E C O N O M I C S B U L L E T I N

Proximity Preservation in an Anonymous Framework

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Abstract

This paper gives a formulation for the condition of preservation of preference proximity which, unlike previous formulations, respects the spirit of anonymity pervading social choice theory. Proximity preservation is however shown to be inconsistent with a very weak condition guaranteeing a minimal non−trivial compensation of pivotal changes.

An earlier version of this paper was presented at the joint Research Seminar of the Department of Economics and the Institute of Public Economics, University of Graz. The author wishes to thank Nick Baigent, Christian Klamler, Johann Mitlöhner, and Ulrich Pferschy, for valuable comments.
Citation: Eckert, Daniel, (2004) "Proximity Preservation in an Anonymous Framework." *Economics Bulletin,* Vol. 4, No. 6 pp. 1−6
Submitted: March 1, 2004. **Accepted:** April 21, 2004.
URL: http://www.economicsbulletin.com/2004/volume4/EB−04D70002A.pdf

1 Introduction

Since Baigent (1987) has established the impossibility of a social aggregation rule which preserves preference proximity and satisfies typical social choice conditions, various impossibility results have shown proximity preservation to be a particularly strong inter-profile consistency condition, as it can be shown to be inconsistent with a very weak no-veto condition and the weakest possible form of non-imposition (Baigent and Eckert 2001, see also Eckert and Lane 2002, and Grafe and Grafe 2001). This strength might however be attributed to the fact that proximity preservation has always been formulated in a way which does not respect the equivalence of profiles which are permutations of each other and hence is not consistent in an obvious way with the spirit of anonymity pervading social choice theory. By contrast, this paper uses a formulation of proximity preservation which not only is consistent with, but even implies anonymity. In this framework an impossibility result involving a weak compensation property is established.

2 Formal framework

Let I denote a countable set of agents and let Θ denote an arbitrary set of characteristics. A profile is an ordered tuple $(x_1, x_2, ..., x_{|I|}) \in \Theta^{|I|}$ determined by an index function $u : I \to \Theta$ assigning a characteristic to every agent. Let $U = \Theta^I$ denote the set of all profiles considered as such index functions. A Social Aggregation Rule (SAR) is a function $f : U \to \Theta$ that assigns, as a group outcome, a characteristic in Θ to profiles in U. The set of characteristics need not be endowed with any particular structure. In particular, the characteristics need not be binary relations. Interpretations are not therefore limited to those that are usual in social choice theory as especially preference relations.

Two conditions central to social choice theory can be formulated with the help of equivalence classes of the set of profiles. A SAR f induces a partition U/f of U into f-level sets. Thus, the elements of U/f are $[u]_f := \{u' \in U \mid f(u') = f(u)\}$, for all $u \in U$. Our first property is a weakening of a property, which was introduced by Wilson (1972) as a substitute for the Arrovian Pareto condition. A SAR f has the **non-imposition** property if $|U/f| > 1$. Thus, SAR's that are not imposed are not constant functions and exhibit some response, possibly minimal, to agent's characteristics.

Another useful partition of U will be denoted by U/Σ, where Σ denotes the set of all permutations on I. The elements of U/Σ are $[u]_\Sigma := \{u' \in U \mid \exists \sigma \in \Sigma : u'(i) = u(\sigma(i)), \forall i \in I\}$, for all $u \in U$.

With the help of these two partitions the familiar property of **anonymity** can easily be formulated as $[u]_\Sigma \subseteq [u]_f$, for all $u \in U$. The invariance of the SAR f with respect to permutations captures the idea of an equal, symmetric treatment of the agents.

Two other partitions will also be used: First, U/i is the family of subsets of U, which consist of profiles which have the same characteristic for agent i. Thus the elements of U/i are

$[u]_i := \{u' \in U : u'(i) = u(i)\}$, for all $i \in I$ and $u \in U$. Second, $U/{-i}$ is the family of subsets of *U*, which consist of profiles that are i-variants of a given profile $u \in U$. Thus, the elements of $U/{-i}$ are $[u]_{-i} := \{u' \in U \mid u'(i) \neq u(i) \wedge u'(j) = u(j), \forall j \in I - \{i\}\}$, for all $i \in I$ and $u \in U$.

3 Proximity preservation

The condition of proximity preservation was introduced by Baigent (1987) into the framework of Arrovian social choice theory in analogy to the condition of continuity in the framework of topological social choice theory initiated by Chichilnisky (1979, 1982). (For surveys on the latter see Lauwers 2000 and Baigent 2004.) The other conditions mainly used in the topological framework are unanimity and anonymity, the former being weaker than the corresponding Pareto condition, while the latter is stronger than the corresponding Arrovian condition of non-dictatorship. Like the continuity condition, proximity preservation can be motivated by the intuitively plausible requirement that smaller mistakes in registering individual characteristics should not result in larger changes in the group outcome than larger mistakes (Kelly 1988). A recent generalization of previous results has shown proximity preservation to be inconsistent with two extremely weak conditions in the spirit of unanimity and anonymity respectively, namely the weakest possible form of non-imposition and a no-veto condition (Baigent and Eckert 2001). All the previous formulations of proximity preservation are however prone to criticism because they are not in an obvious way consistent with the spirit of anonymity, as they do not respect the equivalence of profiles which are permutations of each other. An analogous critique has been raised by Baigent (1985) himself against the use of the product topology in topological social choice.

The condition of proximity preservation has been formulated in various ways in the previous literature. The ordinal formulation in Eckert and Lane (2002) is followed here. Its primitives are distance rankings, i.e. binary relations comparing pairs of objects with respect to the distance between these objects.

Definition: A SAR $f: U \to \Theta$ satisfies **proximity preservation** with respect to a distance ranking $d \subseteq \Theta^2 \times \Theta^2$ for characteristics (with asymmetric part denoted d_a) and a distance ranking $D \subseteq U^2 \times U^2$ for profiles (with asymmetric part denoted D_a) if for all $u, u', v, v' \in U$: $(u,u')D_a(v,v') \Rightarrow \neg\big(f(v), f(v')\big)\, d_a \big(f(u), f(u')\big)$.

Essentially, proximity preservation requires that the ranking of distances among profiles shall not be inverted by the ranking of the corresponding group outcomes.

The inconsistency of proximity preservation with various other social choice conditions has been established for very weak conditions on these distance rankings. In particular, the distance ranking $d \subseteq \Theta^2 \times \Theta^2$ for characteristics is only assumed to fulfill the condition of **other-dissimilarity** which requires that for all $x, y \in \Theta$, $x \neq y$ implies $(x,y)d_a(x,x)$. In this case it will be called a **regular** distance ranking. This condition requires that the distance between identical outcomes be strictly less than the distance between distinct outcomes. This requirement is very weak and does not exclude binary relations on Θ^2 that cannot be represented by any real valued function on Θ^2, let alone by a metric. (See Suppes *et al.* 1989, and Eckert and Lane 2002.)

On the distance ranking $D \subseteq U^2 \times U^2$ for profiles only a monotonicity condition is applied. For all $u,u' \in U$, let $\Delta(u,u') := \{i \in I \mid u(i) \neq u'(i)\}$ denote the set of agents that have different characteristics in profiles u and u'. If the distance ranking D on profiles is such that, for all $u,u',u'' \in U$, $\Delta(u,u') \subset \Delta(u,u'')$ and $\Delta(u,u') \cap \Delta(u',u'') = \varnothing$ implies $(u,u'')D_a(u,u')$, it is called a **monotonic** distance ranking on profiles in Baigent and Eckert 2001. This condition is essentially a ceteris paribus condition which requires that the distance from a given profile increases with the number of components which are different.

This formulation of monotonicity, which is implicit in most previous formulations of proximity preservation, however does obviously not satisfy the spirit of anonymity, as it does not respect the equivalence of profiles, which are permutations of each other. It is easily verified that for all $i \in I$, $u \in U, u' \in [u]_{-i}$ and $u'' \in [u']_i \cap [u]_\Sigma$, $(u,u'')D_a(u,u')$, though u'' is a permutation of u.

Thus we will define an anonymous distance ranking on profiles in terms of other collections of characteristics. Such collections, where the order of the elements does not matter, while repetitions are allowed, are known as **multisets** (see e.g. Stanley 1997). Formally, a multiset *M* on a set *S* is a pair $M = \langle S, N_M \rangle$, where $N_M : S \to \mathbb{N}$ is a function which counts for every element $x \in S$ the number $N_M(x)$ of its occurrences in the mulstiset *M*. While a profile $u \in U$ is an ordered tuple $(x_1, x_2, ..., x_{|I|})$ of elements of Θ, the corresponding multiset, denoted by $\underline{u}$, is the pair $\underline{u} = \langle \Theta, N_{\underline{u}} \rangle$, where, for every characteristic $x \in \Theta$, $N_{\underline{u}}(x)$ counts the number of occurrences of *x* in the multiset corresponding to profile *u*. The relation between any profile $u \in U$ and the corresponding multiset $\underline{u}$ is given, for all $x \in \Theta$, by $N_{\underline{u}}(x) = \left|u^{-1}(x)\right|$. In particular, all profiles $u' \in [u]_\Sigma$ determine the same multiset $\underline{u}$.

Hence we define a **weakly anonymous** distance ranking on profiles by a condition in the spirit of other-dissimilarity which states that for all $u,u',u'' \in U$ and the corresponding multisets $\underline{u},\underline{u}',\underline{u}''$, $\underline{u} = \underline{u}' \neq \underline{u}''$ implies $(u,u'')D_a(u,u')$.[1] With respect to a weakly anonymous distance ranking for profiles and a regular distance ranking for characteristics, proximity preservation is not only consistent with anonymity, but can even be shown to imply anonymity of the SAR.

Lemma. If a SAR $f : U \to \Theta$ satisfies proximity preservation with respect to a regular distance ranking $d \subseteq \Theta^2 \times \Theta^2$ for characteristics and a weakly anonymous distance ranking $D \subseteq U^2 \times U^2$ for profiles, then it satisfies anonymity.

Proof. Consider two profiles $u \in U$ and $u' \in [u]_f - [u]_\Sigma$, and assume to the contrary that there exists a profile $u'' \in [u]_\Sigma - [u]_f$. By weak anonymity of the distance ranking *D* for profiles $(u,u')D_a(u,u'')$, while for any regular distance ranking *d* for characteristics $\left(f(u), f(u'')\right) d_a \left(f(u), f(u')\right)$, which violates proximity preservation. □

[1] This condition is termed weakly anonymous as we only require that the distance between all profiles that are permutations of each other be smaller than the distance between profiles that are not permutations of each other and not that the distance between all profiles that are permutations of each other be minimal.

The framework of multisets can also be used to formulate a monotonicity condition, which is not sensitive to permutations and hence respects the spirit of anonymity. By defining for all multisets M and M' on some set S, $M \cap M' = \langle S, N_{M \cap M'} \rangle$ by $N_{M \cap M'}(x) := \min\{N_M(x), N_{M'}(x)\}$, for all $x \in \Theta$, we can define the multiset difference $M - M' = \langle S, N_{M-M'} \rangle$ by $N_{M-M'}(x) := N_M(x) - N_{M \cap M'}(x)$, for all $x \in \Theta$. Finally, multiset inclusion is defined as $M \subseteq M'$ if $N_M(x) \leq N_{M'}(x)$, for all $x \in \Theta$, and $M \subset M'$ if $M \subseteq M' \wedge \neg M' \subseteq M$.

Definition: A distance ranking $D \subseteq U^2 \times U^2$ for profiles is **monotonic** if for all $u, u', u'' \in U$ and the corresponding multisets $\underline{u}, \underline{u}', \underline{u}''$:
$\underline{u}' - \underline{u} \subset \underline{u}'' - \underline{u} \Rightarrow (u, u'') D_a (u, u')$.

Essentially, this monotonicity condition requires that the distance from a given profile increases with the corresponding multiset differences.

Proximity preservation may now be expressed as follows:

Definition: A SAR $f : U \to \Theta$ satisfies proximity preservation relative to a regular distance ranking $d \subseteq \Theta^2 \times \Theta^2$ for characteristics and a weakly anonymous monotonic distance ranking $D \subseteq U^2 \times U^2$ for profiles if, for all $u, u', u'' \in U$:
$(u, u') D_a (u, u'') \Rightarrow \neg \left(f(u), f(u'') \right) d_a \left(f(u), f(u') \right)$.

4 Minimal non-trivial compensation: an impossibility result

Anonymity guarantees that a pivotal change from some profile $u \in U$ to some profile $u' \in [u]_{-i} - [u]_f$ can be compensated if there exists $j \in I - \{i\}$ such that $u(j) = u'(i)$. In this case a profile $u'' \in [u']_i \cap [u]_\Sigma$ (which is a permutation the original profile u) can be obtained from profile u' by changing in turn individual j's characteristic from $u'(j) = u(j) = u'(i)$ to $u''(j) = u(i)$. This possibility is however trivial (as it is an immediate implication of anonymity) and limited at the same time. We thus formulate a condition of minimal non-trivial compensation to guarantee that pivotal changes can be compensated otherwise than via a permutation of the original profile.

Definition: A SAR $f : U \to \Theta$ satisfies **minimal non-trivial compensation** if it is not the case that for all $u \in U, u' \in [u]_{-i} - [u]_f$, $[u']_i \cap [u]_f - [u]_\Sigma = \varnothing$.

This condition is an extremely weak analogue to the compensation properties familiar from multicriteria decision making, which allow differences on some attribute to be compensated by "sufficiently large" differences on other attributes (see e.g. Bouyssou et al. 1997), and it is hard to imagine a reasonable SAR that would not satisfy it. However, the following theorem establishes its inconsistency with proximity preservation.

Theorem. There is no non-imposed SAR $f : U \to \Theta$ that satisfies the condition of minimal non-trivial compensation and which preserves proximity relative to a regular distance ranking

$d \subseteq \Theta^2 \times \Theta^2$ for characteristics and a weakly anonymous monotonic distance ranking $D \subseteq U^2 \times U^2$ for profiles.

Proof. From non-imposition, there exists a pair of profiles $u, u' \in U$ such that $u' \in [u]_{-i} - [u]_f$ and, for at least one such pair, minimal non-trivial compensation implies that there exists another profile $u'' \notin [u]_\Sigma$ such that $u'' \in [u']_i \cap [u]_f$. By the anonymity of the SAR, which is implied by our formulation of preference proximity, $u'' \notin [u']_\Sigma$, so that it is easily verified that in this case $\underline{u}' - \underline{u} \subset \underline{u}'' - \underline{u}$ holds for the corresponding multisets. By monotonicity it follows that $(u, u'') D_a (u, u')$. On the other hand, $f(u) = f(u'') \neq f(u')$, and hence $\left(f(u), f(u')\right) d_a \left((f(u), f(u'')\right)$. Thus, proximity preservation, relative to all regular distance rankings for characteristics and weakly anonymous monotonic distance rankings for profiles, is violated. □

5 Discussion

Previous formulations of the condition of proximity preservation have failed to respect the equivalence of profiles, which are permutations of each other. From an interpretational point of view, they thus are not consistent in an obvious way with the spirit of anonymity, which pervades social choice theory. This inconsistency can, however, not be considered the reason for the tension of proximity preservation with other conditions for social aggregation rules. Even if the condition of proximity preservation is formulated in a way which implies anonymity, it can be shown to be inconsistent with a very weak condition guaranteeing a minimal non-trivial compensation of pivotal changes.

References

Baigent, N. (1987) "Preference proximity and anonymous social choice" *Quarterly Journal of Economics* **102**, 161-169.

Baigent, N. (1985) "Anonymity and continuous social choice" *Journal of Mathematical Economics* **14**, 1-4

Baigent, N. (2004) "Topological Theories of Social Choice" in: *Handbook of Social Choice and Welfare* by K.J. Arrow, A.K. Sen, and K. Suzumura, Eds., Elsevier: Amsterdam, vol II, chapter 17.

Baigent, N., and D. Eckert (2001) "Abstract aggregations and proximity preservation: An impossibility result" Working Paper Zentrum für Interdisziplinäre Forschung der Universität Bielefeld (The Sciences of Complexity, 2001/052).

Bouyssou, D., M. Pirlot, and Ph. Vincke (1997) "A general model of preference aggregation" in: *Essays in Decision Making* by M.H. Karwan, J. Spronk, and J. Wallenius, Eds., Springer: Berlin, 120-134.

Chichilnisky, G. (1979) "On fixed point theorems and social choice paradoxes" *Economics Letters* **3**, 347-351.

Chichilnisky, G. (1982) "Social aggregation rules and continuity" *Quarterly Journal of Economics* **96**, 337-352.

Eckert, D., and B.D. Lane (2002) "Anonymity, ordinal preference proximity and imposed social choices" *Social Choice and Welfare* **19**, 681-684.

Grafe, F., and J. Grafe (2001) "Social Welfare Functions which preserve distances" *Social Choice and Welfare* **18**, 59-64.

Kelly, J.S. (1988) *Social Choice Theory. An Introduction*, Springer: Heidelberg.

Lauwers, L. (2000) "Topological social choice" *Mathematical Social Sciences* **40**, 1-39.

Stanley, R.P. (1997) *Enumerative Combinatorics*, Cambridge University Press, vol. I.

Suppes, P., D.H. Krantz, R.D. Luce, and A. Tversky (1989) *Foundations of Measurement*, Academic Press: San Diego, vol. II.

Wilson, R.B. (1972) "Social choice theory without the Pareto principle" *Journal of Economic Theory* **5**, 478-486.