E C O N O M I C S   B U L L E T I N

A note on cash−in−advance constraints in continuous time

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Abstract

This paper demonstrates a robust proof of the continuous−time transformations of Stockman's cash−in−advance constraints. When the constraint applies to consumption and capital purchases, monetary growth lowers steady state consumption and capital. When the constraint applies only to consumption purchases, monetary growth is superneutral.

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1. Introduction

Stockman (1981) demonstrates the inflationary implications of two independent cash-in-advance constraints. If the cash-in-advance constraint applies to the purchase of capital and consumption goods, inflation decreases steady state capital and inverts the real sector implications of the Mundell-Tobin effect. However, if the constraint applies strictly to the purchase of consumption goods, inflation does not affect the capital stock. Stockman first derived the cash-in-advance constraint models in discrete-time. This paper transforms the two cash-in-advance constraint models into continuous-time to demonstrate the effects of monetary growth targeting on steady state real sector variables. Monetary growth reduces steady state consumption and capital or generates monetary superneutrality depending on underlying modeling assumptions and whether the constraint applies to consumption and capital purchases or a consumption good alone.

Modeling a cash-in-advance constraint in continuous time is relatively new. However, many papers have used this technique in more complicated monetary models without first demonstrating whether the continuous-time transformation is consistent with the discrete-time model.[1] If this transformation from discrete to continuous-time is not consistent, it is hard to establish whether the new cash-in-advance constraint provides the microeconomic underpinnings to study the underlying causality behind changes in steady state real sector variables (Smithin 2004). This paper supports many recent publications that integrate this transformation by providing a mathematical proof of its robustness.

Section 2 demonstrates the representative agent problem and the steady state effect of monetary growth if the constraint applies to the purchase of both consumption and capital

[1] See Calvo (1987), Wang and Yip (1992), Carlstrom and Fuerst (1995, 1998), Svensson (1985) and Mansoorian and Mohsin (2004).

goods. Section 3 analyzes a special case of the constraint that applies strictly to purchases of a consumption good. The final section offers concluding comments.

2. A Cash-in-Advance Constraint on Consumption and Capital

Stockman models a cash-in-advance constraint that applies to the purchase of capital and consumption goods. In continuous-time, this constraint becomes[2]

$$m_t \geq c_t + \dot{k}_t \tag{1}$$

where c is consumption, m is a stock of real balance holdings and k is capital.[3]

Consider a monetary growth model where real balances do not yield utility but appear in the definition of real wealth, and the asset accumulation identity. Representative agents are infinitely-lived with perfect foresight and complete access to capital. They maximize

$$u_0 = \int_0^\infty u(c_t) e^{-\theta t} dt \tag{2}$$

subject to three flow budget constraints[4]

$$\dot{a}_t = f(k_t) + x_t - c_t - \pi_t m_t \tag{3}$$

$$m_t = c_t + \dot{k}_t \tag{4}$$

$$a_t = m_t + k_t \tag{5}$$

and two stock budget constraints

$$\lim_{t\to\infty} a_t e^{-\int_0^t r_v dv} \geq 0 \tag{6}$$

$$\lim_{t\to\infty} k_t e^{-\int_0^t r_v dv} \geq 0 \tag{7}$$

[2] Since money does not yield utility, the nominal interest rate is positive and (1) holds with strict equality.
[3] Capital depreciation is assumed to be zero throughout with no loss of generality.
[4] Where $u_c > 0$ and $u_{cc} < 0$.

where f is a constant returns to scale production function, x is the value of public sector transfers to hold real balances constant following inflation, a is real wealth, π is the rate of inflation, θ is an exogenous rate of time preference and r is the real interest rate.

Substitute (5) into (4) and (3) to get

$$\dot{k}_t = a_t - k_t - c_t \quad (8)$$

$$\dot{a}_t = f(k_t) + x_t - c_t - \pi(a_t - k_t) \quad (9)$$

Maximizing (2) subject to (8), (9), (6) and (7) yields first order optimality conditions

$$u_c(c) - (\lambda + \gamma) = 0 \quad (10)$$

$$\gamma\theta - \lambda\left(f'(k) + \pi\right) + \gamma = \dot{\gamma} \quad (11)$$

$$\lambda\theta + \lambda\pi - \gamma = \dot{\lambda} \quad (12)$$

$$\lim_{t\to\infty} a_t \lambda_t e^{-\theta t} = 0 \quad (13)$$

$$\lim_{t\to\infty} k_t \gamma_t e^{-\theta t} = 0 \quad (14)$$

where (13) and (14) are transversality conditions and λ and γ are two co-state variables.

In the steady state, from (11) with $\dot{\gamma} = 0$

$$\gamma\theta - \lambda\left(f'(k^*) + \pi\right) + \gamma = 0 \quad (15)$$

from (12) with $\dot{\lambda} = 0$

$$\lambda\theta + \lambda\pi - \gamma = 0 \quad (16)$$

from (8) and (5) with $\dot{k} = 0$

$$c^* = m^* \quad (17)$$

The resource constraint is

$$\dot{k} = f(k) - c \quad (18)$$

From (18) with $\dot{k} = 0$ implies

$$c^* = f(k^*) \tag{19}$$

From (3) and (19) with $\dot{a} = 0$, the result is $x^* = m^*\pi^* = \mu m^*$, which implies

$$\pi^* = \mu \tag{20}$$

where μ is the monetary growth rate.

Linearize (10), (15), (16), (17) and (19) around steady state consumption, capital and real balances (c^*, m^*, k^*) using (20), yields

$$\begin{bmatrix} u_{cc} & 0 & 0 & -1 & -1 \\ 0 & -\lambda f'' & 0 & (1+\theta) & -(f'+\mu) \\ 0 & 0 & 0 & -1 & (\theta+\mu) \\ -1 & f' & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} dc^* \\ dk^* \\ dm^* \\ d\gamma \\ d\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda \\ -\lambda \\ 0 \\ 0 \end{bmatrix} d\mu \tag{21}$$

The determinant of the coefficient matrix in (21) is:

$$|\Delta| = (\lambda f'')(-1-(\theta+\mu)) > 0 \tag{22}$$

The effect of monetary growth on steady state capital and consumption is given by

$$\frac{dk^*}{d\mu} = \frac{dc^*}{d\mu} = \frac{-\lambda(f')^2}{|\Delta|} < 0 \tag{23}$$

Substitution effects raise the opportunity cost of holding real balances, which reduces the initial levels of real wealth, real balances and the rate of time preference but increases savings. Whether the added savings raises or lowers steady state consumption and capital depends on the relative magnitudes of two contradictory wealth effects. First, the cash-in-advance constraint acts as an investment tax that directs the added savings away from the production and demand for steady state real sector variables. However, the decline in real wealth generates wealth effects that convert the additional savings from real balances into steady state capital and consumption through a Mundell-Tobin type process. As the cash-in-advance constraint effect dominates the wealth effect, monetary growth reduces steady state capital and consumption to invert the real implications of the Mundell-Tobin effect.

3. A Cash-in-Advance Constraint on Consumption

The alternative cash-in-advance constraint applies strictly to consumption purchases. A continuous-time transformation of this constraint yields

$$c_t \leq m_t \tag{24}$$

Substitute (5) and (24) into (3) to obtain a single flow budget constraint[5]

$$\dot{a}_t = f(a_t - c_t) + x_t - c_t(1 + \pi_t) \tag{25}$$

Maximize (2) subject to (25) and (6) to get the first order optimality conditions

$$u_c(c) - \phi\left(f'(a - c) + (1 + \pi)\right) = 0 \tag{26}$$

$$\phi\theta - \phi f'(a - c) = \dot{\phi} \tag{27}$$

$$\lim_{t \to \infty} a_t \phi_t e^{-\theta t} = 0 \tag{28}$$

where (28) is the transversality condition and ϕ is the co-state variable.

In the steady state, from (27) with $\dot{\phi} = 0$ yields

$$\theta = f'(k) \tag{29}$$

Linearize (26), (29) and (19) around (c^*, k^*) using (20), yields

$$\begin{bmatrix} u_{cc} & -f''\phi & -\alpha \\ 0 & f'' & 0 \\ -1 & f' & 0 \end{bmatrix} \begin{bmatrix} dc^* \\ dk^* \\ d\phi^* \end{bmatrix} = \begin{bmatrix} \phi \\ 0 \\ 0 \end{bmatrix} d\mu \tag{30}$$

where $\alpha = 1 + \mu + f' > 0$. The determinant of the coefficient matrix in (30) is:

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[5] Money does not yield utility so that the nominal interest rate is positive and (24) holds with strict equality.

$$\left|A\right| = -f''\alpha > 0 \tag{31}$$

The effect of monetary growth on steady state capital and consumption is given by

$$\frac{dk^*}{d\mu} = \frac{dc^*}{d\mu} = \frac{0}{\left|A\right|} \tag{32}$$

Monetary growth raises the real costs of consumption and the opportunity cost of real balance holdings. The demand for money is residually determined so that the substitution effect lowers the demand for consumption goods by raising the initial level of real wealth and savings. However, this generates wealth effects that raise the demand for steady state consumption goods and reduce savings. Therefore, imposing a cash-in-advance constraint solely on purchases of consumption goods produces contradictory substitution and wealth effects. As the two effects are of equal magnitude, monetary growth does not produce any change in the demand for steady state real sector variables and is superneutral.

4. Conclusions

This paper transforms Stockman's cash-in-advance constraint models into continuous time to demonstrate the effect of monetary growth on consumption and capital. When the cash-in-advance constraint is applied to both consumption and capital good expenditures, monetary growth generates an investment tax that reduces the steady state production and demand of both goods to reverse the real sector implications of the Mundell-Tobin effect. However, when the cash-in-advance constraint applies only to purchases of consumption goods, monetary growth does not affect the real sector. Without the investment tax effect, exogenous time preference holds the marginal product of capital and the real interest rate constant. Steady state capital and consumption are not affected by monetary changes and monetary policy is therefore superneutral. This paper provides necessary microeconomic underpinnings for continuous-time cash-in-advance constraints by offering mathematical verification of their robustness.

5. References

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