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On the differentiability of the benefit function

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Abstract

The benefit function, introduced by Luenberger, provides a tool for well−defined cardinal comparisons of different bundles of goods. It also allows to study in an orignal way optimal consumers and firms choices, Pareto−optimality etc... In this note we prove that the benefit function is differentiable under standard conditions. This property is useful in order to study optimal choices.

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1. Introduction

This note studies the differentiability of the benefit function introduced in Luenberger (1992 a,b) and (1995 a). The benefit function is based on a reference bundle g and allows a well-suited cardinal comparison of different bundles of goods. Let a bundle x and a reference utility level α be given. The benefit function $b(x, \alpha)$ measures how many units of g an individual would be willing to give up to move from a utility level α to the point x. There is a corresponding notion in production theory, the Chambers, Chung, Fare's directional distance function. The benefit function has been used in various settings: production theory (Chambers, Chung, Färe (1995), (1998)), consumer theory (Luenberger (1992 a,b, 1995 a, 1996)), risk theory (see e.g. Quiggin and Chambers (1998)), general equilibrium theory (Luenberger 1992 b, 1995 b, 1996). It has also been generalized (see Briec and Gardères (2004)). In these papers, differentiability of the benefit function enables to get interesting results such as: equality of marginal benefits across consumers at an interior Pareto-optimum, equality of marginal benefits with prices at a consumer optimum. Furthermore, Blume-Hudgins and Primont (2003) derive a set of useful restrictions on the first and second derivatives of the directional distance function in order to build an econometric model.

The preceding results are not fully satisfactory since differentiability of the benefit function is assumed. But as benefit function is a derived concept, this amounts to implicitely imposing conditions on the primitives of the models. If the utility function is quasi-concave, then the benefit function is concave and it is differentiable almost everywhere. However, this statement does not provide information about differentiability at a given arbitrary bundle (and even less if the utility function fails to be quasi-concave). Hence, it would be interesting to have conditions on the primitives of the model that ensure differentiability of the benefit function.

In this paper, we shall prove that for any interior bundle x such that $b(x, \alpha)$ is a real number and $x - b(x, \alpha)g$ is in the interior, under a classical regularity condition, $b(.,.)$ is continuously differentiable in a neighborhood of (x, α). We also give the expressions of the partial derivatives in terms of the exogenous variables of the model (to the best of our knowledge these expressions are new). The argument relies upon a simple application of the Implicit Function Theorem. We do not assume quasi-concavity nor concavity. We also show that under some (standard) conditions whenever a bundle x is interior and $b(x, \alpha)$ is a real number, then $x - b(x, \alpha)g$ is indeed an interior point.

2. Setup and Results

Let us summarize the notions used in this note. We assume that a consumer is endowed with a utility function $U : \mathbb{R}^n_{++} \to \mathbb{R}$ that is continuously differentiable on its domain. We let the benefit function $b(.,.)$ be defined as $b : \mathbb{R}^n_+ \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, $(x, \alpha) \mapsto b(x, \alpha) = \sup\{\lambda \in \mathbb{R}; x - \lambda g \in \mathbb{R}^n_{++}, U(x - \lambda g) \geq \alpha\}$, where g is a fixed vector in $\mathbb{R}^n_+ \setminus \{0\}$. If there does not exist λ such that $x - \lambda g \in \mathbb{R}^n_{++}$, and $U(x - \lambda g) \geq \alpha$, we set $b(x, \alpha) = -\infty$. Notice that when the set $\{\lambda \in \mathbb{R}; x - \lambda g \in \mathbb{R}^n_{++}, U(x - \lambda g) \geq \alpha\}$ is non empty, being upper bounded in $\mathbb{R}$, it has a finite supremum. We shall consider a pair (x_0, α_0) for which the following assumption is satisfied.

(H1). *We assume that:* $x_0 \in R^n_{++}$ *and* $x_0 - b(x_0, \alpha_0)g \in R^n_{++}$. *We also suppose that* $\langle \nabla U(x_0 - b(x_0, \alpha_0)g), g\rangle \neq 0$ *and that* g *is locally good at* x_0, *i.e.: there is a neighborhood* W_{x_0}*of* x_0 *such that for all* x *in* W_{x_0}, *for all positive* β, *one has:* $U(x + \beta g) > U(x)$.

Note that when $U(.)$ is strictly increasing (i.e. $x \geq y, x \neq y$, implies $U(x) > U(y)$), then g is locally good at each x.

Proposition 1. *Assume H1. Then there exist two neighborhoods of* x_0 *and* α_0, V_{x_0} *and* V_{α_0} *respectively, such that the benefit function* $b(.,.)$ *is continuously differentiable at each point* (x, α) *of* $V_{x_0} \times V_{\alpha_0}$. *Moreover, one has:* $\nabla_x b(x, \alpha) = \frac{\nabla U(x - b(x,\alpha)g)}{\langle \nabla U(x - b(x,\alpha)g), g\rangle}$ *and* $\nabla_\alpha b(x, \alpha) = -\frac{1}{\langle \nabla U(x - b(x,\alpha)g), g\rangle}$.

Proof. We shall first show that $U(x_0 - b(x_0, \alpha_0)g) = \alpha_0$, and then we shall apply the Implicit Function Theorem to obtain our result.
Let us show that $U(x_0 - b(x_0, \alpha_0)g) = \alpha_0$. Since $b(x_0, \alpha_0) = \sup\{\lambda \in \mathbb{R}; x_0 - \lambda g \in \mathbb{R}^n_{++}, U(x_0 - \lambda g) \geq \alpha_0\}$, there is a non-decreasing sequence $(\lambda_n)_n$ which goes to $b(x_0, \alpha_0)$ such that for all n, $U(x_0 - \lambda_n g) \geq \alpha_0$. By continuity, one has $U(x_0 - b(x_0, \alpha_0)g) \geq \alpha_0$. Suppose that $U(x_0 - b(x_0, \alpha_0)g) > \alpha_0$. Then, since $x_0 - b(x_0, \alpha_0)g$ is in $\mathbb{R}^n_{++}$ and $U(.)$ is continuous, there would exist $\lambda > b(x_0, \alpha_0)$ such that: $U(x_0 - \lambda g) > \alpha_0$, which contradicts the definition of $b(x_0, \alpha_0)$.

Now, under H1 there exist neighborhoods $V'_{x_0} \subset W_{x_0}$ and $V'_{b(x_0,\alpha_0)}$ of x_0 and $b(x_0, \alpha_0)$ respectively such that: for all (x, λ) in $V'_{x_0} \times V'_{b(x_0,\alpha_0)}$, $x - \lambda g$ is in $\mathbb{R}^n_{++}$. Let us define the function $H : V'_{x_0} \times V'_{b(x_0,\alpha_0)} \to \mathbb{R}$, $(x, \lambda) \mapsto H(x, \lambda) = U(x - \lambda g)$. One has $H(x_0, b(x_0, \alpha_0)) = U(x_0 - b(x_0, \alpha_0)g) = \alpha_0$.

Hence, since $\langle \nabla U(x_0 - b(x_0, \alpha_0)g), g\rangle \neq 0$, one can apply the Implicit Function Theorem to H (e.g. Florenzano and Levan (2002) (Theorem A.4.1, page 146)). There exist neighborhoods $V_{x_0} \subset V'_{x_0}$, $V_{b(x_0,\alpha_0)} \subset V'_{b(x_0,\alpha_0)}$, V_{α_0} of x_0, $b(x_0, \alpha_0)$ and α_0 respectively; a function $\psi : V_{x_0} \times V_{\alpha_0} \to V_{b(x_0,\alpha_0)}$, such that: for all $x \in V_{x_0}$, $\lambda \in V_{b(x_0,\alpha_0)}$, $\alpha \in V_{\alpha_0}$, $H(x, \lambda) = U(x - \lambda g) = \alpha \Leftrightarrow \lambda = \psi(x, \alpha)$. Moreover, ψ is continuously differentiable on $V_{x_0} \times V_{\alpha_0}$.

It remains to show that for all (x, α) in $V_{x_0} \times V_{\alpha_0}$, $b(x, \alpha) = \psi(x, \alpha)$. So let (x, α) be in $V_{x_0} \times V_{\alpha_0}$. We have $\psi(x, \alpha)$ is in $V_{b(x_0,\alpha_0)}$ and satisfies: $x - \psi(x, \alpha)g \in \mathbb{R}^n_{++}$ and $U(x - \psi(x, \alpha)g) = \alpha$. This implies that the set $\{\lambda \in \mathbb{R}; x - \lambda g \in \mathbb{R}^n_{++}, U(x - \lambda g) \geq \alpha\}$ is non empty. So $b(x, \alpha)$ is finite and $b(x, \alpha) \geq \psi(x, \alpha)$. Suppose that $b(x, \alpha) > \psi(x, \alpha)$. We have $\alpha = U(x - \psi(x, \alpha)g) = U(x - b(x, \alpha)g + (b(x, \alpha) - \psi(x, \alpha))g) > U(x - b(x, \alpha)g)$ since g is locally good by assumption. But this contradicts the definition of $b(x, \alpha)$. Thus $b(x, \alpha) = \psi(x, \alpha)$.

Finally, since the partial derivatives of $b(.,.)$ are locally that of $\psi(.,.)$, the Implicit Function Theorem yields also:

$$\nabla_x b(x,\alpha) = \frac{\nabla U(x - b(x,\alpha)g)}{\langle \nabla U(x - b(x,\alpha)g), g\rangle}$$
$$\nabla_\alpha b(x,\alpha) = -\frac{1}{\langle \nabla U(x - b(x,\alpha)g), g\rangle}$$

This ends the proof. Q.E.D.

The previous proposition raises an immediate question: when is $x - b(x,\alpha)g$ a point in $\mathbb{R}^n_{++}$? We have already mentionned that this is true when $b(x,\alpha)$ is a non-positive real number. In order to give an answer to this question, let us first assume:

(H2). *For all* $x \in \mathbb{R}^n_{++}$, $\overline{\{z : U(z) \geq U(x)\}} \subset \mathbb{R}^n_{++}$, *that is: the closure of the set of the bundles giving a utility level at least as great as* $U(x)$ *is in the interior of* $\mathbb{R}^n_{+}$.

This assumption was introduced by Debreu (1972) and is rather standard in the theory of general equilibrium (see e.g. Magill and Quinzi (1998), page 50).

Proposition 2. *Assume H2. Also assume that* x *is in* $\mathbb{R}^n_{++}$, *that* $b(x,\alpha)$ *is a real number and that there is* $y \in \mathbb{R}^n_{++}$ *such that* $U(y) = \alpha$. *Then* $x - b(x,\alpha)g$ *is in* $\mathbb{R}^n_{++}$ *and* $U(x - b(x,\alpha)g) = \alpha$.

Proof. It suffices to consider the case $b(x,\alpha) > 0$ (i.e $U(x) \geq \alpha$), otherwise, $x - b(x,\alpha)g$ is always in $\mathbb{R}^n_{++}$. Since $b(x,\alpha)$ is finite, there is a non-decreasing sequence of real numbers $(\lambda_n)_n$ that goes to $b(x,\alpha)$ such that for all n, $x - \lambda_n g \in \mathbb{R}^n_{++}$, and $U(x - \lambda_n g) \geq \alpha$. Clearly $(x - \lambda_n g)_n$ goes to $x - b(x,\alpha)g$ and since there is $y \in \mathbb{R}^n_{++}$ such that $U(y) = \alpha$, assumption H2 implies that $x - b(x,\alpha)g \in R^n_{++}$. The rest of the proof is similar to the beginning of the proof of proposition 1. Q.E.D.

This result is interesting since Luenberger takes $\mathbb{R}^n_+ \times \mathcal{U}$ instead of $\mathbb{R}^n_+ \times \mathbb{R}$ as the definition set of $b(.,.)$; $\mathcal{U}$ being the range of $U(.)$. Hence, using Luenberger's definition of $b(.,.)$, assuming H2 and $\langle \nabla U(x - b(x,\alpha)g), g\rangle \neq 0$ would yield differentiability. We shall now introduce an assumption that implies H2 and makes the requirement that α is in the range of $U(.)$ unnecessary.

(H3). *Let* x *be a boundary point of* $\mathbb{R}^n_{++}$.
Then $\lim_{z \in \mathbb{R}^n_{++} \to x} U(z) = -\infty$.

An assumption similar to H3 has been used in the context of Optimal Growth Theory (e.g. McKenzie (1986), assumption (I) page 1285, or McKenzie (2002), Assumption 5, page 249). This assumption implies H2. Suppose not. Then there would exist an x in $\mathbb{R}^n_{++}$ and a sequence $(x_n)_n$ in $\mathbb{R}^n_{++}$ converging to a boundary point z such that for all n, $U(x_n) \geq U(x)$. Then $\lim_{n\to\infty} U(x_n) = -\infty \geq U(x)$ which is impossible.

Note that H3 is indeed stronger than H2 (e.g., $U : (x, y) \in \mathbb{R}^2_{++} \mapsto U(x, y) = xy$ does satisfy H2 but not H3).

Using H3 yields an interesting differentiability result.

Proposition 3. *Assume H3 and that g is good, i.e.: for all x in $\mathbb{R}^n_{++}$, for all positive β, one has $U(x + \beta g) > U(x)$. Let x be in $\mathbb{R}^n_{++}$ and $b(x, \alpha)$ be a real number. Then $x - b(x, \alpha)g$ is in $\mathbb{R}^n_{++}$ and if $\langle \nabla U(x - b(x, \alpha)g), g\rangle \neq 0$, $b(., .)$ is continuously differentiable in a neighborhood of (x, α).*

Proof. Again, the only interesting case is when $b(x, \alpha) > 0$. Let x be in $\mathbb{R}^n_{++}$ and assume that $b(x, \alpha)$ is a real number. Suppose that $x - b(x, \alpha)g$ is in the boundary of R^n_+. By definition of $b(x, \alpha)$, there is a non-decreasing sequence $(\lambda_n)_n$ that goes to $b(x, \alpha)$ such that $U(x - \lambda_n g) \geq \alpha$. Then, under H3, $\lim_{n\to\infty} U(x - \lambda_n g) = -\infty \geq \alpha$, which is impossible. Hence, $x - b(x, \alpha)g$ is an interior point. Since g is good, proposition 1 yields the result. Q.E.D.

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