E c o n o m i c s   B u l l e t i n

A meaningful two−person bargaining solution based on ordinal preferences

Jozsef Sakovics
Edinburgh School of Economics

Abstract

In this note I argue that the traditional argument proving the non−existence of two−person ordinal bargaining solutions is misleading, and also provide an example of such a solution.

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1 Introduction

Bargaining solutions based on ordinal preferences are of interest for two reasons. On one hand, the cardinal preferences of the players may be difficult to elicit, leaving us having to do with ordinal ones. On the other hand, we may want the solution to be robust to certain variations (say, in attitudes towards risk) in the cardinal preferences, leading us to the use of ordinal ones instead. Notwithstanding this interest, there are no known reasonable ordinal bargaining solutions. The ultimate cause for this hiatus in bargaining theory is Shapley's (1969) famous non-existence proof (for the two-person[1] bargaining problem). In my opinion, we have given up all too quickly. In this note, I scrutinize Shapley's argument and propose an alternative interpretation of it, which is more robust and also opens the way for a two-person, ordinal bargaining solution. For completeness' sake, I also propose such a solution, together with a non-cooperative implementation.

2 Shapley's impossibility result

Take a compact and strictly convex bargaining set, S, and a disagreement point, d, (in utility space, of course). Assume that there exists an ordinal bargaining solution, $B(S,d) \in S$.

First, note that, since the solution is based on ordinal preferences, any monotonic (that is, order-preserving) transformation applied to the utilities should leave the solution unchanged, in terms of physical outcome. In other words, the "new" solution in utility terms should correspond to the transformed utility of the "old" solution. Let the extreme points of the (individually rational part of the) Pareto frontier be denoted by, a and b. Now, we can apply the following sequence of two monotonic transformations (one for each player's utility) to all the points in S that strictly dominate the coordinate-wise minimum[2] of a and b: first, move the points to the right, maintaining the order with respect to the extreme point, which is not moving. Next move the points down, in such a way that the final bargaining set looks identical to the old one. Figure 1 depicts these transformations. Shapley then argues that the solution (in terms of utilities) of the transformed problem should be the same as in the original problem (since the new problem is identical to the old one). However, for any

[1] For bargaining games with more players Shapley's result does not follow. See e.g. Shubik (1982) and Safra and Samet (2002).

[2] Sometimes called the "meet."

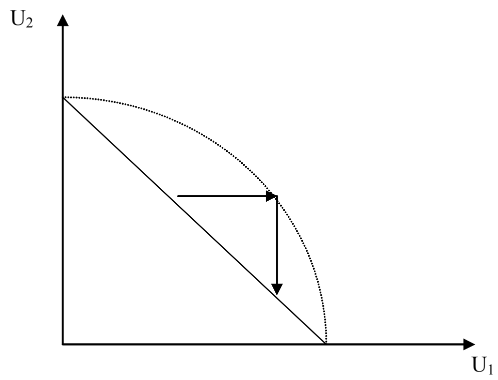

Figure 1:

(feasible and strictly individually rational) solution, following the transformations the original utility vector will correspond to a different physical outcome from the original one. Contradiction.[3]

3 Are we ranking the outcomes or their utilities?

I believe that Shapley's result is wholly misleading. Note, that it hinges on the assertion that the (twice) transformed utilities lead to the very same problem that we started out with. I would like to convince you otherwise.

Consider the following (discrete[4]) bargaining problem: there are five possible locations (say, for an airport between Glasgow and Edinburgh) denoted by A, B, C, D, E. Based on preliminary reports, Glasgow and Edinburgh city councils' preferences are antagonistic. Glasgow's preferences are: $u_g(A) > u_g(B) > u_g(C) = u_g(D) = u_g(E)$, while Edinburgh's preferences are: $u_e(A) < u_e(B) < u_e(C) = u_e(D) = u_e(E)$. There are two possible ways of approaching this problem – in the absence of information on the intensities of preferences – in order to find a "fair" compromise. One is to say that effectively there are only three possible outcomes, since C, D and E are "equivalent". Consequently, the fair solution is B. Alternatively, we can say that there are five candidates, so the solution should be (one) in the middle: C, D or E.

Let me introduce a realistic procedure[5] to determine the solution: the players take

[3]See Osborne and Rubinstein (1990), p. 25, for a slight modification of this argument.

[4]The discreteness of the example is only used to simplify the exposition and has nothing to do with the underlying argument.

[5]This is a simpler version of the veto game in Anbarci (1993).

turns in vetoing an alternative. The outcome of this mechanism clearly supports the latter interpretation. This is not surprising, since the rules of the game are in terms of outcomes rather than utilities. But that IS the point: bargaining parties think in terms of alternatives, NOT the associated utilities.[6]

Alternatively, assume that further studies have broken the indifference and the updated preferences are: $u_g(A) > u_g(B) > u_g(C) > u_g(D) > u_g(E)$, and $u_e(A) < u_e(B) < u_e(C) < u_e(D) < u_e(E)$. Now, either interpretation would lead to C as the solution. That is, just by breaking the tie between C, D and E,the solution would move from B to C, according to the "utility-centric" view. But, is it reasonable that a reordering of outcomes that do not include the solution – so that the rank of the solution is unchanged for both players – should have an effect on it? I claim that it isn't.[7] Note that the "axiom" I am advocating is in the spirit of – but much weaker than – Nash's Independence of Irrelevant Alternatives, since I do not ask for all those alternatives not to matter, just their internal ranking not to make a difference.

Once we accept that bargaining is over the physical outcomes and not their utilities, we can keep working in utility space, but keeping track of the number of physical outcomes that give the same utility to both[8] players. In other words, it is not sufficient to define a bargaining problem in terms of the support of the utility distribution – as it is usually done – rather we need to take into account the "density" of physical outcomes in utility space. As a consequence, we have already refuted Shapley's argument, since the bargaining problem after the two utility transformations will not be identical to the original one, since we have moved "utility mass" down and to the right.

The above observation should really not come as a surprise, since for a finite number of agreements it is easy to see (and well known) that Shapley's trick does not

[6]My arguments are similar to Sen's (1979) and Roemer's (1986) critique of the "welfarist" (i.e. utility based) nature of axiomatic bargaining theory. On one hand, I feel the need to repeat these in a game theory journal, since the message seems not to have filtered across (witness the Osborne-Rubinstein (1990) treatment of Shapley's result). On the other hand, there are two differences between my argument and theirs. First, I am not appealing to any subjective motive (like "social justice"), just to common sense. Second, I am not rejecting the idea of carrying out the analysis in utility space, I am only requiring that we make sure to "carry over" all the necessary information.

[7]Yet another way of making the same argument is to require that the solution should satisfy continuity to small perturbations. That is, the solution to $u_g(A) > u_g(B) > u_g(C) = u_g(D) = u_g(E)$, $u_e(A) < u_e(B) < u_e(C) = u_e(D) = u_e(E)$ should be consistent with the solution to $\lim_{\varepsilon\to 0} \{u_g(A) > u_g(B) > u_g(C) + \varepsilon > u_g(D) > u_g(E) - \varepsilon, u_e(A) < u_e(B) < u_e(C) - \varepsilon < u_e(D) < u_e(E) + \varepsilon\}$. However, continuity in ordinal utilities is of limited meaning, so I am not stressing this argument.

[8]If the indifference is only for one player, one of the two outcomes is Pareto dominated.

work even according to the old interpretation: there is no way of using a monotone transformation to reach the same Pareto frontier. Now, why should we expect that the existence of an ordinal solution depend on the number of agreements? Note that the very justification of using the continuum approach is that it is the limit of the discrete one. Consequently, we need continuity in the solution (and especially in its existence) at the limit as the number of alternatives grows towards infinity.

4 Constructing a solution

The above argument may lead you to believe that I am advocating a radically new definition of the bargaining problem. For the purposes of the current paper I am not.[9] Notice that all we need to do is not to throw away information that we have already got. Any real-life bargaining problem – obviously – is defined in terms of the available physical alternatives. The modeler should simply not lose sight of the density of the feasible agreements corresponding to each utility vector.

In order to construct an ordinal solution, recall, that in an ordinal world of negotiation, what identifies an agreement is its *rank* according to bargainers' preferences. Therefore, I propose the following candidate for a "reasonable" ordinal bargaining solution:

Definition 1 *(Ordinal Bargaining Solution) Take the set of Pareto efficient (physical) agreements most preferred by Player 1 among those that she prefers to at most 50% of all Pareto efficient (physical) agreements. Similarly, take the set of Pareto efficient (physical) agreements most preferred by Player 2 among those that he prefers to at most 50% of all Pareto efficient (physical) agreements. If these coincide, that is the solution. Otherwise, pick Player 1's preferred outcome among them.*

This is a well-defined, and obviously ordinal, solution – even[10] when there are a continuum of feasible agreements. Note as well, that while the agreement selected by this solution may not be unique, both players are indifferent between all solutions. That is, the utility vector is uniquely determined. Finally, it is implemented by the veto game introduced in the previous section.

[9] See Esteban and Sákovics (2003) for a proposal to redefine the nature of "the" bargaining problem.

[10] In fact, especially, since we have no tie-breakingworries when there are a continuum of feasible agreements.

5 Conclusion

The purpose of this note was to draw attention to the dangers of forgetting that even though we carry out our analysis in utility space, bargaining is conducted over physical agreements. This observation happens to overthrow the widely held belief that (meaningful) two-person ordinal bargaining solutions are not feasible. I have made no effort here to come up with a full-fledged analysis of ordinal two-person bargaining problems. The example of such a solution that I provided is just that: an example.

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