E C O N O M I C S   B U L L E T I N

Matching buyers and sellers

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Abstract

This note analyzes the repeated interaction among buyers of a homogeneous good, in a setting of imperfect buyer mobility. The buyers are assumed to play a dynamic game of imperfect information: at each stage every buyer chooses which seller to visit without knowing the current and past choices of the other buyers. A norm of conditional loyalty might prevail, according to which buyers keep loyal if previously served. Under generalized conditional loyalty, an efficient allocation is certainly reached in a finite number of stages. There is a clear case for boundedly rational buyers to keep conditionally loyal. And, most importantly, for the two−seller case we are able to establish adherence to a strategy of conditional loyalty as an "assessment equilibrium" of the dynamic buyer game.

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1 Introduction

Buyers often develop attachment to sellers in such a way that the market of similar goods is to some extent segmented into the particular markets of the different sellers. This feature has commonly been related to product differentiation (Chamberlin, 1962, p. 69) or, more recently, to switching costs arising from brand-specific learning (Klemperer, 1987). The present paper shows that, even under product homogeneity, in a setting of imperfect mobility the concern of buyers about service prospects is another possible reason for loyalty. Other recent works also point to the possible emergence of loyalty in a similar setting, though by a different methodology. Kirman and Vriend (2001) and Goldman, Kraus, and Shehory (2004)[1] adopt a computational approach with automated buyers and sellers. Kirman and Vriend discover a significant pattern of conditional loyalty in the actual evolution of a market where buyers and sellers behave adaptively. Goldman et al. make simulations with alternative strategy profiles and find "experimental equilibria" in which the buyers are conditionally loyal.

In the model below, there are n sellers that supply with fixed and equal capacities a homogeneous good at the same, exogenously given price. Total capacity equals m, the number of buyers. At each stage every buyer demands one unit and chooses independently which of the n sellers to visit without knowing the choices previously made by the other buyers. Thus we are concerned with a dynamic game of imperfect information that is played by the m buyers when prices are set equal by the n sellers. The interesting question is whether one can expect, without any coordination, that some efficient allocation of buyers will nonetheless be reached. In spite of there being many such allocations, this turns out to be a concrete possibility. The buyers might obey a norm that prescribes keeping loyal if previously served and switching to any seller never tried before if rationed. When this norm of conditional loyalty prevails, then an efficient allocation will certainly be reached in a finite number of stages. Two possible explanations are given for adherence to such a norm. The first one is in terms of boundedly rational buyers who derive expectations about service prospects from their most recent experience with any seller already tried. Alternatively, conditional loyalty may look rewarding to buyers who recognize their strategic interdependence. More specifically, looking for equilibria of the dynamic buyer game we develop a notion of "assessment equilibrium" that is closely

[1]Kirman and Vriend (p. 464) and Goldman et al. (pp. 331-332) discuss differences in methodology and assumptions with our earlier work on the topic (De Francesco, 1996 and 1998).

related to Kreps and Wilson's (1987) sequential equilibrium. In fact, for the two-seller case we are able to prove that it is an assessment equilibrium for the m buyers to adhere to a strategy incorporating the norm of conditional loyalty.

2 The static buyer game

Let $\mathcal{M} = \{1, ..., h, ..., m\}$ and $\mathcal{N} = \{1, ..., i, ..., n\}$ be the set of buyers and sellers, respectively, with $m/n = \lambda$ integer. The sellers produce the same indivisible good at some price $p < 1$, 1 being the buyer's reservation price. Every buyer demands one unit and every seller has capacity λ, hence total capacity and demand are both equal to m. The buyers choose simultaneously which seller to visit. In this section this choice is made once and for all. Thus the pure strategy set available to each buyer is $\mathcal{S}_h = \{s_h\} = \mathcal{N}$, where $s_h = i$ is the action of visiting seller i by buyer h. This reflects the assumption that switching seller is too costly for a buyer who gets rationed. A pure strategy profile for the m buyers is written as $s = (s_1, ..., s_h, ..., s_m)$ and a strategy profile for all $k \neq h$ is written as s_{-h}. Any s leads to a vector of demands for the n sellers, $m = (m_1, ..., m_i, ..., m_n)$, where $m_i = \#\{h : s_h = i\}$. Also, $\widehat{m}_i = \#\{k \neq h : s_k = i\}$ is the number of all buyers but h who visit seller i. Production takes place on demand, hence seller i produces $\min\{m_i, \lambda\}$. If $m_i > \lambda$, then the seller chooses randomly which buyers to serve. Denote by $\pi_h(s)$ buyer h's service probability at s: then $\pi_h(s) = \min\{1, \lambda/(\widehat{m}_i + 1)\}$ for any $h : s_h = i$. The buyers are risk neutral, so each h seeks to maximize expected surplus $(1-p)\pi_h(s_h, s_{-h})$ and hence $\pi_h(s_h, s_{-h})$. The pure strategy equilibria (PSEs) of the static game are easily found.

Proposition 1 *Any strategy profile $s^* : m_i^* = \lambda$ for any $i \in \mathcal{N}$ is a PSE.*

Proof. In the Appendix. ■

PSEs are efficient ($\pi_h(s^*) = 1$). There is a serious selection problem, though: their number is $\Pi_{k=0}^{n-1} \begin{pmatrix} m - k\lambda \\ \lambda \end{pmatrix}$, hence rapidly increasing in n and m. Not surprisingly, recent analyses of Bertrand-Edgeworth competition (Peters, 1984 and 2000, Deneckere and Peck, 1995, Burdett, Shi, and Wright, 2001) have assumed that, at the set prices, a mixed strategy equilibrium of the buyer game is played when such an equilibrium exists. A mixed strategy is a vector $\sigma = (\upsilon_1, ..., \upsilon_i, ..., \upsilon_n)$, where υ_i is the probability that seller i is visited by the buyer. A mixed strategy equilibrium (MSE) is easily found.

Proposition 2 *All buyers playing* $\sigma^* = (\frac{1}{n}, ..., \frac{1}{n})$ *is a MSE.*

Proof. In the Appendix. ■

The MSE is clearly inefficient: mismatchings between demands and capacities occur with positive probability so that $\pi_h(\sigma^*) < 1$.

3 The dynamic buyer game

The buyers are now assumed to take repeat decisions over a large number of stages. At each stage $t = 1, ..., T$, every buyer demands one unit and chooses which seller to visit. This incorporates imperfect mobility in a simple way: though too costly within a single stage, switching seller is costless from one stage to the next. No buyer can observe the actions previously taken by others, hence we have a dynamic game of imperfect information. For simplicity, the buyers care only about current payoffs. As in the static game, any rationing at a seller takes place randomly among forthcoming buyers.

The most immediate solutions of the dynamic game are repeat playing of any equilibrium of the static game. Then the coordination problem sorrounding PSEs would suggest that repetition of the MSE of the static game gives a better prediction of the actual evolution of play. Yet a different behavior might in principle prevail: rather than choosing the seller randomly at every stage, the buyers might obey a norm of "conditional loyalty".

DEFINITION 1 According to the **norm of conditional loyalty**, at any $t > 1$ the buyer is loyal if served at $t - 1$, otherwise switching to any seller he has never tried before (so long as there are any).

This norm is akin to a "reduced strategy" (as defined, e. g., by Osborne and Rubinstein, 1994, p. 94): it contains prescriptions to be adhered to *so long as* the buyer has kept conditionally loyal over the past. More precisely, it is *part of* a reduced strategy: its prescriptions are also incomplete in another respect, since the rationed buyer is not told what to do in the event he has been rationed by all sellers over the past.

Denote by $m(t) = (m_1(t), ..., m_n(t))$ the vector of the demands at stage t and by $\mu(m(t))$ the probability distribution of $m(t)$, as valued at the start of the game. Quite remarkably, with all buyers conditionally loyal an efficient allocation will certainly be reached in a finite number of stages.

Proposition 3 *Let* $\tau \equiv (m - \lambda + 1)(n - 2) + 1$. *Then, under generalized conditional loyalty: (i)* $\mu(m(t) = (\lambda, ..., \lambda)) = 1$ *for all* $t > \tau$;

(ii) $\mu(m(t) = (\lambda, ..., \lambda)) < 1$ *for all* $t \leq \tau$.

Proof. (i) With all buyers conditionally loyal, if $m_i(t-1) \leq \lambda$ then $m_i(t) \geq m_i(t-1)$. Thus no fewer buyers are served at t than at $t-1$. Also, it can be $m(t) \neq (\lambda, ..., \lambda)$ only if $m(t-1) \neq (\lambda, ..., \lambda)$. Most important, any buyer can at most be rationed $n-1$ times. To see this, suppose that, by some stage $t-1$, buyer h has been rationed $n-1$ times thus far and denote by i the seller still to be tried by h. Note that $\widehat{m}_j(t-1) \geq \lambda$ for any $j \neq i$. Since $\widehat{m}_i(t-1) = m - 1 - \sum_{j \neq i} \widehat{m}_j(t-1)$, we then have $\widehat{m}_i(t-1) \leq m - 1 - (n-1)\lambda = \lambda - 1$. It follows from $\widehat{m}_j(t-1) \geq \lambda$ that $\widehat{m}_j(t) \geq \lambda$, hence $\widehat{m}_i(t) < \lambda$. Thus h cannot be rationed any more. Assume now that just one buyer is rationed at $t = 1$: e. g., $m_1(1) = \lambda + 1$, $m_2(1) = ... = m_{n-1}(1) = \lambda$, and $m_n(1) = \lambda - 1$. Note that the $\lambda - 1$ buyers who visit seller n at $t = 1$ do not risk being rationed at any $t > 1$; the buyers exposed to such a risk are the remaining $m - \lambda + 1$ ones. Thus it cannot be that $m(t) \neq (\lambda, ..., \lambda)$ at $t > (m - \lambda + 1)(n-2) + 1$, otherwise at least one buyer would have been rationed more that $n-1$ times.[2]

(ii) In the Appendix. ■

In light of Prop. 3, the coordination needed to achieve some efficient allocation would gradually arise if the buyers were conditionally loyal. For example, one can immediately check that, with two sellers, $\tau = 1$ regardless of the number of buyers: in the $m \times 2$ case, two stages are enough for an efficient allocation to be reached with unit probability.

All the above leads to the question of whether the buyers may have an individual incentive to be conditionally loyal. Let us begin with buyers who neglect their strategic interaction. More specifically, we refer to a boundedly rational buyer as one whose expectations are derived from his most recent experience with every seller already tried.

DEFINITION 2 **A boundedly rational buyer** expects he would currently be served (rationed) at seller i if he was so on his last visit to i.

Then the following result is easily established.

Proposition 4 *At any* $t > 1$ *boundedly rational buyers adhere to the norm of conditional loyalty.*

Proof. Let i be the seller visited by buyer h at the preceding stage $t-1$. It follows immediately from Definition 2 that h keeps loyal to i if previously

[2] *A fortiori* it would be so if several buyers were rationed at $t = 1$.

served. If rationed at $t-1$, buyer h moves away from i while avoiding coming back to any $j \neq i$ he visited at some earlier stage (he was rationed on his last visit to any such j, otherwise he would still be there at $t-1$). ■

We now turn to fully rational buyers who optimize in the face of their opponents' strategies. Before doing this, let us clarify the main difference in approach from Goldman et al. (2004). In their model automated buyers and sellers interact repeatedly, each with a number of available actions at each stage. In particular, a buyer might place his order randomly among the sellers or condition his current choice on past service experience. A large number of simulations are run for alternative strategy profiles, using each agent's average payoff as an estimate of his expected payoff. Then it is checked whether there are gains from a unilateral deviation: if not, then the strategy profile under consideration is an "experimental equilibrium". One result is that it is part of an "experimental equilibrium" for all the buyers to keep loyal if previously served and to switch seller if rationed.

One motivation for their empirical approach is the difficulty of finding an analytical solution for this game. It is worth, though, to explicitly address this issue in order to see how much progress can be done in this classical direction and where the main difficulties are.

To solve our dynamic game we propose developing a notion of equilibrium that is closely related to Kreps and Wilson's (1987) sequential equilibrium. Denote by $h_i^s(t)$ ($h_i^r(t)$) the event that buyer h is served (rationed) by seller i at stage t. At each date t - just before stage t is played - h is at an information set, $H(t)$. This is a $(t-1)$-component vector describing h's past experience. For example, $H(3) = (h_3^r(1), h_4^s(2))$ states that h was rationed by seller 3 at $t = 1$ and served by seller 4 at $t = 2$. Conditional on $H(t)$ buyer h has a belief on the past, that is, a probability distribution over histories of the game thus far. An "assessment" is a profile of behavioral strategies together with a system of beliefs. Our "assessment equilibrium"[3] is an assessment that meets the two basic requirements of "sequential rationality" and "structural consistency". The former stipulates that strategies be mutual best responses: at each $H(t)$, adhering to the equilibrium strategy is an optimal response when all $k \neq h$ are henceforth adhering to the equilibrium strategy. Sequential rationality holds at any $H(t)$ on the equilibrium path - any $H(t)$ that occurs with positive probability when the buyers have always adhered to the equilibrium strategy - as well as at any $H(t)$ off the equilibrium path. "Structural consistency" is about coherence of beliefs with

[3] Binmore (1992) defines so a weakened version of sequential equilibrium.

strategies, requiring that beliefs be derived using Bayes' rule.[4] If $H(t)$ is on the equilibrium path, then beliefs are derived by Bayes' rule and the assumption that all $k \neq h$ have always adhered to the equilibrium strategy; at $H(t)$ off the equilibrium path, beliefs are derived by Bayes' rule under some alternative assumption about the strategies played by the other buyers.

In the remainder we are mainly concerned with the $m \times 2$ case. Then the norm of conditional loyalty is incorporated into the following strategy, denoted by $\Theta_{m\times 2}$.

DEFINITION 3 $\Theta_{m\times 2}$ makes the following prescriptions: at $t = 1$ choose either seller with probability $\frac{1}{2}$; at any $H(t)$ arising at $t > 1$, keep loyal if served at $t-1$ and switch seller if rationed.

Establishing conditional loyalty as an assessment equilibrium is quite simple.

Proposition 5 *In the $m \times 2$ case, along with consistent beliefs it is an assessment equilibrium for every buyer to adhere to $\Theta_{m\times 2}$.*

Proof. Optimality of $\Theta_{m\times 2}$ at $t = 1$ is immediate. As regards $t > 1$, consider first any information set $H(t) = (..., h_1^r(t-1))$ (buyer h rationed by seller 1 at the last stage). Then h infers that $\widehat{m}_1(t-1) \geq \lambda$ and $\widehat{m}_2(t-1) < \lambda$. Since all $k \neq h$ are held to obey $\Theta_{m\times 2}$ at t, h expects $\widehat{m}_1(t) = \lambda$ and $\widehat{m}_2(t) = \lambda - 1$. Consequently, h's subjective service probability is one if switching seller as prescribed by $\Theta_{m\times 2}$, while being $\lambda/(\lambda+1) = m/(m+2)$ if trying again seller 1. Turn now to any $H(t) = (..., h_1^s(t-1))$. This is consistent with both $\widehat{m}_1(t-1) \geq \lambda$ and $\widehat{m}_1(t-1) < \lambda$.[5] Whatever the case may be, h expects $\widehat{m}_1(t) = \lambda - 1$ and $\widehat{m}_2(t) = \lambda$. Therefore, h's service probability is one if loyal and $m/(m+2)$ if switching to seller 2. ■

The above proof is so simple, first, because a complete action plan incorporating the norm of conditional loyalty is easy to draw and, second, because subjective service probabilities are easily determined at any information set, be it on or off the equilibrium path. Both tasks are much more complex to accomplish in the $m \times n$ case. Quite intuitively, the strategy to be checked as

[4] We do not impose Kreps and Wilson's requirement of "consistency". (For doubts about the cogency of this requirement, see Osborne and Rubistein, 1994, pp. 224-225).

[5] Incidentally, it is worth recalling that $\tau = 1$ in the $m \times 2$ case. Consequently, any $H(t) = (..., h_1^s(t-1))$ is on the equilibrium path if and only if $H(t) = (..., h_1^s(2), h_1^s(3), ..., h_1^s(t-1))$. At any such $H(t)$, structural consistency requires that h's belief at date t be $\widehat{m}_1(2) = ... = \widehat{m}_1(t-1) = \lambda - 1$.

an assessment equilibrium for the general case would contain the following subset of prescriptions - call it $\Theta_{m\times n}^{(on)}$ - at information sets on the equilibrium path: "at $t = 1$ play σ^*; at $t > 1$, keep loyal if served at $t - 1$ and otherwise visit with equal probability any seller never tried before."[6] At the present state of our research it remains a conjecture - still to be verified for the general $m \times n$ case - that adherence to $\Theta_{m\times n}^{(on)}$ is an assessment equilibrium.[7] To conclude this note, we give just a few hints for such a conjecture. Suppose $H(2) = (h_i^r(1))$. Then, assuming the other buyers are adhering to $\Theta_{m\times n}^{(on)}$, buyer h expects $\widehat{m}_i(2) \geq \lambda$. Since $\sum_{j\neq i} \widehat{m}_j(2) = \lambda n - 1 - \widehat{m}_i(2)$, this means that $\sum_{j\neq i} \widehat{m}_j(2) < (n-1)\lambda$: unlike seller i, remaining sellers *as a whole* will be visited by less buyers than their total capacity can accomodate. This is a hint for h's service prospects being better if switching to any $j \neq i$. Suppose next $H(2) = (h_i^s(1))$. Of course, this is consistent with $\widehat{m}_i(1) \lesseqgtr \lambda$ as well as with $\widehat{m}_j(1) \lesseqgtr \lambda$ for any $j \neq i$. Then it is immediately inferred that the number of $k \neq h$ who got served is $\min\{\widehat{m}_i(1), \lambda - 1\} \leq \lambda - 1$ at seller i and $\min\{\widehat{m}_j(1), \lambda\} \leq \lambda$ at any $j \neq i$. Consequently, buyer h anticipates that, at $t = 2$, the number of *loyal* buyers inherited from the preceding stage is certainly not higher than $\lambda - 1$ for seller i and certainly not higher than λ for any $j \neq i$.[8] This is is a hint for h's service prospects being better if keeping loyal to i at $t = 2$.

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[4] De Francesco, M. A., 1998, "The emergence of customer markets in a dynamic buyer game", *Quaderni del Dipartimento di Economia Politica*, No. 225, Università di Siena.

[6] Note that, at the relevant information sets, $\Theta_{m\times n}^{(on)}$ is a complete action plan: as seen in the proof of part (i) of Prop. 3, under generalized conditional loyalty the rationed buyer has always some seller not yet tried before.

[7] Though cumbersome, constructive proofs can easily be provided for small m and n. For example, we have such a proof for the 3×3 case.

[8] Of course, the number of buyers forthcoming to a seller may well exceed the number of loyal buyers that seller is inheriting from the preceding stage.

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APPENDIX

Proof of Prop. 1. For any s^* and h, s^*_{-h} is such that $\widehat{m}^*_i = \lambda$ for all $i \in \mathcal{N}$ but one, denoted by j, where $\widehat{m}^*_j = \lambda - 1$. Buyer h 's best response to s^*_{-h} is clearly $s_h = s^*_h = j$ because $\pi_h(s^*_h, s^*_{-h}) = 1$ whereas $\pi_h(s_h \neq j, s^*_{-h}) = \lambda/(\lambda+1) = m/(m+n)$. Also, no strategy profile s° implying $m_i{}^\circ \neq \lambda$ for some i can be an equilibrium. At s°, $m_i{}^\circ < \lambda$ and $m_j{}^\circ > \lambda$ for some $i, j \in \mathcal{N}$. Any $h : s_h{}^\circ = j$ has clearly failed to make a best response because $\pi_h(s_h = i, s_{-h}{}^\circ) = 1$. ■

Proof of Prop. 2. Suppose $\sigma_k = \sigma^*$ for all $k \neq h$. Then the probability that $\widehat{m}_i$ out of these buyers choose seller i is $\begin{pmatrix} m-1 \\ \widehat{m}_i \end{pmatrix} \left(\frac{1}{n}\right)^{\widehat{m}_i} \left(\frac{n-1}{n}\right)^{m-1-\widehat{m}_i}$.

Denote by $\pi_h(s_h = i, \sigma^*_{-h})$ buyer h's service probability at i when all $k \neq h$ play σ^*. Then

$$\pi_h(s_h = i, \sigma^*_{-h}) = \sum_{\widehat{m}_i=0}^{m-1} \binom{m-1}{\widehat{m}_i} \left(\frac{1}{n}\right)^{\widehat{m}_i} \left(\frac{n-1}{n}\right)^{m-1-\widehat{m}_i} \min\left(1, \frac{\lambda}{\widehat{m}_i + 1}\right).$$

Since $\pi_h(s_h = i, \sigma^*_{-h})$ is the same for any $i \in \mathcal{N}$, then buyer h does his best as well by playing $\sigma_h = \sigma^*$. This applies to all $h \in \mathcal{M}$. ■

Proof of part (ii) of Prop. 3. The proof is constructive, showing that it may be $m(t) \neq (\lambda, ..., \lambda)$ for any $t \leq \tau$. Let just one buyer be rationed at $t = 1$: without loss of generality, $m_1(1) = \lambda + 1$, $m_i(1) = \lambda$ for $i = 2, ..., n-1$, and $m_n(1) = \lambda - 1$. For notational convenience, the buyers are labelled increasingly according to their allocation at $t = 1$. Thus we have the following allocation at $t = 1$:

buyers 1 to $\lambda + 1$ at seller 1;
buyers $(i-1)\lambda + 2$ to $i\lambda + 1$ at seller $i = 2, ..., n-1$;
buyers $(n-1)\lambda + 2$ to m at seller n.

The buyers to follow (henceforth, the "relevant" buyers) are clearly buyers 1 to $(n-1)\lambda+1$: with all buyers conditionally loyal, those visiting seller n at $t = 1$ could never be rationed. Note that $m(t) \neq (\lambda, ..., \lambda)$ so long as the buyer who got rationed at $t-1$ has switched to some seller $i \neq n$ at t so that $m_i(t) = \lambda + 1$. Now we design circumstances so as to delay, as far as possible, switching to seller n by the buyer who gets rationed by seller i. First, who is rationed among the $\lambda + 1$ buyers forthcoming to seller i? It is assumed that, by chance, it is rationed the buyer who over the past has been rationed the lowest number of times. In case of ties, the buyer with the lowest label is assumed to be rationed (this is just for notational simplicity). Second, which seller does the rationed buyer switch to? The odds are assumed to be that seller $i + 1$ is selected if $i = 1, ..., n-2$, while seller 1 is selected if $i = n - 1$: this, of course, so long as it does not violate the requirement (in Definition 1) that a rationed buyer switches to any seller he has never tried before.

Now we study the evolution of play under the above circumstances. At $t = 1$ seller 1 rations buyer 1; at $t = 2$ buyer 1 switches to seller 2 where buyer $\lambda+2$ is rationed; at $t = 3$ buyer $\lambda+2$ switches to seller 3 where buyer $2\lambda + 2$ is rationed, and so on up to stage $n-1$ where buyer $(n-2)\lambda + 2$ is rationed by seller $n-1$. At stage n this buyer switches to seller 1 and then a similar sequence of stages takes place along which: seller 1 rations buyer

2 who at stage $n+1$ switches to seller 2 where buyer $\lambda+3$ is rationed, and so on. By now, one understands that by stage $t = m - \lambda + 1$ all the relevant buyers (whose number is precisely $m - \lambda + 1$) have been rationed once over the past. Also, they are allocated as follows at stage $t = (m - \lambda + 1) + 1$:

buyers $(n-2)\lambda + 2$ to $(n-1)\lambda + 1$ at seller 1;
buyers 1 to $\lambda + 1$ at seller 2;
buyers $(i-2)\lambda + 2$ to $(i-1)\lambda + 1$ at seller $i = 3, ..., n-1$.

So, the buyers who visited seller $i = 1, ..., n-2$ at $t = 1$ are now visiting seller $i+1$, and the buyers who visited seller $n-1$ are now visiting seller 1. Stage $t = (m - \lambda + 1) + 1$ is the first of a second series of $m - \lambda + 1$ stages at the end of which all the relevant buyers have been rationed twice over the past. Similarly as above, at stage $t = 2(m - \lambda + 1) + 1$ we find:

buyers $(n-3)\lambda + 2$ to $(n-2)\lambda + 1$ at seller 1;
buyers $(n-2)\lambda + 2$ to $(n-1)\lambda + 1$ at seller 2;
buyers 1 to $\lambda + 1$ at seller 3;
buyers $(i-3)\lambda + 2$ to $(i-2)\lambda + 1$ at seller $i = 4, ..., n-1$.

So, the buyers who at $t = (m - \lambda + 1) + 1$ visited seller $i = 1, ..., n-2$ are now visiting seller $i+1$, and the buyers who visited seller $n-1$ are now visiting seller 1. The game proceeds likewise until date $\tau = (m - \lambda + 1)(n-2)$ is reached. Then at stage τ we have:

buyers $\lambda + 2$ to $2\lambda + 1$ at seller 1;
buyers $i\lambda + 2$ to $(i+1)\lambda + 1$ at seller i, for any $i = 2, ..., n-2$;
buyers 1 to $\lambda + 1$ at seller $n-1$.

Thus we are done since one buyer is still rationed at stage τ. Of course, $m(\tau + 1) = (\lambda, ..., \lambda)$:[9] indeed, by date τ all buyers have been rationed $n-2$ times over the past. Consequently, buyer 1 (the one rationed by seller $n-1$ at τ) has to switch to seller n at $t = \tau + 1$. ■

[9] As it must be according to part (i) of Prop. 3.