E C O N O M I C S B U L L E T I N

Interest−rate rule and multiple equilibria with endogenous growth

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Abstract

This paper examines the role of interest−rate feedback rule in a monetary endogenous growth model in which money is introduced via a cash−in−advance constraint and long−run growth is sustained by external increasing returns. It is shown that dynamic properties as well as the balanced−growth characterization are highly sensitive not only to the degree of increasing returns but also to the interest−rate feedback rule adopted by the monetary authority. In particular, the conditions for indeterminacy of equilibrium depends heavily upon whether the interest−rate rule is active or passive.

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1 Introduction

In the last several years, many authors have demonstrated that the interest-rate feedback policy advocated by Taylor (1993) may generate multiple equilibria so that it would amplify business fluctuations rather than work as a stabilizing anchor. The well-cited investigations on this topic such as Benhabib et al. (2001a and b) examine real indeterminacy under the interest-rate feedback rule in simple production economies without capital. Very recently, several authors pointed out that indeterminacy under the interest-rate feedback rule is difficult to hold, if the model economy allows capital accumulation. For example, Dupor (2001) and Meng and Yip (2002) introduce the interest-rate feedback rule into the neoclassical monetary growth models and conclude that indeterminacy may only hold by assuming restrictive conditions.[1] The key idea of such a claim is that introducing capital stock narrows the range of parameter values that generate indeterminacy. As a result, the interest-rate feedback rule that yields multiple equilibria in the economy without capital would not be the source of indeterminacy in the economy with capital accumulation.[2]

This paper reconsiders the relationship between the interest-rate rule and indeterminacy of equilibrium in the context of an endogenously growing economy. Existing studies on multiple equilibria in the real business cycle models have shown that indeterminacy conditions for the endogenous growth models may be substantially different from those for the neoclassical (exogenous) growth models. This is particularly true for the models with market distortions in which endogenous growth is sustained by increasing returns to scale. We show that this finding in the real business cycle literature still holds in the monetary endogenous growth model. In contrast to the case of the neoclassical growth model, the interest-rate feedback rule easily produces indeterminacy in our endogenous growth setting. In addition, the dynamic behavior as well as the balanced-growth characterization of the model economy are highly sensitive to whether or not the interest-rate feedback scheme follows the Taylor principle.

2 The Model

In this paper we use an endogenous growth version of the model presented by Benhabib and Farmer (1994) that has been the prototype analytical framework to investigate indeterminacy in the real business cycle theory. We introduce

[1] See also Carlstrom and Fuerst (2001b).

[2] This conclusion somewhat resembles the results concerning the indeterminacy issue in the money-in-the-utility function models under the *constant money growth* rule. It is well known that if the utility function is not separable between consumption and real money balances and if the cross derivative of the utility function exhibits a negative sign, then multiple converging equilibria hold in exchange economies: see, for example, Obstfeld (1983). Indeterminacy, however, will not emerge if capital formation is introduced into the model.

money into the base model via a cash-in-advance constraint. There is a continuum of infinitely-lived households whose number is normalized to one. The representative family solves the following optimizing problem:

$$\max \int_0^\infty \left[\log c - \frac{l^{1+\gamma}}{1+\gamma}\right] e^{-\rho t} dt, \quad \gamma > 0,\ \rho > 0$$

subject to

$$\dot{a} = ra + wl - c - im,$$

$$a = k + m,$$

$$c \leq m,$$

where c is consumption, l labor supply, r the real rate of return, w the real wage rate, i the nominal interest rate, m real money balances, k capital holding, and a is the total wealth of the household. We assume that the cash-in-advance constraint applies to consumption spending alone.

To derive the optimization conditions for the household, we set up the current-value Hamiltonian function such that

$$\mathcal{H} = \log c - \frac{l^{1+\gamma}}{1+\gamma} + \lambda\left(ra + wl + \tau - c - im\right) + \zeta\left(m - c\right) + \eta\left(a - k - m\right),$$

where λ denotes the costate variable of a, and ζ and η are Lagrange multipliers. The necessary conditions for an optimum involve

$$\partial\mathcal{H}/\partial c = 1/c - (\lambda + \zeta) = 0, \tag{1}$$

$$\partial\mathcal{H}/\partial l = -l^\gamma + \lambda w = 0, \tag{2}$$

$$\partial\mathcal{H}/\partial m = -\lambda i + \zeta - \eta = 0, \tag{3}$$

$$\partial\mathcal{H}/\partial k = -\eta = 0, \tag{4}$$

$$\zeta\left(m - c\right) = 0, \ \ m - c \geq 0, \ \ \zeta \geq 0, \tag{5}$$

$$\dot{\lambda} = \lambda\left(\rho - r\right) - \eta, \tag{6}$$

together with the transversality condition, $\lim_{t\to\infty} a\left(t\right)\lambda\left(t\right)e^{-\rho t} = 0$, and the initial condition on the total wealth, $a\left(0\right) = k\left(0\right) + m\left(0\right).$

Equations (1) and (2) yield

$$cl^\gamma = \frac{w}{1+i}. \tag{7}$$

This equation shows that the marginal rate of substitution between consumption and labor supply is equal to the real wage rate in terms of the effective price of consumption, that is, one plus the nominal interest rate (i.e. the opportunity cost of holding an additional unit of money).

Following Benhabib and Farmer (1994), the production technology of an individual firm is specified as

$$y = k^a l^{1-a} \bar{k}^{\alpha - a} \bar{l}^{\beta + a - 1}, \;\; 0 < a < 1, \; 0 < a < \alpha, \; \beta > 1 - a,$$

where $\bar{k}$ and $\bar{l}$ represent external effects associated with capital and labor of the economy at large. Each firm takes those externalities as given. Thus profit maximization in the competitive markets equates the marginal products of private capital and labor to the real rent and the real wage, respectively:

$$r = a k^{a-1} l^{1-a} \bar{k}^{\alpha - a} \bar{l}^{\beta + a - 1},$$

$$w = (1-a) \, k^a l^{-a} \bar{k}^{\alpha - a} \bar{l}^{\beta + a - 1}.$$

Normalizing the number of firms to one, in a symmetric equilibrium it holds that $k = \bar{k}$ and $\bar{l} = l$. Accordingly, the social level of production and factor prices that involve external effects can be written as $y = k^{\alpha} l^{\beta}$, $r = a k^{\alpha - 1} l^{\beta}$ and $w = (1-a) \, k^{\alpha} l^{\beta - 1}$. In this paper we assume that $\alpha = 1$ so that endogenous growth can be sustained in the long-term equilibrium. Given this assumption, the aggregate production function is

$$y = k l^{\beta}, \tag{8}$$

and the rate of return and the real wage rate are respectively expressed by

$$r = a l^{\beta}, \tag{9}$$

$$w = (1-a) \, k l^{\beta - 1}. \tag{10}$$

We assume that the monetary authority controls the nominal interest rate according to the feedback rule under which the nominal interest rate is positively related to the current rate of inflation. This rule is formulated as follows:

$$i = \phi(\pi), \;\; \phi' > 0, \;\; \phi(0) \geq 0. \tag{11}$$

The condition $\phi(0) \geq 0$ means that the nominal interest rate is kept nonnegative in the absence of inflation. By use of (9) and (11), the Fisher condition, $i = r + \pi$, is given by

$$\phi(\pi) = a l^{\beta} + \pi.$$

As a result, the relation between the rate of inflation and the equilibrium level of employment is written as

$$\pi = \pi(l), \;\; \pi'(l) = a \beta l^{\beta - 1} / (\phi' - 1), \tag{12}$$

so that

$$\operatorname{sign} \pi'(l) = \operatorname{sign} \; (\phi' - 1).$$

If the monetary authority raises the nominal interest rate by more than (less than) 1% when the rate of inflation rises by 1%, the control rule is said to be active (passive).[3] Thus (12) means that if the interest-rate feedback rule is active ($\phi' > 1$), the equilibrium rate of inflation increases with the level of employment, while if the rule is passive ($\phi' < 1$), the rate of inflation decreases with l.

Finally, the market equilibrium condition for final goods is

$$y = c + \dot{k}. \tag{13}$$

For simplicity, we assume that capital never depreciates.

3 Equilibrium Dynamics and Indeterminacy

By use of (9), (10) and (12) , (7) is rewritten as

$$\frac{c}{k} = \frac{(1-a)\, l^{\beta-1-\gamma}}{1 + al^{\beta} + \pi\,(l)}. \tag{14}$$

Denoting $\Lambda\,(l) \equiv 1 + al^{\beta} + \pi\,(l)$, the above equation gives

$$\frac{\dot{c}}{c} - \frac{\dot{k}}{k} = \left[(\beta - 1 - \gamma) - \frac{\Lambda'\,(l)\, l}{\Lambda\,(l)}\right] \frac{\dot{l}}{l} \tag{15}$$

Notice that from (1), (3) and (4) we obtain $1/c = \lambda\,(1 + r + \pi) = \lambda\Lambda\,(l)$. As a result,

$$\frac{\dot{c}}{c} = -\frac{\dot{\lambda}}{\lambda} - \frac{\Lambda'\,(l)\, l}{\Lambda\,(l)} \frac{\dot{l}}{l} = al^{\beta} - \rho - \frac{\Lambda'\,(l)\, l}{\Lambda\,(l)} \frac{\dot{l}}{l} \tag{16}$$

On the other hand, (13) and (14) yield

$$\frac{\dot{k}}{k} = l^{\beta} - \frac{(1-a)\, l^{\beta-\gamma-1}}{\Lambda\,(l)}. \tag{17}$$

Substituting (16) and (17) into (15) and solving for $\dot{l}/l$, we obtain a complete dynamical system:

$$\dot{l} = \frac{l}{\gamma + 1 - \beta} \left[(1-a)\, l^{\beta} + \rho - \frac{(1-a)\, l^{\beta-\gamma-1}}{\Lambda\,(l)}\right]. \tag{18}$$

For analytical convenience, we specify (11) by the following linear feedback rule:

$$\phi\,(\pi) = \phi\pi + \varepsilon, \quad \phi > 0, \quad \varepsilon \geq 0. \tag{19}$$

[3] The 'active' and 'passive' monetary policies defined above are somewhat different from those in Leeper (1991) who explicitly considers the interaction between monetary and fiscal authorities. More precisely, the active policy defined here follows the Taylor principle, while the passive policy follows the anti-Taylor principle.

Then (11) becomes

$$\pi\left(l\right)=\frac{al^{\beta}-\varepsilon}{\phi-1}.$$

Let us denote $x\equiv l^{\beta}$. Then (18) can be rewritten as

$$\dot{x}=\frac{\beta x}{\left(\gamma+1-\beta\right)R\left(x\right)}\left[F\left(x\right)-G\left(x\right)\right], \tag{20}$$

where

$$\begin{aligned}
R\left(x\right) &\equiv 1+ax+\frac{ax-\varepsilon}{\phi-1}>0,\\
F\left(x\right) &\equiv \left[\left(1-a\right)x+\rho\right]\left[\frac{\phi ax}{\phi-1}+\frac{\phi-\varepsilon-1}{\phi-1}\right],\\
G\left(x\right) &\equiv \left(1-a\right)x^{\frac{\beta-\gamma-1}{\beta}}.
\end{aligned}$$

Note that since we are concerned with the case where the nominal interest rate has a nonnegative value, $R\left(x\right)$ has a positive sign. Obviously, the dynamic behavior of l exactly corresponds to the behavior of x.

Provided that $\gamma+1-\beta\neq0$, balanced-growth is attained when $G\left(x\right)=F\left(x\right).$ Figure 1 (a) and (b) depict the relationship between $F\left(x\right)$ and $G\left(x\right)$ functions under $\beta<\gamma+1$. As Figure 1 (a) shows, in this case if $\phi>1$, the steady-state level of x (so the steady state value of l) is uniquely given. In contrast, if $\phi<1$, then there may exist dual steady states: see Figure 1 (b). On the other hand, Figure 2 (a), (b) and (c) show the cases under $\beta>\gamma+1$. If $\phi>\varepsilon+1$, then $F\left(x\right)$ has a positive intercept on the vertical axis so that there may be two steady state solutions: see Figure 2 (a). In contrast, as Figure 2 (b) and (c) demonstrate, either if $1<\phi<1+\varepsilon$ or if $\phi<1$, the steady state is uniquely determined.

Since the initial value of $x\left(=l^{\beta}\right)$ is not predetermined, the balanced-growth equilibrium is locally indeterminate if and only if (20) is stable around the steady state. Therefore, considering that $R\left(x\right)>0$, if

$$\frac{F'\left(x\right)-G'\left(x\right)}{\gamma+1-\beta}<0 \tag{21}$$

holds at the steady state, indeterminacy emerges around the balanced-growth path. Otherwise, the economy always stays on the balanced growth path so that determinacy of equilibrium holds. Thus in the case of $\beta<\gamma+1$, indeterminacy emerges if $F'\left(x\right)<G'\left(x\right).$ Conversely, in the case of $\beta>\gamma+1$ indeterminacy holds if $F'\left(x\right)>G'\left(x\right)$ at the steady state.

First, assume that $\beta<\gamma+1$. Figure 1 (a) shows that $F'\left(x\right)>G'\left(x\right)$ is satisfied at the steady state. Therefore, the unique balanced-growth equilibrium under $\beta<\gamma+1$ is determinate. On the other hand, noting that the balanced-growth rate is given by $\dot{c}/c=ax-\rho$, Figure 1 (b) reveals that in the presence

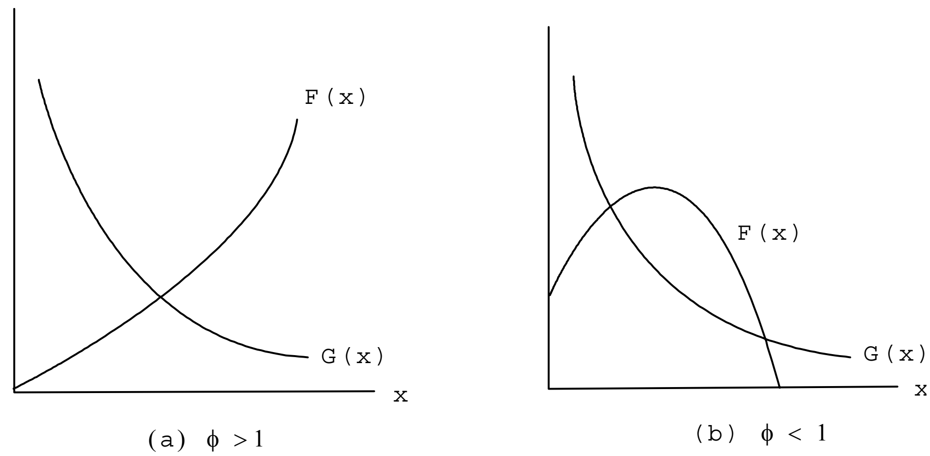

Figure 1: $\beta < \gamma + 1$

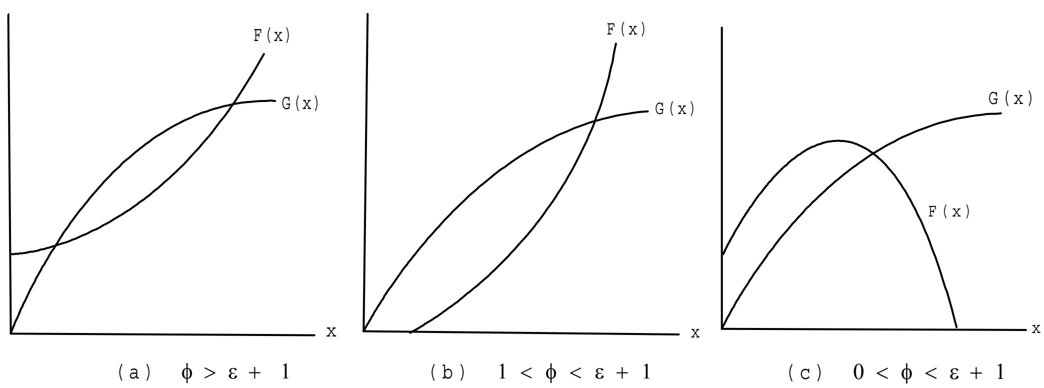

Figure 2: $\beta > \gamma + 1$

of dual balanced-growth paths, $F'(x) < G'(x)$ is satisfied at the high-growth steady state, while $F'(x) < G'(x)$ at the low-growth steady state. Hence, if there are dual balanced-growth equilibria under $\beta < \gamma + 1$, the low-growth steady state is determinate and the high-growth steady state is indeterminate. Next, consider the case of $\beta > \gamma + 1$. We see that in Figure 2 (a) $F'(x) < G'(x)$ at the low-growth steady state, while $F'(x) > G'(x)$ at the high- growth steady state. Again, the low-growth steady state is locally determinate, while the high-growth steady state is locally indeterminate. If $1 < \phi < \varepsilon + 1$, Figure 2 (b) shows that $F'(x) > G'(x)$ holds at the uniquely given steady state. Therefore, the balanced-growth path is indeterminate in this case. Finally, in view of Figure 2 (c), if $\phi < 1$, then $F'(x) < G'(x)$ and thus the unique balanced-growth path is determinate.

To sum up, the above argument, we have obtained the following propositions:

Proposition 1 *Suppose that the interest-rate rule is given by (19) and that* $\beta < \gamma + 1$. *Then if the interest-rate rule is active* $(i.e., \phi > 1)$ *there is a unique balanced-growth path that exhibits determinacy. If the interest-rate rule is passive* $(i.e., \phi < 1)$, *then there may exist two balance growth paths at most, one of which with a lower growth rate is locally determinate and the other with a higher growth rate is locally indeterminate.*

Proposition 2 *Suppose that the interest-rate feedback rule is given by (19) and that* $\beta > \gamma + 1$. *Then if the interest-rate rule is passive* $(i.e., \phi < 1)$ *there is a unique balance-growth path that exhibits determinacy. If the interest-rate rule is active enough to satisfy* $\phi > \varepsilon + 1$, *there are two balanced-growth path at most, one of which with a lower growth rate is determinate and the other with a higher growth rate is indeterminate. If the interest-rate rule is active but* $1 < \phi < \varepsilon + 1$, *then the balance growth path is unique and exhibits indeterminacy.*

In order to obtain intuitive implications of the above results, let us consider the case of $\beta < \gamma + 1$ as an example. Suppose that the economy initially stays on a balanced-growth path. That is, at the outset consumption, capital and real money balances grow at a common, constant rate and the employment level stays constant over time. Now assume that there is a sunspot-driven, anticipated increase in consumption demand and hence the consumption-capital ratio, c/k, is expected to rise. First, assume that $\phi > 1$. In view of (12) and (14), an increase in c/k is associated with a fall in l if $\beta < \gamma + 1$ and $\phi > 1$.[4] Since under the active control rule $(\phi > 1)$ the rate of inflation will decline as a result of the fall in l, the liquidity constraint on consumption becomes weaker. This accelerates consumption growth and then from (17) capital formation will be depressed. Consequently, c/k increases further, which means that the balanced-growth equilibrium exhibits instability. To avoid diverging behavior of the economy, the initial expected increase in consumption should not be self-fulfilled, so that the economy cannot diverge from the initial position. Thus the initial balanced-growth equilibrium satisfies determinacy. On the other hand, if $\phi < 1$, the initial decrease in l raises the equilibrium rate of inflation, which increases

[4] Notice that (14) is rewritten as

$$\frac{c}{k} l^{\gamma} = \frac{\beta (1-a)}{1+\phi(\pi)} l^{\beta-1}.$$

Given the nominal interest rate, $\phi(\pi)$, and the consumption-capital ratio, c/k, the left-hand side of the above equation represents the labor supply curve and the right-hand side expresses the labor demand curve. If $\beta < \gamma + 1$, the labor supply curve is steeper than the labor demand curve and thus an increase in c/k produces an upward shift of the labor supply curve. Hence, the equilibrium rate of employment, l, will decrease when there is a rise in the expected level of consumption.

the nominal interest rate. This effect raises the opportunity cost of consumption, so that consumption growth is not necessarily accelerated. Rather, it will be lowered if the rise in the nominal interest rate is large enough. Remember that there are two balanced-growth paths under $\beta < \gamma + 1$ and $\phi < 1$. The stable behavior holds in the high-growth steady state, while stability is not satisfied on the low-growth steady state. Therefore, local indeterminacy is established on the balanced-growth path with a higher growth rate.

4 Conclusion

We have shown that the dynamic property of the economy around the balanced-growth path is highly sensitive not only to the degree of external increasing returns but also to the specification of interest-rate feedback rule selected by the monetary authority. This conclusion is in contrast to the indeterminacy results obtained in the model under the constant money growth policy. In fact, Fukuda (1996) and Itaya and Mino (2003a) find that in the present model the determinacy/indeterminacy conditions are the same as those in the model without money, if the growth rate of nominal money supply is kept constant. Thus our finding reveals that monetary policy rule may have a profound impact on long-term performance of a monetary economy.

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