbseq}(y)$. But for all $\epsilon > 0$ and for all n, we have

$$u_1(x_1^n - \epsilon y_1) = 1 - \frac{1}{n - \epsilon + 1} < u_1(x_1^n) = 1 - \frac{1}{n+1}.$$

(Note that the back-up argument doesn't hold.) Thus, inconsequential arbitrage is not satisfied .□

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