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Citation: Taylor, Robert, (2005) "On the limiting behaviour of augmented seasonal unit root tests." *Economics Bulletin,* Vol. 3, No. 3 pp. 1−10
Submitted: October 22, 2004. **Accepted:** January 17, 2005.
URL: http://www.economicsbulletin.com/2005/volume3/EB−04C20032A.pdf

On the Limiting Behaviour of Augmented Seasonal Unit Roots Tests

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October 2004

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In a recent paper, Taylor (2003) has shown that the seasonal unit root tests of Dickey *et al.* (1984) [DHF] have non-degenerate limiting distributions for series which admit unit roots at any of the zero or seasonal frequencies. In this note we go a stage further and show that the standard practice of augmenting the DHF regression with lagged dependent variables alters the limiting distributions of the DHF statistics in the above scenario. Associated Monte Carlo evidence shows that this may either increase or decrease (possibly even *below* the nominal level) the rejection frequencies of the tests, relative to the unaugmented case.

Keywords: seasonal unit root tests; lag augmentation.
JEL Code: C2.

1 Introduction

Taylor (2003) has shown that the Dickey *et al.* (1984) [DHF] seasonal unit root test statistics have non-degenerate limiting distributions when applied to data generated by processes with unit roots at either the zero or seasonal frequencies. This result confirms the validity of a conjecture made, on the basis of Monte Carlo simulation results, by Ghysels *et al.* (1994) [GLN] that, and in contrast to the seasonal unit root tests of Hylleberg *et al.* (1990) [HEGY], '... the DHF test may not separate unit roots at each frequency ... ' *op cit.* p.432. We provide a brief review of the DHF and HEGY tests and of the key results in Taylor (2003) in Section 2.

GLN also observed from their simulations that the power of the DHF tests against a pure random walk declines as one augments the test regression with lagged dependent variables. In Section 3 we demonstrate that, unlike the HEGY tests, in this scenario lag augmentation effects a shift in the large sample distributions of the DHF statistics.

This does not alter the key result of Taylor (2003) that the DHF statistics do not diverge, but associated Monte Carlo evidence provided demonstrates that in most cases the probability of rejecting the null decreases relative to the unaugmented case. Our numerical results also show that the effects of lag augmentation depend both on the periodicity of the data and on deterministic components. We find many cases where lag augmentation reduces the empirical rejection frequency *below* the nominal level, but we do find some cases where augmentation can increase the empirical rejection frequencies *above* those seen in the unaugmented case. Our main result is proved in an Appendix.

2 Seasonal Unit Root Tests

Consider the *seasonal* time series process, $\{x_t\}$, observed with periodicity S, $S > 1$,

$$\alpha(L)(x_t - \mu_t) = \epsilon_t, \;\; \epsilon_t \sim IID(0, \sigma^2), \; t = 1, \ldots, T, \tag{1}$$

where $\alpha(L) \equiv 1 - \sum_{j=1}^{S} \alpha_j L^j$, L the usual lag operator. The roots of $\alpha(z) = 0$ lie either outside or (distinctly) on the unit circle at some or all of the zero and seasonal spectral frequencies, $\exp(\pm i2\pi k/S)$, $k = 0, ..., [S/2]$, $[\cdot]$ denoting the integer part. We consider three cases, indexed by ξ, for the deterministic kernel μ_t: (i) $\xi = 0$, $\mu_t = 0$; (ii) $\xi = 1$, $\mu_t = \sum_{s=1}^{S} D_{s,t}\gamma_s$, and (iii) $\xi = 2$, $\mu_t = \sum_{s=1}^{S} D_{s,t}\gamma_s + \sum_{s=1}^{S} D_{s,t}\beta_s t$, where $D_{s,t} = 1$ if t lies in season s, $s = 1, \ldots, S$, zero otherwise.

DHF restrict (1) to case where $\alpha(z) \equiv (1 - \rho_S z^S)$, so one may re-write (1) as

$$\Delta_S x_t = (\rho_S - 1)x_{t-S} + \mu_t^* + \epsilon_t, \qquad t = 1, \ldots, T, \tag{2}$$

where $\mu_t^* \equiv \alpha(1)\mu_t$. In the context of (2), DHF consider testing the null hypothesis $H_0 : \rho_S = 1$ against the one-sided stationary alternative $H_1 : |\rho_S| < 1$, *via* the regression statistics from (2),

$$T(\hat{\rho}_S - 1) = T\frac{\sum_{t=1}^{T} x_{t-S}^{\xi}(x_t^{\xi} - x_{t-S}^{\xi})}{\sum_{t=1}^{T}(x_{t-S}^{\xi})^2} \tag{3}$$

$$t_{\rho_S=1} = \frac{\hat{\rho}_S - 1}{\sqrt{\hat{\sigma}^2 / \sum_{t=1}^{T}(x_{t-S}^{\xi})^2}}, \tag{4}$$

where $x_t^0 \equiv x_t$, and x_t^1 and x_t^2 are the OLS residuals from the regression of x_t on seasonal intercepts, and seasonal intercepts and seasonal trends, respectively, and $\hat{\sigma}^2 = T^{-1}\sum_{t=1}^{T}(x_t^{\xi} - \hat{\rho}_S x_{t-S}^{\xi})^2$. Under H_0, it is well-known that

$$T(\hat{\rho}_S - 1) \Rightarrow \frac{S\sum_{s=1}^{S}\int_0^1 W_s^{\xi}(r)dW_s^{\xi}(r)}{\sum_{s=1}^{S}\int_0^1 W_s^{\xi}(r)^2 dr} \equiv \eta_{\xi}(S) \tag{5}$$

$$t_{\rho_S=1} \Rightarrow \frac{\sum_{s=1}^{S}\int_0^1 W_s^{\xi}(r)dW_s^{\xi}(r)}{\left(\sum_{s=1}^{S}\int_0^1 W_s^{\xi}(r)^2 dr\right)^{1/2}} \equiv \tau_{\xi}(S), \tag{6}$$

where $\Rightarrow$ denotes weak convergence, and the $W_s^\xi(r)$, $s = 1, ..., S$, are S independent standard, de-meaned, and de-meaned and de-trended Brownian motion processes for $\xi = 0$, $\xi = 1$ and $\xi = 2$ respectively, while, under H_1, $t_{\rho_S=1}$ and $T(\hat{\rho}_S - 1)$ both diverge to minus infinity.

An alternative procedure, suggested by Smith and Taylor (1999) and HEGY, is based on the *unrestricted* linear reparameterisation of (1)

$$\Delta_S x_t = \mu_t^* + \sum_{j=0}^{S-1} \pi_j x_{j,t-1} + \epsilon_t, \; t = 1, \ldots, T \tag{7}$$

omitting $\pi_{S/2} x_{S/2,t-1}$ if S is odd, and where, corresponding to $\omega_k = 2\pi k/S$, $k = 0, ..., [S/2]$, $x_{0,t} \equiv \sum_{j=0}^{S-1} x_{t-j}$, $x_{S/2,t} \equiv \sum_{j=0}^{S-1} \cos[(j+1)\pi] x_{t-j}$, $x_{k,t} \equiv \sum_{j=0}^{S-1} \cos[(j+1)\omega_k] x_{t-j}$ and $x_{S-k,t} \equiv -\sum_{j=0}^{S-1} \sin[(j+1)\omega_k] x_{t-j}$, $k = 1, ..., S^*$, where $S^* = (S/2) - 1$ (if S is even) and $[S/2]$ (if S is odd). The null hypothesis H_0 implies $\pi_j = 0$, $j = 0, \ldots, S-1$, in (7). Consequently, the regression F-test, $F_{0\ldots[S/2]}$, for the exclusion of the regressors $\{x_{j,t-1}\}_{j=0}^{S-1}$ from (7) provides a test of H_0. However, unlike the DHF statistics the alternative hypothesis, is of stationarity at *at least one* (i.e., not necessarily all) of the zero and seasonal frequencies, the parameters $\{\pi_j\}_{j=0}^{S-1}$ determining which frequencies admit unit root behaviour. Consequently, one may test for unit roots at the zero and seasonal frequencies using t- and F-statistics from (7); see Smith and Taylor (1999) and HEGY for details. Representations for the limiting null distributions of these statistics are provided in, *inter alia*, Smith and Taylor (1999).

Taylor (2003) considers the case where $\{x_t\}$ follows a simple random walk

$$x_t = x_{t-1} + \mu_t^* + \epsilon_t, \quad \epsilon_t \sim IID(0, \sigma^2), \;\; t = 1, \ldots, T, \tag{8}$$

so that (2) is misspecified, and demonstrates that in this case

$$T(\hat{\rho}_S - 1) \;\; \Rightarrow \;\; \frac{S \int_0^1 W^\xi(r) dW^\xi(r)}{\int_0^1 W^\xi(r)^2 dr} \equiv \eta_\xi^*(S) \tag{9}$$

$$t_{\rho_S=1} \;\; \Rightarrow \;\; \frac{\sqrt{S} \int_0^1 W^\xi(r) dW^\xi(r)}{\left(\int_0^1 W^\xi(r)^2 dr\right)^{1/2}} \equiv \tau_\xi^*(S), \tag{10}$$

where $W^\xi(r)$ is a standard, de-meaned, and de-meaned and de-trended Brownian motion for $\xi = 0$, $\xi = 1$ and $\xi = 2$, respectively. A consequence of this is that neither $T(\hat{\rho}_S - 1)$ nor $t_{\rho_S=1}$ will diverge to minus infinity. In contrast, the HEGY statistic $F_{0\ldots[S/2]}$ from (7) diverges to plus infinity. Indeed, only the t-statistic for $\pi_0 = 0$ from (7), (which tests solely for a unit root at the zero frequency) will not diverge in this case, since $\pi_0 = 0$.

3 Including Lagged Dependent Variables

Focusing on quarterly data ($S = 4$), GLN also investigated, for $\xi = 1$, the effects of augmenting (2) with the lagged dependent variables $\{\Delta_S x_{t-j}\}_{j=1}^p$. Their simulation results, computed for $p = 4$ and $p = 8$, suggest that the empirical rejection frequencies of both of the DHF tests against a random walk process decline, relative to the case of $p = 0$, as p is increased. GLN also observe a decline in finite sample power when the quarterly HEGY regression (7) is similarly augmented. However, we now show that in the case of the DHF statistics, this is not a purely finite sample effect, and that the inclusion of the lagged dependent variables acts to alter the limiting distributions of the DHF statistics when the data is generated as a random walk. In contrast, the HEGY statistics are asymptotically unaffected by such augmentation; cf. Burridge and Taylor (2001). For reasons of exposition we shall focus on the case of $p = 1$, which is predictive for the key result that the limiting distributions of the DHF statistics differ for $p > 0$ *vis-à-vis* $p = 0$. We will use Monte Carlo experimentation to illustrate not only the finite sample effects of $p = 1$, but also larger values of p. Moreover, although we focus on the case where $\{x_t\}$ is generated as a random walk with IID innovations, the same qualitative conclusions are drawn if any of the zero and seasonal frequency unit roots $z = \exp\{i2\pi k/S\}$, $k = 0, \ldots, S-1$ solve the characteristic equation $\alpha(z) = 0$ and under weaker conditions on the innovations.

We therefore consider the behaviour of the OLS estimators and regression t-statistics from the first-order augmented regression equation

$$\Delta_S x_t = (\rho_S - 1)x_{t-S} + \mu_t^* + \phi\Delta_S x_{t-1} + \epsilon_t, \tag{11}$$

when the process $\{x_t\}$ is generated according to (8).

Theorem 1 *If the process $\{x_t\}$ is generated according to (8), then the DHF statistics $t_{\rho_S=1}$ and $T(\hat{\rho}_S - 1)$, together with the OLS estimator $\hat{\phi}$, obtained from OLS estimation of (11) have the following large sample properties*

$$T(\hat{\rho}_S - 1) \Rightarrow \frac{\left(\int_0^1 W^{\xi}(r)dW^{\xi}(r)\right) + (1-S)}{\int_0^1 W^{\xi}(r)^2 dr} \equiv \eta_{\xi}^{**}(S) \tag{12}$$

$$t_{\rho_S=1} \Rightarrow \frac{\sqrt{\frac{S}{2S-1}}\left(\int_0^1 W^{\xi}(r)dW^{\xi}(r)\right) + (1-S)}{\left(\int_0^1 W^{\xi}(r)^2 dr\right)^{1/2}} \equiv \tau_{\xi}^{**}(S) \tag{13}$$

$$\hat{\phi} \to^p \frac{S-1}{S} \tag{14}$$

$$T^{1/2}\hat{\phi} \to +\infty. \tag{15}$$

Remark 1: Theorem 1 demonstrates that the presence of $\Delta_S x_{t-1}$ in (11) alters the limiting distributions of the $t_{\rho_S=1}$ and $T(\hat{\rho}_S - 1)$ statistics from the form given in (10) and (9) respectively, appropriate for $p = 0$. This contrasts sharply with the HEGY tests of Section 2 whose limiting distributions are unaffected by the addition of lagged dependent variables to (7); cf. Burridge and Taylor (2001). To understand why, consider first the (scaled) second moment matrix $\mathbf{R} \equiv (\mathbf{D}_T(\mathbf{X}'\mathbf{X})\mathbf{D}_T)$. It is clear from results (16), (18) and (17) of Lemma 1 of the Appendix that $\mathbf{R}$ is asymptotically (block) diagonal between the lagged regressor, x_{t-S}, and the lagged dependent variable, $\Delta_S x_{t-1}$, of (11). For the limiting distributions of the $t_{\rho S=1}$ and $T(\hat{\rho}_S - 1)$ statistics to be unaffected by the inclusion of the lagged dependent regressor in (11), we also require that $\mathbf{D}_T\mathbf{X}'\mathbf{y}$ is $O_p(1)$; as it is for the corresponding vector from the augmented HEGY regression. It is clear from (17) and (19) that while the first element of this vector is $O_p(1)$, the second is $O_p(T^{1/2})$. Consequently, $\mathbf{D}_T\mathbf{X}'\mathbf{y}$ is $O_p(T^{1/2})$, not $O_p(1)$.

Remark 2: A consequence of Remark 1 is that the limiting distribution of the $T(\hat{\rho}_S - 1)$ and the right member of (9) are related *via*

$$T(\hat{\rho}_S - 1) \Rightarrow \frac{1}{S}\eta_\xi^*(S) + \frac{1-S}{\int_0^1 W^\xi(r)^2 dr}.$$

Although the scaling of $\eta_\xi^*(S)$ by S^{-1} will clearly cause tail shrinkage, the additional term $(1-S)(\int_0^1 W^\xi(r)^2 dr)^{-1}$ takes negative values with probability one which will off-set this effect, hence the relative position of the lower tails of the distributions of $\eta_\xi^*(S)$ and $\eta_\xi^{**}(S)$ is not immediately obvious and will clearly vary across both S and ξ. Similar comments also apply to the $t_{\rho S=1}$ statistic, since $\tau_\xi^{**}(S) \equiv (2S-1)^{-1/2}\tau_\xi^*(S) + (1-S)(\int_0^1 W^\xi(r)^2 dr)^{-1/2}$.

Remark 3: Theorem 1 also shows that the unscaled OLS coefficient estimator on the lagged dependent variable converges in probability to $S^{-1}(S-1)$, while $T^{1/2}\hat{\phi}$ is $O_p(T^{1/2})$. The latter contrasts with the case where a lagged dependent variable is included in (7), where standard *root-T* asymptotic normality applies. Moreover, the associated t-statistic for testing $\phi = 0$ against $\phi \neq 0$ diverges to $+\infty$, at rate $T^{1/2}$. This follows immediately from (14) and results (16) and (18) from the Appendix.

Table 1 reports the Monte Carlo simulated rejection frequencies of the DHF $t_{\rho S=1}$ and $T(\hat{\rho}_S - 1)$ and HEGY $F_{0\ldots[S/2]}$ tests against the random walk DGP, where (2) and (7) are augmented with the lagged dependent variables $\{\Delta_S x_{t-j}\}_{j=1}^p$. We report results for the cases of $p=1$, $p=S$ and $p=2S$; other values of p were also considered but qualitatively added little to what is reported, while results for $p=0$ are reported in Table 3.1 of Taylor (2003). In all cases, the tests were run using the .05 level critical values obtained by Monte Carlo simulation from the seasonal random walk $\Delta_S x_t = \epsilon_t \sim IN(0,1)$, $x_0 = \cdots = x_{1-S} = 0$, for the given values of T, S and ξ. All simulations were programmed using the RNDN random number generator of Gauss 3.2 on a Pentium II micro-computer over 100,000 replications.

It is clear from Table 1 that the empirical rejection frequencies of the $t_{\rho S=1}$ and $T(\hat{\rho}_S - 1)$ tests, and to a lesser extent the HEGY $F_{0\ldots[S/2]}$ test, depend on S, the periodicity of the data, and on ξ and p. In the case of $F_{0\ldots[S/2]}$ such dependence vanishes rapidly with T, reflecting the consistency of the test. Contrastingly, and as predicted by Theorem 1, the rejection frequencies for the two DHF tests seem to depend little on T. For all values of S and ξ considered, the rejection frequencies of the $t_{\rho S=1}$ and $T(\hat{\rho}_S - 1)$ tests decrease as p is increased. Generally speaking, the effects of p on the rejection frequencies of the DHF tests increase with ξ, this effect also appears to interact positively with S, being most pronounced for $S = 12$. For $\xi > 0$, the rejection frequencies of the DHF tests lie below the nominal level for all S whenever $p = 2S$. For $S = 4$ and $S = 12$ this is also true for $p = S$, while for $S = 12$ it is also true for $p = 1$.

Table 1: Empirical Rejection Frequencies (Nominal 0.05 level) of the $T(\hat{\rho}_S - 1)$, $t_{\rho_S=1}$ and $F_{0\ldots[S/2]}$ Seasonal Unit Root Tests
DGP: $(1 - L)x_t = \epsilon_t \sim IN(0, 1)$, $t = 1, \ldots, T$
Test Regressions (2) and (7) augmented by $\{\Delta_S x_{t-j}\}_{j=1}^{p}$.

S	T/S	p	$\xi = 0$			$\xi = 1$			$\xi = 2$		
			$T(\hat{\rho}_S - 1)$	$t_{\rho_S=1}$	$F_{0\ldots[S/2]}$	$T(\hat{\rho}_S - 1)$	$t_{\rho_S=1}$	$F_{0\ldots[S/2]}$	$T(\hat{\rho}_S - 1)$	$t_{\rho_S=1}$	$F_{0\ldots[S/2]}$
2	25	1	.178	.256	1.000	.136	.185	.980	.128	.225	.908
		S	.087	.091	.982	.062	.050	.760	.052	.039	.524
		$2S$	.079	.081	.739	.049	.040	.349	.050	.028	.198
	50	1	.176	.254	1.000	.141	.199	1.000	.135	.255	1.000
		S	.092	.093	1.000	.058	.051	1.000	.044	.036	.992
		$2S$	.072	.076	.999	.044	.036	.926	.031	.020	.773
	100	1	.175	.257	1.000	.142	.202	1.000	.141	.260	1.000
		S	.085	.091	1.000	.050	.047	1.000	.035	.031	1.000
		$2S$	.074	.078	1.000	.040	.032	1.000	.024	.019	1.000
4	25	1	.215	.355	1.000	.100	.160	1.000	.065	.154	1.000
		S	.110	.134	1.000	.046	.032	1.000	.027	.014	.992
		$2S$	.085	.104	1.000	.035	.016	.933	.021	.006	.683
	50	1	.229	.364	1.000	.098	.167	1.000	.061	.161	1.000
		S	.105	.133	1.000	.032	.023	1.000	.014	.009	1.000
		$2S$	.080	.100	1.000	.018	.010	1.000	.007	.003	1.000
	100	1	.218	.359	1.000	.098	.162	1.000	.055	.163	1.000
		S	.103	.125	1.000	.028	.023	1.000	.009	.006	1.000
		$2S$	.079	.105	1.000	.013	.010	1.000	.004	.002	1.000
12	25	1	.193	.429	1.000	.015	.032	1.000	.003	.009	1.000
		S	.095	.187	1.000	.007	.002	1.000	.002	.000	1.000
		$2S$	.068	.145	1.000	.003	.001	1.000	.001	.000	.999
	50	1	.190	.430	1.000	.012	.032	1.000	.001	.007	1.000
		S	.089	.175	1.000	.003	.001	1.000	.000	.000	1.000
		$2S$	.059	.134	1.000	.001	.000	1.000	.000	.000	1.000
	100	1	.191	.426	1.000	.010	.031	1.000	.001	.008	1.000
		S	.088	.176	1.000	.002	.001	1.000	.000	.000	1.000
		$2S$	.057	.135	1.000	.001	.001	1.000	.000	.000	1.000

Our results for $S = 4$, $\xi = 1$ and $p \in \{S, 2S\}$ are similar to those reported in GLN. In contrast, for $p = 1$, in both the quarterly ($S = 4$) and biannual ($S = 2$), but not the monthly ($S = 12$), cases there are many examples where the DHF tests, $t_{\rho_S=1}$ and $T(\hat{\rho}_S - 1)$, display *larger* rejection frequencies than the corresponding outcomes for $p = 0$ reported in Table 3.1 of Taylor (2003). For example, for $S = 2$ and $\xi = 2$ we see from Table 1 that the rejection frequency of the $t_{\rho_S=1}$ test for $T = 50$ is 0.225 for $p = 1$; this is almost double the corresponding rejection frequency of 0.131 reported for $p = 0$ in Table 3.1 of Taylor (2003). The case of $p = 1$ is therefore very interesting: here one may observe either an increase *or* a decrease in the rejection frequencies of the DHF tests, relative to the unaugmented case.

Appendix: Proof of Theorem 1

We prove results for $\xi = 0$ throughout. The results for $\xi = 1$ and $\xi = 2$ follow in the same way, replacing the standard Brownian motion $W^0(r)$ by its de-meaned or de-meaned and de-trended counterparts, $W^1(r)$ and $W^2(r)$, respectively. We assume that $T^{-1/2}x_{-j} \to^p 0$, $j = 0, ..., S-1$, but this may be dropped for both $\xi = 1$ and $\xi = 2$. In order to prove the results stated in Theorem 1 we first present some key results in a preparatory Lemma.

Lemma 1 *Under the conditions of Theorem 1*

$$T^{-2}\sum x_{t-S}^2 \Rightarrow \sigma^2 \int_0^1 W^0(r)^2 dr \tag{16}$$

$$T^{-1}\sum x_{t-S}\Delta_S x_{t-j} \Rightarrow \sigma^2\left(S\int_0^1 W^0(r)dW^0(r) + j\right),\ j = 0, 1, \tag{17}$$

$$T^{-1}\sum(\Delta_S x_{t-j})^2 \to^p S\sigma^2,\ j = 0, 1 \tag{18}$$

$$T^{-1}\sum \Delta_S x_t \Delta_S x_{t-1} \to^p (S-1)\sigma^2, \tag{19}$$

where $W^0(r)$ is a standard Brownian motion process, $r \in [0, 1]$.

Proof of Lemma 1
Proof of (16): See Proposition 17.1, part (e), of Hamilton (1994,p.486).
Proof of (17): Notice first that $T^{-1}\sum x_{t-S}\Delta_S x_{t-j} = T^{-1}\sum x_{t-S}(\epsilon_{t-j} + \cdots + \epsilon_{t-S+1-j})$ $= T^{-1}[\sum x_{t-S}\epsilon_{t-j} + \cdots + \sum x_{t-S}\epsilon_{t-S+1-j}]$, $j = 0, 1$. From Lemma 1 of Hall (1989,p.52), we have that

$$T^{-1}\sum x_{t-S}\epsilon_{t-k} \Rightarrow \sigma^2\int_0^1 W^0(r)dW^0(r),\ k = 0, \ldots S-1, \tag{20}$$

and the result for $j = 0$ therefore follows directly using applications of the Continuous Mapping Theorem (CMT). For $j = 1$,

$$\begin{aligned} T^{-1}\sum x_{t-S}\epsilon_{t-S} \equiv T^{-1}\sum(x_{t-S-1} + \epsilon_{t-S})\epsilon_{t-S} &= T^{-1}\sum x_{t-S-1}\epsilon_{t-S} + T^{-1}\sum \epsilon_{t-S}^2 \\ &\Rightarrow \sigma^2\left(\int_0^1 W^0(r)dW^0(r) + 1\right) \end{aligned} \tag{21}$$

and so the result for $j = 1$ follows from (20), (21) and applications of the CMT.
Proof of (18): Noting that $T^{-1}\sum(\Delta_S x_{t-j})^2 \equiv T^{-1}\sum(\epsilon_{t-j} + \cdots + \epsilon_{t-S+1-j})^2$, $j = 0, 1$, (18) follows from the assumption that $\epsilon_t \sim IID(0, \sigma^2)$.

Proof of (19): On noting that $T^{-1}\sum \Delta_S x_t \Delta_S x_{t-1} \equiv T^{-1}\sum (u_t + \cdots + u_{t-S+1})(u_{t-1} + \cdots + u_{t-S})$, (19) follows from the assumption that $\epsilon_t \sim IID(0, \sigma^2)$. $\square$

Proof of (12): Defining the scaling matrix $\mathbf{D}_T = \text{diag}\{T^{-1}, T^{-1/2}\}$, the scaled OLS estimator from (11) may be written as $\mathbf{D}_T^{-1}\hat{\beta} = (\mathbf{D}_T(\mathbf{X}'\mathbf{X})\mathbf{D}_T)^{-1}\mathbf{D}_T\mathbf{X}'\mathbf{y}$ where the $T \times 2$ regressor matrix $\mathbf{X}$ collects together the sample observations of x_{t-S} and $\Delta_S x_{t-1}$, in the standard way, and $\mathbf{y}$ collects together the T observations on $\Delta_4 x_t$. The scaled OLS estimator may be written as

$$\mathbf{D}_T^{-1}\hat{\beta} = \delta^{-1}\begin{bmatrix} T^{-1}\sum(\Delta_S x_{t-1})^2 & -T^{-3/2}\sum x_{t-S}\Delta_S x_{t-1} \\ -T^{-3/2}\sum x_{t-S}\Delta_S x_{t-1} & T^{-2}\sum x_{t-S}^2 \end{bmatrix}\begin{bmatrix} T^{-1}\sum \Delta_S x_t x_{t-S} \\ T^{-1/2}\sum \Delta_S x_t \Delta_S x_{t-1} \end{bmatrix} \quad (22)$$

where $\delta = \left(T^{-2}\sum x_{t-S}^2\right)\left(T^{-1}\sum(\Delta_S x_{t-1})^2\right) - T^{-3}\left(\sum x_{t-S}\Delta_S x_{t-1}\right)^2$, with first element:

$$\begin{aligned} T(\hat{\rho}_S - 1) &= \delta^{-1}\Big[\Big(T^{-1}\sum(\Delta_S x_{t-1})^2\Big)\Big(T^{-1}\sum \Delta_S x_t x_{t-S}\Big) \\ &- \Big(T^{-1/2}\sum \Delta_S x_t \Delta_S x_{t-1}\Big)\Big(T^{-3/2}\sum x_{t-S}\Delta_S x_{t-1}\Big)\Big]. \end{aligned} \quad (23)$$

Now using the identity, $(T^{-1/2}\sum \Delta_S x_t \Delta_S x_{t-1})(T^{-3/2}\sum x_{t-S}\Delta_S x_{t-1}) \equiv (T^{-1}\sum \Delta_S x_t \Delta_S x_{t-1})$ $(T^{-1}\sum x_{t-S}\Delta_S x_{t-1})$, (12) follows from Lemma 1 and applications of the CMT.

Proof of (13): First observe from the results in (12) and (14) that $\hat{\sigma}^2$ is asymptotically equal to $T^{-1}\sum\left(\Delta_S x_t - \frac{S-1}{S}\Delta_S x_{t-1}\right)^2$. Noting that $T^{-1}\sum(\Delta_S x_t - \frac{S-1}{S}\Delta_S x_{t-1})^2 \equiv T^{-1}\sum(\epsilon_t + \frac{1}{S}\epsilon_{t-1} + \ldots + \frac{1}{S}\epsilon_{t-S+1} - \frac{S-1}{S}\epsilon_{t-S})^2$, we see immediately that $\hat{\sigma}^2 \to^p \sigma^2\left[1 + \frac{S-1}{S^2} + \frac{(S-1)^2}{S^2}\right] = \sigma^2\left[\frac{2S-1}{S}\right]$, from which, using (12), Lemma 1 and applications of the CMT, (13) follows.

Proof of (14): Multiplying the second element of (22) through by $T^{-1/2}$, we have that $\hat{\phi} = T^{-1/2}\delta^{-1}[(-T^{-3/2}\sum x_{t-S}\Delta_S x_{t-1})(T^{-1}\sum \Delta_S x_t x_{t-S}) + (T^{-2}\sum x_{t-S}^2)(T^{-1/2}\sum \Delta_S x_t \Delta_S x_{t-1})]$. Using Lemma 1 and applications of the CMT the stated result follows.

Proof of (15): From (22), the numerator of $T^{1/2}\hat{\phi}$ is $(-T^{-3/2}\sum x_{t-S}\Delta_S x_{t-1})(T^{-1}\sum \Delta_S x_t x_{t-S})$ $+(T^{-2}\sum x_{t-S}^2)(T^{-1/2}\sum \Delta_S x_t \Delta_S x_{t-1})$. From foregoing results it is seen that the first of these terms is $O_p(1)$. However, $T^{-1/2}\sum \Delta_S x_t \Delta_S x_{t-1}$, and hence the second term, is $O_p(T^{1/2})$ and positive. Consequently, $T^{1/2}\hat{\phi}$ is also $O_p(T^{1/2})$ and positive, as stated.

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