E C O N O M I C S   B U L L E T I N

Kemeny's rule and Slater's rule: A binary comparison

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Abstract

The purpose of this paper is to provide a binary comparison of two distance−based preference aggregation rules, Slater's rule and Kemeny's rule. It will be shown that for certain lists of individual preferences the outcomes will be antagonistic in the sense that what is considered best according to one rule is considered worst according to the other rule.

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1 Introduction

The purpose of this paper is to provide a binary comparison of two distance-based preference aggregation rules, Slater's rule and Kemeny's rule. Both rules avoid the problem of voting cycles, and therefore, they are used in the literature to extend the simple majority rule as it is pointed out by Fishburn (1977). Moreover, not only do those voting rules guarantee a choice but even a transitive social preference.

Inherent in both voting rules is the idea of being "close" to simple majority rule. Intuitively, ranking the alternatives according to Kemeny's rule can be seen as the best compromise in the sense that on average it gives the "closest" social preference to the individual preferences. The social preference derived from Slater's rule is intuitively close to simple majority rule as it minimizes the distance to the simple majority relation. To be more precise, Kemeny's rule is the composition of simple majority rule with the function that assigns the "closest" unanimous preference profile to any preference profile. Slater's rule is the composition of the function that assigns the "closest" weak order to any complete binary relation with the simple majority rule. Such "closest" social preferences need not necessarily be unique. This paper shows that there exist preference profiles such that the Slater winner, i.e. the top alternative in the social preference derived from the Slater rule, is the Kemeny loser, i.e. the bottom alternative in the social preference derived from the Kemeny rule, and vice versa. Hence, despite this common underlying idea of "closeness" to simple majority rule, for certain preference profiles the outcomes of the two rules will be antagonistic in the sense that what is considered best according to one rule is considered worst according to the other rule.

The significance of those voting rules lies, on the one hand, in the fact that both rules are used to overcome the problems of voting cycles and intransitivities associated with simple majority rule (Fishburn, 1977). On the other hand, the richer information provided by voting rules, which instead of selecting just one alternative, rank all alternatives, has obvious advantages.[1] For example, recruitment committees rank candidates to avoid having to call for a new meeting in case higher ranked candidates drop out later. Therefore, further insight into the binary relationship between the Slater rule and the Kemeny rule will be useful in evaluating the differences in the social preferences that might occur.

Comparisons of preference aggregation rules in general take place in various different frameworks. Fishburn (1977) analyses different social choice functions using an axiomatic framework. His analysis and comparison is based on whether or not they satisfy certain reasonable conditions for social choice functions. Laffond et al. (1995) use a set-theoretical framework. They compare tournament choice correspondences[2] providing results on the inclusion/disjunction relations between different correspondences. For pairs of choice correspondences they determine whether the selected alternatives of one are also always selected by the other, whether there is always a non-empty intersection, or whether for some tournaments there is an empty intersection. In contrast, Ratliff (2001, 2002) compares Dodgson's rule to Kemeny's rule and the Borda rule in a framework which is binary and distance-based, i.e. considers binary relations obtained from voting rules with respect to the distance between them. He shows that there is no consistency between the Dodgson rule and the other two rules in the sense that the Dodgson winner can be found in any position of the

[1] Such rules which assign an alternative from the entire set of alternatives to any preference profile, are called voting schemes in Gibbard (1973). This should not be confused with social choice functions which assign to any pair consisting of a preference profile and a subset of the set of alternatives a non-empty subset of that set. Hence social choice functions have more structure than the voting rules used in this paper.

[2] A tournament social choice correspondence assigns a subset of the set of alternatives to any binary relation over the set of alternatives.

social preferences derived from Kemeny's rule and Borda's rule. It is the latter approach which will be followed in this paper.
The paper is structured as follows. The next section states the formal framework. Sections 3 and 4 introduce the Slater rule and the Kemeny rule. Finally, the main theorems are presented and proved in section 5.

2 Formal Framework

Let X denote a finite set of $m \geq 4$ alternatives and I denote a finite set of $n \geq 2$ individuals. A preference relation $R \subseteq X^2$ is a binary relation on X. For all $x, y \in X$, the weak preference of x over y will be denoted by $x \succsim_R y$, the symmetric and asymmetric parts of R will be written as $\sim_R$ and $\succ_R$ respectively.[3] For any set $T \subseteq X$, $R \mid T = \{(x,y) \in R : x, y \in T\}$ is the restriction of R to T. Let $\mathcal{B}$ be the set of all complete binary relations on X, $\mathcal{W} \subset \mathcal{B}$ the set of all weak orders (complete and transitive binary relations) on X and $\mathcal{L} \subset \mathcal{W}$ the set of all linear orders (complete, transitive and asymmetric binary relations) on X. A preference profile will be denoted by $u = (R_1^u, R_2^u, ..., R_n^u) \in \mathcal{W}^n$ where $R_i^u \in \mathcal{W}$ is individual i's preference on X in preference profile u. For all $x_j, x_k \in X$, the majority margin of x_j over x_k in profile $u \in \mathcal{W}^n$ is denoted by $a_{j,k}^u = \left|\{i \in I : x_j \succ_{R_i^u} x_k\}\right| - \left|\{i \in I : x_k \succ_{R_i^u} x_j\}\right|$.[4] We define simple majority rule as a function $v : \mathcal{W}^n \to \mathcal{B}$ such that for all $u \in \mathcal{W}^n$ and all $x_j, x_k \in X$, $x_j \succsim_{v(u)} x_k$ if and only if $a_{j,k}^u \geq 0$. That is, an alternative x_j is at least as good as alternative x_k if and only if there are at least as many individuals preferring x_j over x_k than there are individuals preferring x_k over x_j.
Finally, use will be made of concepts which measure the distance between binary relations and preference profiles, respectively. Let $\mathbb{R}$ be the set of all real numbers. The Kemeny distance between two binary relations $R, R' \in \mathcal{B}$ is given by half the cardinality of their symmetric difference[5], i.e $\delta(R, R') = \frac{|(R-R') \cup (R'-R)|}{2}$. Distance on the set of preference profiles will be measured by the distance function $d : \mathcal{W}^n \times \mathcal{W}^n \to \mathbb{R}_+$ which is such that for all $u, u' \in \mathcal{W}^n$, $d(u, u') = \sum_{i=1}^{n} \delta(R_i^u, R_i^{u'})$.

3 Slater's Rule

To preserve the attractiveness of the simple majority rule even in cases of intransitive simple majority relations, Slater (1961) suggested assigning the weak order which is of minimal distance to the simple majority relation. I.e., Slater's rule is the composition of the function that assigns the closest weak order to the simple majority relation relative to the Kemeny

[3] Subscripts will be dropped whenever there is no danger of confusion.
[4] Whenever there is no danger of confusion the superscript will be dropped.
[5] The division by 2 is for the convenience of being able to talk about distance values and numbers of pairwise switches interchangeably.

distance function with simple majority rule.[6] The Slater ranking will thus be defined as follows:

Definition 1: For all profiles $u \in \mathcal{W}^n$, $S \in \mathcal{W}$ is the *Slater ranking* if and only if for all $R \in \mathcal{W}$, $\delta\left(v(u), S\right) \leq \delta\left(v(u), R\right)$.

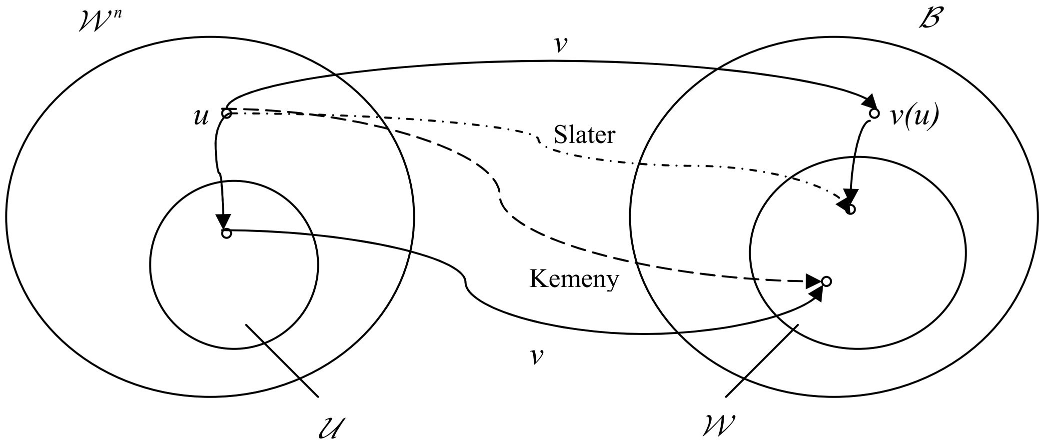

Figure 1

The intuition behind Slater's rule can also be seen in Figure 1, where in addition to the notation defined above, $\mathcal{U} \subset \mathcal{W}^n$ is the set of all unanimous preference profiles, i.e. $u \in \mathcal{U}$ if and only if for some $R \in \mathcal{W}$, $R_i^u = R$ for all $i \in I$. Figure 1 provides an example of simple majority rule leading to a non-transitive outcome. Given the preference profile $u \in \mathcal{W}^n$, the simple majority rule provides a non-transitive social preference $v(u)$, i.e. $v(u) \notin \mathcal{W}$. Hence, Slater's way to overcome such problems is by assigning the closest preference relation in $\mathcal{W}$ to any $v(u)$.

4 Kemeny Ranking

Kemeny's (1959) approach is focusing on the domain of *v*. In particular, if $v(u) \notin \mathcal{W}$, then the Kemeny rule selects the social preference $v(u')$, where $u' \in \mathcal{U}$ minimizes the Kemeny distance between the preference profile $u \in \mathcal{W}^n$ and all preference profiles $u'' \in \mathcal{U}$.[7] This can also be seen in Figure 1. The Kemeny ranking is now defined as follows:

Definition 2: Let $\overline{u} = (K, K, ..., K) \in \mathcal{U}$. For all $u \in \mathcal{W}^n$, $K \in \mathcal{W}$ is the *Kemeny ranking* if and only if for all $u' \in \mathcal{U}$, $d\left(u, \overline{u}\right) \leq d\left(u, u'\right)$.

Saari and Merlin (2000) suggested a more convenient way to derive the Kemeny ranking based only on information about the pairwise margins. To be more precise, let, for all

[6] That distance minimization relative to different distance functions can characterize different aggregation rules can be seen in Baigent and Klamler (2004) who provide a distance characterisation of the transitive closure rule.
[7] Young and Levenglick (1978) provide a detailed discussion of the Kemeny rule and prove that the Kemeny rule is the unique rule that is neutral, consistent, and Condorcet.

$u \in \mathcal{W}^n$, and all $R \in \mathcal{W}$, $\mathcal{C}_R^u = \{\{x,y\} \subset X : v(u) \mid \{x,y\} \neq R \mid \{x,y\}\}$ be the set of all pairs such that the preference over these pairs is different in *v(u)* and *R*. Then for all $u \in \mathcal{W}^n$, the function $g^u : \mathcal{W} \to \mathbb{R}$ is defined such that for all $R \in \mathcal{W}$, $g^u(R) = \sum_{\{x_j,x_k\} \in \mathcal{C}_R^u} \left|a_{j,k}^u\right|$. This means that the function g^u counts the margins over pairs of alternatives whose preference is different in *v(u)* and *R*.

Lemma 1: (Saari and Merlin, 2000): For all $u \in \mathcal{W}^n$, $K \in \mathcal{W}$ is the *Kemeny ranking* if and only if for all $R \in \mathcal{W}$, $g^u(K) \leq g^u(R)$.

5 Results

In this section we will show that for some preference profiles Slater's rule and Kemeny's rule will result in antagonistic outcomes in the sense that what is considered best according to one rule is considered worst according to the other rule.

The following example will provide some intuition for the results that will be proved below.

Example 1: Consider $X = \{x_1, x_2, x_3, x_4\}$, $|I| = 9$, and the following preference profile $u \in \mathcal{W}^9$ given in Table 1, where alternatives in higher rows are preferred to alternatives in lower rows and the numbers in the first row determine how many voters have each ranking.

3	1	2	2	1
x_1	x_2	x_3	x_4	x_4
x_2	x_3	x_2	x_1	x_3
x_3	x_4	x_4	x_2	x_2
x_4	x_1	x_1	x_3	x_1

Table 1: Preference Profile

This preference profile is such that the pairwise tallies and margins are as stated in Table 2. Obviously, there is a cycle including all alternatives in *X*. Since the preference relation *S* which is such that for all pairs of alternatives $(x_i, x_j) \neq (x_1, x_4)$, $x_i \succsim_S x_j$ if and only if $x_i \succsim_{v(u)} x_j$ and $x_1 \succ_S x_4$ has a Kemeny distance of 1 and any other transitive preference relation has a strictly higher Kemeny distance, we can conclude that x_1 is the Slater winner and x_4 is the Slater loser. From Table 2 we can also determine the Kemeny ranking using Saari and Merlin's method. For the preference relation *S* we get a value of $g^u(S) = 3$, as only one switch in the pair x_1, x_4 has to be made, giving a margin of 3. Consider the preference relation *K*, $x_2 \succ_K x_3 \succ_K x_4 \succ_K x_1$. This leads to a value $g^u(K) = 2$. As any other weak order will definitely have a value larger than 3, *K* is the Kemeny ranking and x_1 the Kemeny loser.

	Tallies	Margins		Tallies	Margins
$x_1 \succ x_2$	5,4	1	$x_2 \succ x_3$	6,3	3
$x_1 \succ x_3$	5,4	1	$x_2 \succ x_4$	6,3	3
$x_1 \succ x_4$	3,6	-3	$x_3 \succ x_4$	6,3	3

Table 2: Pairwise margins

Hence, this example shows the existence of antagonistic outcomes of Slater's rule and Kemeny's rule. We generalize this observation in the following two theorems.

Theorem 1: If there are at least four alternatives, then there exist preference profiles such that the unique Slater winner is the unique Kemeny loser.

Theorem 2: If there are at least four alternatives, then there exist preference profiles such that the unique Kemeny winner is the unique Slater loser.

As the following results will depend on creating particular preference profiles, we will make use of a theorem in Saari (1995).[8] Let $\mathbb{Z}$ be the set of all integers. Given a preference profile $u \in \mathcal{W}^n$, the vector of pairwise majority margins will be denoted by $w^u = \left(a^u_{1,2},...,a^u_{j,k},...,a^u_{m-1,m}\right) \in \mathbb{Z}^{\binom{m}{2}}$ where $j,k \in \{1,2,...,m\}$, $j<k$.

Lemma 2: For any $z \in \mathbb{Z}^{\binom{m}{2}}$ with all entries having the same parity, there exists a preference profile $u \in \mathcal{L}^n$ such that $w^u = z$.[9]

Proof of Theorem 1: Let $u \in \mathcal{W}^n$ be such that $a^u_{1,j} = l$ for all $j \in \{2,3,...,m-1\}$, $a^u_{j,k} = q$ for all $j,k \in \{2,3,...,m\}$, $j<k$, and $a^u_{1,m} = -h$. Obviously, given a preference profile $u \in \mathcal{W}^n$ which satisfies the stated conditions, applying the majority rule leads to a non-transitive social preference relation of the form $x_1 \succ x_2 \succ ... \succ x_{m-1} \succ x_m \succ x_1$.

<u>Slater part:</u> The unique Slater ranking $S \in \mathcal{W}$ is $x_1 \succ_S x_2 \succ_S ... \succ_S x_{m-1} \succ_S x_m$ and $\delta(v(u),S)=1$. Any preference relation with x_1 above x_m but with x_1 not on top will obviously be of larger Kemeny distance from the simple majority relation. In any preference relation with x_m above x_1, x_m has to be moved above at least $m-2-s$ alternatives and x_1 has to be moved below at least s alternatives ($2 \le s \le m-2$). It is obvious that there have to be at least $m-2$ switches, which guarantees that, for $m \ge 4$, x_1 will be the Slater winner for any $l,h \in \mathbb{Z}_+$.

<u>Kemeny part:</u> Consider the three preference relations $R,R',R'' \in \mathcal{W}$ such that $x_1 \succ_R x_2 \succ_R ... \succ_R x_{m-1} \succ_R x_m$, $x_2 \succ_{R'} x_3 \succ_{R'} ... \succ_{R'} x_{m-1} \succ_{R'} x_m \succ_{R'} x_1$ and $x_m \succ_{R''} x_1 \succ_{R''} x_2 \succ_{R''} ... \succ_{R''} x_{m-1}$. It follows that $g^u(R)=h$, $g^u(R')=l(m-2)$ and $g^u(R'')=q(m-2)$. It is obvious that any other preference relation $\overline{R} \in \mathcal{W}$ has

[8] A similar version of that theorem can be found in the literature on tournaments by Debord (1987). Extensions of that lemma can be found in Ratliff (2001) and Klamler (2002).

[9] For a proof of lemma 2 refer to Saari (1995) or Ratliff (2001),

$g^u(\bar{R}) > \min\{h, l(m-2), q(m-2)\}$. Therefore only R, R' and/or R'' could be closest to u in the Kemeny sense. Now it is clear that x_1, the Slater winner, is the Kemeny loser in R'. But R' is the unique Kemeny ranking whenever $l(m-2) < h$ and $l < q$. By Lemma 2 such a preference profile exists and this proves the theorem. □

Proof of Theorem 2: Let the preference profile be such as in the proof of Theorem 1.
Slater part: As before the unique Slater ranking $S \in \mathcal{W}$ is $x_1 \succ_S x_2 \succ_S \ldots \succ_S x_{m-1} \succ_S x_m$ and x_m is the Slater loser.
Kemeny part: Consider, as before, the three preference relations R, R', R'' which could be the only possible Kemeny rankings. Now it is clear that x_m, the Slater loser, is the Kemeny winner in ranking R''. But R'' is the unique Kemeny ranking whenever $q(m-2) < h$ and $q < l$. By Lemma 2 such a preference profile exists and this proves the theorem. □

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