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ON WESNER'S METHOD OF SEARCHING FOR CHAOS ON LOW FREQUENCY

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Abstract. An alternative to Wesner's method of detecting deterministic behavior and chaos in small sample sets is presented. This new method is applied to analyze the dynamics of several stock prices.

1. Introduction

It is widely believed that the laws of the efficient market hypothesis (partly developed by E. Fama [4] in 1965) govern the financial markets. One of the assertions of this theory is that the dynamics of the commodities and stock prices are of random nature (this goes back to Bachelier [2]). There is no claim that the market would be 100% efficient all the time. It is possible that some price can follow a deterministic process over a short period of time before it returns to a stochastic behavior. This motivated numerous strategies to search for deterministic chaos in economics and financial time series.

A chaotic system is a deterministic system that exhibits sensitive dependence on initial conditions. A new empirical method of searching for chaos in small data sets was developed by N. Wesner in a series of recent papers [11, 12].

In this note, we will comment on the mathematical background of Wesner's method and present an alternative strategy for searching for determinism in low frequency financial time series. We will also present the results of numerical experiments on chaotic and stochastic systems illustrating the applicability of the method.

2. Mathematical background

In this section we briefly review some notions on dynamical systems. For details, see [1, 6] and the references listed therein. A dynamical system consists of a set of possible states, modelled as a subset X of a Euclidean space (called the phase space), and a continuous map $F : X \to X$. The orbit of a point $x \in X$ is the set $\{x, F(x), F^2(x), \ldots, F^t(x), \ldots\}$, where F^t ($t \in \mathbb{N}$) denotes the t-th iterate of F. An orbit is called periodic provided $F^t(x) = x$ for some $t \geq 1$. For almost every orbit, one can compute the Lyapunov exponents, measuring the exponential convergence/divergence rates of orbits that are small perturbations of the initial orbit. The convergence/divergence rate depends on the direction of the perturbation. The number of Lyapunov exponents equals the dimension of the phase space. Negative Lyapunov exponents account for convergent nearby orbits, while positive Lyapunov exponents account for divergent nearby orbits.

An orbit is called chaotic if it is neither periodic nor asymptotic to a periodic orbit, and has a positive Lyapunov exponent. A set $S \subseteq X$ is said to be invariant provided $F(S) \subseteq S$. The restriction $F : S \to S$ induces a dynamical system on S.

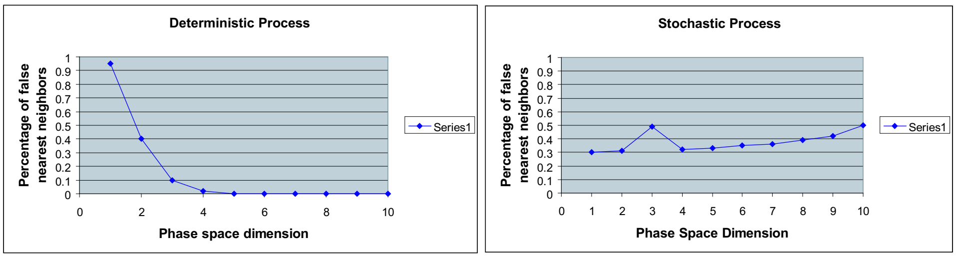

FIGURE 1. Dependence of proportion of false nearest neighbors on the reconstructed phase space dimension in the cases of deterministic and stochastic processes.

The invariant set S is a chaotic set provided that it is the forward limit set of a chaotic orbit which itself is contained in its forward limit set. In the sequel we will assume that S is a chaotic set for F.

In practice, one does not have prior knowledge of a process $F : S \to S$, and often can only measure an observable quantity $h(F(x)) \in \mathbb{R}$ associated to it. The successive measurements $s(t) = h(F^t(x))$ of the observable, starting with an initial time t_0 and at time intervals of τ from one another, produce a time series

$$(s(t_0), s(t_0 + \tau), s(t_0 + 2\tau), \ldots, s(t_0 + (i-1)\tau), \ldots).$$

In the sequel, we will chose units of time such that $\tau = 1$.

It is possible to reconstruct the qualitative behavior of the original process F only from its time series. In this order, we consider a reconstructed phase space consisting of delay coordinate vectors of the type

$$y(t) = [s(t), s(t-1), s(t-2), \ldots, s(t-(m-1))],$$

where m, the dimension of reconstructed phase space, is chosen conveniently large. We consider a mapping G on the reconstructed phase space defined by

$$G(y(t)) = y(t+1).$$

A theorem due to Takens and extended by Sauer, Yorke, and Casdagli states that for generic h and τ, the reconstructed phase space constitutes an embedded copy (with no self-crossings) of S in $\mathbb{R}^m$, provided m is chosen sufficiently large. Moreover, the action of G on the reconstructed phase space reproduces the action of F on S.

An important aspect of this construction is how big the dimension m has to be chosen. One way to estimate m is through the proportion of false nearest neighbors (see [8]). For each point $y(t)$ in the reconstructed phase space, one searches for its nearest neighbor $y(t')$ in $\mathbb{R}^m$. The nearest neighbor of $y(t)$ is a point $y(t') \neq y(t)$ such that

$$d(y(t), y(t')) \leq d(y(t), y(t'')) \quad \text{for all } t'' \neq t.$$

Typically, only one such $y(t')$ exists. The idea of the false nearest neighbors method is that, if m is less than the embedding dimension, $y(t')$ may appear to be the nearest neighbor of $y(t)$ in $\mathbb{R}^m$ only due to the self-crossings still present in the reconstructed phase space, and the pair $(y(t), y(t'))$ may not correspond to a pair of nearest neighbors in S. As m reaches the minimal embedding dimension and the self-crossing in the reconstructed phase space are eliminated, the proportion of false nearest neighbors approaches zero. See Figure 1.

3. Wesner's method

In [11, 12], Wesner introduces a new method to detect deterministic chaos in small data sets, based on the method of nearest neighbors (see [5, 7]). Nearest neighbors may not remain nearest neighbors under iteration, as their mutual distances change according to the Lyapunov exponents. Typically, if $y(t')$ is the nearest neighbor of $y(t)$, then $G(y(t')) = y(t'+1)$ remains the nearest neighbor of $G(y(t)) = y(t+1)$ provided that the Lyapunov exponent at $y(t)$ in the direction of $y(t') - y(t)$ is non-positive. In this case, $y(t')$ is a persistent nearest neighbor of $y(t)$. The proportion of persistent nearest neighbors (PPNN) is given by

$$PPNN = \frac{\text{number of persistent nearest neighbors}}{N - m + 1},$$

where N is the number of observations in the time series and $N - m + 1$ is the number of m-delay coordinate vectors that can be formed from the time series. The proportion of persistent nearest neighbors typically grows with the embedding dimension. The number

$$D = \frac{PPNN}{m}$$

is introduced in [11, 12] as a measure of determinism. Based on numerical experiments, Wesner proposes a value of $D \geq 0.1$ as characteristic for deterministic processes in contrast to a value of $D < 0.1$ as characteristic for stochastic processes.

The choice of the benchmark value of 0.1 in Wesner's method seems arbitrary. There are indeed examples of low dimensional deterministic systems for which Wesner's method does not detect chaos. One such example will be discussed in the next section. Also, it is not clear why Wesner's formula requires a division by the phase space dimension m (other than keeping the numbers bounded). We would also like to remark that even when determinism is found through this procedure, it is no guarantee that there is chaos. For example, an irrational rotation of the circle, which is usually regarded as non-chaotic, leads to $D = 1 > 0.1$.

4. An alternative to Wesner's method

We propose an alternative method for detecting determinism and chaos based on the proportion of persistent nearest neighbors (PPNN). This proportion accounts for the non-divergent nearby trajectories in the system, and reflects the existence of non-positive Lyapunov exponents. Numerical experiments for deterministic systems show that PPNN typically grow for a while with the dimension m of the reconstructed phase space, until they enter a plateau regime that spreads out over some range of dimensions. After they leave the plateau regime, the PPNN either continue to grow or start decreasing at some point. As a first example we compute PPNN for an x-coordinate time series of the Henon Map:

Dimension	PPNN	Dimension	PPNN	Dimension	PPNN
2	0.4914	9	0.4689	16	0.5504
3	0.4854	10	0.4910	17	0.5661
4	0.4888	11	0.4972	18	0.5904
5	0.4810	12	0.5172	19	0.5947
6	0.4860	13	0.5382	20	0.6160
7	0.4795	14	0.5502		
8	0.4832	15	0.5415		

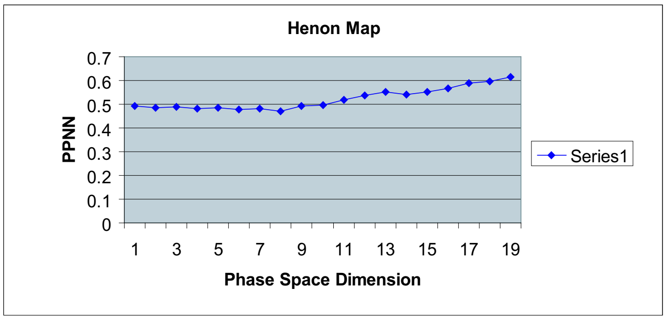

FIGURE 2. Dependence of PPNN on the phase space dimension in the case of the Henon Map.

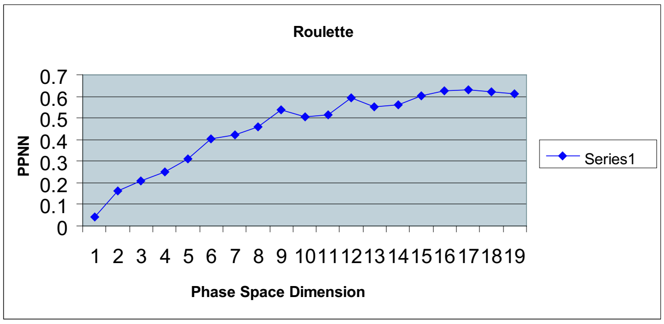

FIGURE 3. Dependence of PPNN on the phase space dimension in the case of a roulette.

We note that for $4 \leq m \leq 10$ the corresponding PPNN values stay approximately constant at 0.4826. See Figure 2. We applied linear regression to the PPNN values and found the slope -0.0007 of the regression line, which is away from 0 by less than the standard error of the estimate 0.0014. The fact that $PPNN$ stabilizes (temporarily) at a level 0.4826 less than 1 show that, besides non-positive Lyapunov exponents, there must exist positive Lyapunov exponents, and so chaotic dynamics.

In contrast, for stochastic processes, the PPNN typically keep growing with respect to the phase space dimension. When the dimensions m become very large, the PPNN start to decrease. This behavior is illustrated in Figure 3 for the roulette wheel of the online Casino Tropez.

We verified these types of behavior for other classes of systems. We found that the dependence of PPNN on the dimension of the reconstructed phase space is almost complementary to the dependence of the proportion of false nearest neighbors on the dimension of the reconstructed phase space. Based on numerical experiments, we propose the following criterion for determinism and chaos:

Criterion: A short time series ($N \leq 1000$) is declared deterministic if there exists a range of dimensions $m_1 \leq m \leq m_2$ with $m_2 - m_1 + 1 \geq 5$ such that, within this range, the values of PPNN stay at an approximately constant level. If this level is lower than 1, the time series is declared chaotic.

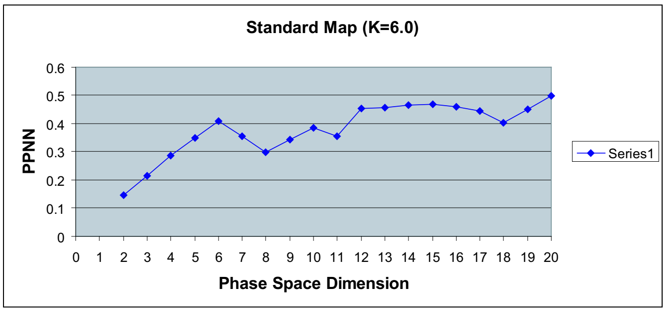

FIGURE 4. Dependence of PPNN on the phase space dimension in the case of the Standard Map (K=6.0).

In order to check that the values of PPNN stay at approximately constant level, we apply linear regression. We want the slope of the regression line to be very close to 0. In our experiments, the value of the slope was of the order of 10^{-3} and was smaller than the standard error of the estimate. The size of 5 for the range of dimensions in the above was devised experimentally for short time series ($N \leq 1000$). We note that for large dimensions and small data ($m \approx 20$ and $N \approx 200$) the PPNN start to decrease in both chaotic and stochastic cases.

In practice, we identify this type of behavior on the graph of PPNN versus the phase space dimension.

As another example, we consider the standard map, given by

$$x_{n+1} = x_n + y_{n+1} \pmod{2\pi}, \quad y_{n+1} = y_n + K\sin(x_n).$$

For $K = 0$ the dynamics is non-chaotic. When the parameter $K > 0$ increases, the dynamics becomes more and more random-like (although it always remains deterministic). See [9] for details. The table below shows PPNN and Wesner's number computed for some orbit of the Standard Map in the case when $K = 6$. For the time series, we considered the x-coordinate of the orbit. We note that Wesner's number falls consistently below the benchmark of 0.1. Thus Wesner's method does not detect deterministic chaos in this example. In the same time, the values of the PPNN exhibit the typical behavior of a deterministic system in the view of our criterion: the values of PPNN for $12 \leq m \leq 17$ stay almost constant at 0.4579. See Figure 4. Linear regression applied to this data gives a slope of -0.0011, which differs from 0 by less than the standard error of the estimate 0.0020.

Dimension	PPNN	Wesner	Dimension	PPNN	Wesner
2	0.1472	0.0736	12	0.4545	0.0378
3	0.2142	0.0714	13	0.4569	0.0351
4	0.2871	0.0717	14	0.4648	0.0332
5	0.3505	0.0701	15	0.4673	0.0311
6	0.4093	0.0682	16	0.4590	0.0286
7	0.3541	0.0505	17	0.4450	0.0261
8	0.2984	0.0373	18	0.4033	0.0224
9	0.3421	0.0380	19	0.45	0.0236
10	0.3862	0.0386	20	0.4972	0.0248
11	0.3563	0.0323			

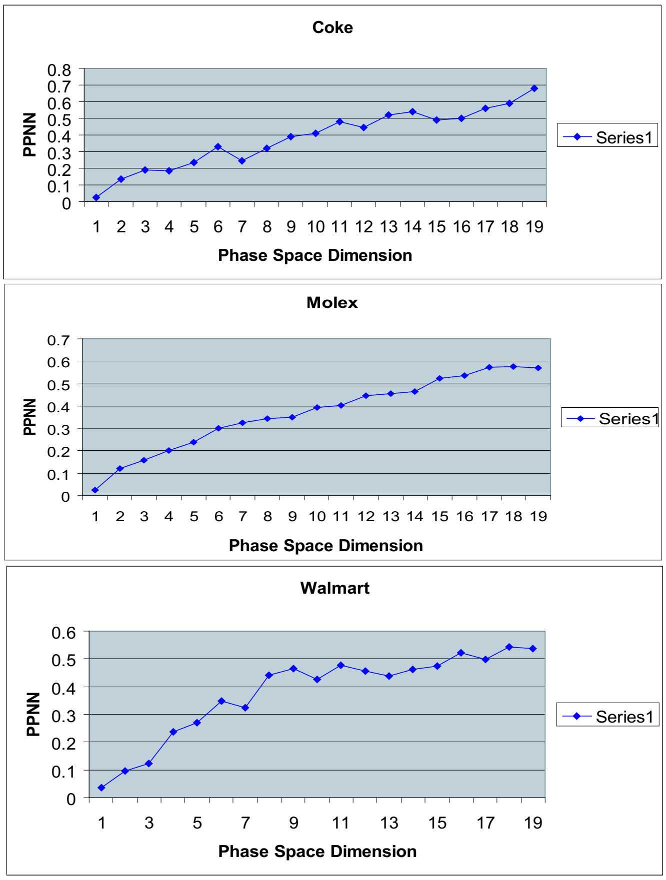

FIGURE 5. Dependence of PPNN on the phase space dimension for several stocks.

5. Numerical experiments

Using our alternative method for detection of determinism, we analyzed several stock prices: stocks with high beta coefficient (BioTech, FlyI, Genta, Molex), and stocks with low beta coefficient (ATT, Coke, GE, SBC, Wal-Mart). The stock data was obtained from Media General Financial Services. For the time series, we used the daily changes of the stock prices for up to two years. We did not find conclusive evidence of determinism in these stocks. Some of the corresponding graphs are shown in Figure 5.

Among these stocks, Wal-Mart was the closest to exhibit a deterministic type of behavior, with respect to our criterion (for $11 \leq m \leq 15$). This could be related to some of the results in [3], where time cycles that appear to govern Wal-Mart stock price fluctuations were found through the method of moving averages.

6. Conclusions and future work

We proposed a new method for detecting deterministic behavior and chaos in short time series that is applicable to economic and financial time series. Our method is an alternative to Wesner's method for detection of determinism. It performed well with small data sets and detected determinism in situations where the original method failed. Our method is supported by numerical experiments. One should however be aware of the limited reliability of such methods, since recurrent behavior, even if exists, may not be detectable in small data sets.

We plan to refine this method and to enhance its mathematical foundation. We also plan to apply these ideas in analyzing other types of time series, such as the oil price futures. In [10], it is claimed that evidence of chaos was found in the daily oil price futures from 1983–2003, using the method of Brock, Dechert, LeBaron, and Scheinkman (BDS), Lyapunov numbers, and neural networks tests. Our preliminary tests using the method of nearest neighbors resulted in no evidence of determinism. A detailed analysis of this data will make the subject of a future project.

References

[1] K. Alligood, T. Sauer and J. Yorke, **Chaos: An introduction to Dynamical Systems**, Springer Verlag, 1996.

[2] L. Bachelier, *Théorie de la speculation*, Annales Scientifiques de l'Ecole Normale Superieure 3 (1900), pp. 21-86.

[3] C. Droke, **Moving Averages Simplified**, Marketplace Books, 2001.

[4] E. Fama, *Random walks in stock market prices*, Financial Analysts Journal 21 (1965), pp. 34-109.

[5] J.D. Farmer and J.J. Sidorowich, *Predicting chaotic time series*, Physical Review Letters 59 (1987), pp. 845-848.

[6] M. Gidea and C. Niculescu, **Chaotic Dynamical Systems: An Introduction**, Universitaria Press, 2002.

[7] D.T. Kaplan, *Exceptional Events as Evidence for Determinism*, Physica D 73 (1994), pp. 38-48

[8] M. Kennel, R. Brown and H. Abarbanel, *Determining embedding dimension for phase-space reconstruction using a geometrical construction*, Physical Review A 45 (1992), pp. 3403-3411.

[9] J. Meiss, *Standard map*, in **Encyclopedia of Nonlinear Science**, (ed. Alwyn Scott), Routledge, 2004.

[10] S. Moshiria and F. Foroutan, *Testing for deterministic chaos in futures oil price; Does neural network lead to better forecast?*, (2004), preprint.

[11] N. Wesner, *Searching for chaos on low frequency*, Economics Bulletin 3 (2004), pp. 1-8.

[12] N. Wesner, *Detecting Low Dimensional Chaos in Small Noisy Sample Sets*, (2005), preprint.

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