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Measures of downside risk

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Abstract

The paper characterizes a family of downside risk measures. They depend on a target value and a parameter reflecting the attitude towards downside risk. The indicators are probability weighted −order means of possible shortfalls. They form a subclass of the measures intro¬duced by Stone (1973) and are related to the measures proposed by Fishburn (1977). The axiomatization is based on some properties which are desirable and appropriate for the measurement of risk.

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1. Introduction

A decision maker faces uncertainty almost everywhere: Future values of prices, returns or rates of return are in general uncertain. Therefore she has to decide how to deal with this issue. Given a probability distribution over outcomes one often assumes that the decision maker's preferences with respect to inherent risk are reflected by a von Neumann-Morgenstern utility function and that she maximizes expected utility. Another possibility is to evaluate risk directly by a risk measure and then to take into account risk or to minimize risk measured by such an index for a given level of return (see e.g. Sarin/Weber (1993) for a survey and Gootveld and Hallerbuch (1999) for an application of risk-value models). A priori the concept of risk is symmetrical: Good and bad outcomes may be risky. Experimental studies (Unser (2002)) demonstrate that one is often only interested in an evaluation of those outcomes which do not meet a target value: outcomes with values smaller than the target value are viewed as risky, outcomes, whose values are larger, are interpreted as nonrisky. In this case only downside risk is relevant.

The objective of this paper is to characterize a family of downside risk measures. They depend on two parameters, the target value t and a parameter ε reflecting the decision maker's attitude towards downside risk. These measures are given by

$$\rho_t^\varepsilon = \left(\sum_{x_i < t} p_i \left(t - x_i \right)^\varepsilon \right)^{1/\varepsilon} \quad \text{for } \varepsilon > 0 \tag{1}$$

where p_i and x_i are the probability and, respectively, value of outcome *i*. The indicator ρ_t^ε is a probability weighted ε-order mean of the respective shortfalls. This family is well known and forms a subclass of the measures introduced by Stone (1973). It is also related to the $\alpha - t$ measures defined in Fishburn (1977). In the last decade a number of papers has been published which discuss some properties of risk measures explicitly (see e.g. Pedersen and Satchell (1998), Artzner et al. (1999), Szegö (2002)) and survey the literature. But they do not present any characterization of the risk measures considered. Furthermore, sometimes the number parameters used is confusing: Pedersen and Satchell present a general class of measures depending on five (!) parameters. Then it seems to be impossible to comprehend their meaning and their interaction.

We axiomatize the family (1) in two steps: At first a representative risky outcome is characterized. In the second step the measure of downside risk is derived as the shortfall of this representative outcome. The representative outcome can then be interpreted as certainty equivalent defined w.r.t. to the risk measure. The axioms employed are formulated for the representative risky outcome and consider the risk inherent in probability distributions directly.

The properties imposed are necessary or desirable for downside risk measures: one property restricts the analysis to downside risk. Others postulate monotonicity in probabilities and outcomes. The particular functional structure is implied by three axioms. A substitution axiom allows us to take into account compound distributions in a simple way. Two more axioms consider equal absolute and, respectively, proportional changes in all outcome values and the target value. They require that the representative risky outcome is changed in the same way. The characterization reveals the essential properties of the family (1) and allows us to discuss the underlying value judgments.

The measurement of downside risk is formally related to the measurement of poverty (see Breitmeyer, Hakenes, and Pfingsten (2004)). The target value could be interpreted as poverty line and the shortfall of an outcome as poverty gap. There is a formal relationship between the family (1) and the Foster, Greer, and Thorbecke (FGT) family of poverty measures whose orderings are characterized in Ebert and Moyes (2002). In fact, corresponding measures are *ordinally* equivalent. But their functional structure and the measures are different: The FGT-measures are additively decomposable, whereas the measures of downside risk are not, but the latter satisfy an aggregation property which is only appropriate for the measurement of risk. Risk and poverty are different concepts. Therefore a distinct axiomatization proves to be necessary.

The paper is organized as follows: Section 2 presents the framework. In section 3 the relevant axioms are introduced and discussed. Section 4 derives the family of risk measures, relates them to other measures treated in the literature, and investigates their properties. Section 5 concludes.

2. Framework

The framework[1] used is discrete. We describe a random variable by its probability distribution over outcomes $(\boldsymbol{x},\boldsymbol{p}) = ((x_1, p_1,), (x_2, p_2), ..., (x_n, p_n))$ where $n \geq 1$ denotes the number of possible outcomes. (x_i, p_i) means that p_i is the probability of the outcome (value) $x_i \in \mathbb{R}$. Probabilities have to be nonnegative $(p_i \geq 0)$ and their sum has to be equal to unity: $\Sigma_i\, p_i = 1$. Furthermore, for generality two outcome values may be identical $(x_i = x_j \text{ for } i \neq j)$. The expected outcome is denoted by $\mu(\boldsymbol{x},\boldsymbol{p}) = \Sigma_i\, p_i x_i$. We do not interpret the variable *x* in a definite manner. *x* could for example be the return of an investment, a rate of return etc. The set of admissible distributions is denoted by $X = \{(\boldsymbol{x},\boldsymbol{p}) | (\boldsymbol{x},\boldsymbol{p}) \text{ probability distribution, } n \geq 1\}$. The number of outcomes can be arbitrary.

There are some particular cases: $(\boldsymbol{x},\boldsymbol{p}) = ((x,1)) =: (x,1)$ gives outcome[2] *x* probability 1. It is certain. Similarly, $(x_i, p_i) = (x_i, 0)$ means that there is no uncertainty about x_i. It cannot occur. Whenever $0 < p_i < 1$ there is some uncertainty left which can be interpreted as benchmark or required outcome.

We suppose that there is a target value $t \in \mathbb{R}$. It is in general given exogenously and allows us to distinguish two ranges: any outcome $x_i \geq t$ is nonrisky and desirable. On the other hand any $x_i < t$ is undesirable and risky. Then the difference $t - x_i$ for a risky outcome x_i denotes the (positive) deviation from the target *t*. It is called the shortfall of x_i. These definitions demonstrate that we are interested in *downside risk* incorporated in below target outcomes.[3]

For risk measurement we assume that for every *t* there is a weak order defined on the set of distributions *X*. It ranks distributions by their risk ("$(\boldsymbol{x},\boldsymbol{p})$ is at least as risky as $(\boldsymbol{y},\boldsymbol{q})$ given the target value *t*"). We want to derive a risk measure ρ_t representing this ordering. According to our idea of risk, ρ_t measures downside risk given *t* and can be defined as a function $\rho_t : X \rightarrow \mathbb{R}_+$ for $t \in \mathbb{R}$. The (class of) measure(s) ρ_t will be determined in two steps. For $t \in \mathbb{R}$ we will at first introduce a Representative risky outcome $R_t(\boldsymbol{x},\boldsymbol{p})$ for any $(\boldsymbol{x},\boldsymbol{p}) \in X$. It is negatively related to downside risk and is to represent the underlying ordering. The

1 Cf. Fishburn (1984) for the terminology.

2 If possible we will simplify the notation used and drop brackets.

3 Stone (1973) considers a three parameter family of measures. He introduces another parameter determining the change of deviations from the target which are to be taken into account. For the measurement of downside risk this parameter is a priori identified with the target value.

Representative risky outcome is derived and characterized by a set of (fully cardinal) properties. In the second step we consider the “representative shortfall” and define a corresponding risk measure as the shortfall of $R_t(\boldsymbol{x},\boldsymbol{p})$, i.e. by $\rho_t(\boldsymbol{x},\boldsymbol{p}) := t - R_t(\boldsymbol{x},\boldsymbol{p})$. It is always nonnegative since $R_t(\boldsymbol{x},\boldsymbol{p}) \le t$ given the axioms we impose. Finally, it will turn out that the outcome value $R_t(\boldsymbol{x},\boldsymbol{p})$ *given with certainty* is as risky as $(\boldsymbol{x},\boldsymbol{p})$ when t is the target value. Thus the Representative risky outcome is an analogue to the certainty equivalent in the expected utility framework.

3. Properties

In the following we present a number of properties of the Representative risky outcome R_t. For the proof of the results in section 4 it is sufficient to formulate most axioms only for two or three outcomes. The indicators characterized below will satisfy the corresponding properties for an arbitrary number of outcomes $n \ge 3$.

At first we restrict the range of outcome values relevant for risk measurement.

Axiom 1 (Range)

$$R_t\left((x,p),(y,1-p)\right) = R_t\left((x,p),(t,1-p)\right)$$

for all x, y s.t. $y \ge t$ *and all* p, $0 \le p \le 1$.

Since outcomes $y \ge t$ are nonrisky their (exact) value does not play a role and can be replaced by t. Nevertheless, as we will see below, the probability of this kind of outcome will be taken into account.

In the definition of a distribution $(\boldsymbol{x},\boldsymbol{p})$ we have not imposed any condition on the numbering of outcomes. Indeed, it should not be important:

Axiom 2 (Symmetry)

$$R_t(\boldsymbol{x},\boldsymbol{p}) = R_t\left(\boldsymbol{x}^\pi,\boldsymbol{p}^\pi\right) \text{ for all } (\boldsymbol{x},\boldsymbol{p}) \in X$$

where π *is a permutation of* $\{1,\dots,n\}$ *and* $\left(\boldsymbol{x}^\pi,\boldsymbol{p}^\pi\right) = \left(\left(x_{\pi(1)},p_{\pi(1)}\right),\dots,\left(x_{\pi(n)},p_{\pi(n)}\right)\right)$.

This means that $(\boldsymbol{x},\boldsymbol{p})$ can be interpreted as distribution in the ‘statistical sense’. Only the outcome values and their probabilities count. The way they are numbered is irrelevant.
For an evaluation of risk the risky outcome values are crucial. We introduce

Axiom 3 (Monotonicity in *x*)

$$R_t\left((x,p),(y,1-p)\right) < R_t\left((x',p),(y,1-p)\right)$$

for all x, x', y s.t. $x < x' \le t$ *and all* p, $0 < p \le 1$.

Here the outcome $x < t$ has positive probability. Then any increase in x is an improvement. It lowers the risk and therefore increases the Representative risky outcome R_t.

We postulate an analogous reaction of R_t if the probability of the higher risky (!) outcome is increased.

Axiom 4 (Monotonicity in *p*)

$$R_t\left((x,p),(y,1-p)\right) < R_t\left((x,p'),(y,1-p')\right)$$

for all x, y s.t. $y < x \le t$ *and all* $p, p' \ge 0$ s.t. $0 \le p < p' \le 1$.

Here again risk is reduced and thus the Representative risky outcome increases.

The next property postulates that there is no distinction between equally valued outcomes and provides a normalization rule for R_t:

Axiom 5 (Normalization)

$$R_t\left(\left(x,p\right),\left(x,1-p\right)\right)=x$$

for all $x\le t$ *and* p s.t. $0\le p\le 1$.

The distribution $\left(\left(x,p\right),\left(x,1-p\right)\right)$ is equivalent to $\left(x,1\right)$ and – since there is no uncertainty – the Representative risky outcome should be equal to *x*.

Similarly, there is no uncertainty about *y* in $\left(\left(\boldsymbol{x},\boldsymbol{p}\right),\left(y,0\right)\right)$. Its probability is zero. Therefore it can be neglected:

Axiom 6 (Irrelevance)

$$R_t\left(\left(x,p\right),\left(x',1-p\right)\right)=R_t\left(\left(x,p\right),\left(x',1-p\right),\left(y,0\right)\right)$$

for all x,x',y *and* p, $0\le p\le 1$.

Now we introduce a substitution or aggregation property. It requires that R_t can be determined stepwise. Then also compound probability distributions can be considered:

Axiom 7 (Substitution)

$$R_t\left(\left(x,p\right),\left(x',p'\right),\left(y,1-\left(p+p'\right)\right)\right)$$

$$=R_t\left(\left[R_t\left(\left(x,\frac{p}{p+p'}\right),\left(x',\frac{p'}{p+p'}\right)\right),p+p'\right],\left(y,1-\left(p+p'\right)\right)\right)$$

for all $x,x',y\in\mathbb{R}$ *and* p,p' s.t. $p\ge 0,\quad p'\ge 0$ *and* $\left(p+p'\right)\le 1$.

It is formulated for three outcomes. Here at first the conditional distribution $\left(\left(x,\frac{p}{p+p'}\right),\left(x',\frac{p'}{p+p'}\right)\right)$ is considered and its Representative risky outcome $\tilde{R}_t$ is determined. It is clear that – given the entire distribution – its probability is $\left(p+p'\right)$. Axiom 7 allows us to substitute $\left(x,p\right),\left(x',p'\right)$ by the corresponding Representative risky outcome $\tilde{R}_t$ and its probability $\left(p+p'\right)$, i.e. by $\left(\tilde{R}_t,p+p'\right)$.

Up to now only a given target value has been examined. Finally we introduce two properties which link the Representative risky outcomes for different target values. At the same time they put further conditions on the cardinalization of R_t. We propose

Axiom 8 (Linear homogeneity)

$$R_{\lambda t}\left(\left(\lambda x,p\right),\left(\lambda y,1-p\right)\right)=\lambda R_t\left(\left(x,p\right),\left(y,1-p\right)\right)$$

for all $x,y\in\mathbb{R}$, *all* p s.t. $0\le p\le 1$, *and all* $\lambda>0$.

It requires that equal proportional changes in the target value and outcome values change the Representative risky outcome by the same proportion. Equal absolute changes are dealt with in

Axiom 9 (Translatability)

$$R_{t+\alpha}\left(\left(x+\alpha,p\right),\left(y+\alpha,1-p\right)\right)=R_t\left(\left(x,p\right),\left(y,1-p\right)\right)+\alpha$$

for all $x,y,\in\mathbb{R}$, *all* p s.t. $0\le p\le 1$, *and* $\alpha\in\mathbb{R}$.

If the same amount is simultaneously added to the target value and all outcome values the Representative risky outcome has to be altered in the same way.

4. Risk measures

Up to now a set of desirable properties has been introduced. In the next step we examine their implications. We establish

Proposition 1

R_0 *satisfies Axioms 1-8 for* $t=0$ *if and only if there is* $\varepsilon>0$ *such that*

$$R_0\left(\boldsymbol{x},\boldsymbol{p}\right)=-\left(\sum_{x_i<0}p_i\left(-x_i\right)^{\varepsilon}\right)^{1/\varepsilon}\quad \textit{for all } \left(\boldsymbol{x},\boldsymbol{p}\right)\in X. \tag{2}$$

It turns out that R_0 is a weighted linear homogeneous function of all losses $-x_i$ where $x_i<0$. The measures with $\varepsilon\le 0$ are excluded, since they are not monotonic if there is an outcome x_i such that $x_i=t$. The proof (see Appendix) demonstrates that Axiom 1 allows us to concentrate on all $\left(\boldsymbol{x}^t,\boldsymbol{p}\right)$ where $x_i^t=\min\left(x_i,t\right)$. I.e. every distribution $\left(\boldsymbol{x},\boldsymbol{p}\right)$ can be truncated at t (since $x_i\ge t$ is nonrisky). Furthermore, Axioms 2-7 require that R_0 is a (probability) weighted quasi-linear mean. Axiom 8 implies linear homogeneity.

Next we extend the result to R_t for $t\ne 0$ by using Axiom 9. We obtain:

$$R_0\left(\left(x-t,p\right),\left(y-t,1-p\right)\right)=R_{t-t}\left(\left(x-t,p\right),\left(y-t,1-p\right)\right)$$

$$=R_t\left(\left(x,p\right),\left(y,1-p\right)\right)-t\quad \text{for } x,y \text{ and } 0\le p\le 1$$

and therefore

Proposition 2

R_t *satisfies Axiom 1-9 for all t if and only if there is* $\varepsilon>0$ *such that*

$$R_t\left(\boldsymbol{x},\boldsymbol{p}\right)=R_t^{\varepsilon}\left(\boldsymbol{x},\boldsymbol{p}\right):=t-\left(\sum_{x_i<t}p_i\left(t-x_i\right)^{\varepsilon}\right)^{1/\varepsilon}\quad \textit{for all } \left(\boldsymbol{x},\boldsymbol{p}\right)\in X. \tag{3}$$

The magnitude of the Representative risky outcome R_t^{ε} is essentially determined by *t* and the (probability) weighted ε-order mean of shortfalls[4]: the higher this mean, the lower the Representative risky outcome. By definition we always have $R_t^{\varepsilon}\left(\boldsymbol{x},\boldsymbol{p}\right)\le t$.

Now we are able to discuss the corresponding risk measures. We define

$$\rho_t^{\varepsilon}\left(\boldsymbol{x},\boldsymbol{p}\right):=t-R_t^{\varepsilon}\left(\boldsymbol{x},\boldsymbol{p}\right)=\left(\sum_{x_i<t}p_i\left(t-x_i\right)^{\varepsilon}\right)^{1/\varepsilon}\quad \text{for all } \left(\boldsymbol{x},\boldsymbol{p}\right)\in X \text{ and } \varepsilon>0. \tag{4}$$

[4] Ebert and Moyes (2002) present a characterization of a similar concept, the equivalent societal income for poverty orderings. Their framework is different. It is completely ordinal. Furthermore the individuals involved are equal (i.e. $p_i=1/n$).

The (downside) risk measure ρ_t^ε is equal to the weighted ε-order mean of all shortfalls. The measures ρ_t^ε form a subclass of the class of risk measures proposed by Stone (1973) and depend on two parameters, the target value t and ε. The latter reflects the attitude towards downside risk.[5] The Representative risky outcome R_t^ε can be interpreted as 'certainty equivalent' since $\rho_t^\varepsilon\left(R_t^\varepsilon(\boldsymbol{x},\boldsymbol{p}),1\right)=\rho_t^\varepsilon(\boldsymbol{x},\boldsymbol{p})$. ρ_t^ε is related to the $\alpha-t$ measure considered by Fishburn (1977) (for $\alpha=\varepsilon$) which is given by $\left(\rho_t^\varepsilon(\boldsymbol{x},\boldsymbol{p})\right)^\varepsilon$ and which has been also considered in Stone (1973). If we set t equal to the (weighted) expected outcome $\mu(\boldsymbol{x},\boldsymbol{p})$ the measure ρ_t^ε is the ε^{th} root of the lower partial moment LPM_ε of order ε. The risk measure $\rho_{\mu(\boldsymbol{x},\boldsymbol{p})}^1(\boldsymbol{x},\boldsymbol{p})$ is identical to the lower partial moment $LPM_1(\boldsymbol{x},\boldsymbol{p})$, the expected loss (measured w.r.t. to the expected outcome). For $\varepsilon=2$ we obtain the semistandard deviation $\rho_{\mu(\boldsymbol{x},\boldsymbol{p})}^2(\boldsymbol{x},\boldsymbol{p})$ (cf. also Pedersen and Satchell (1998) who define an extended family of risk measures).

The risk measure $\rho_t^\varepsilon(\boldsymbol{x},\boldsymbol{p})$ possesses a number of attractive properties:

(i) $\rho_t^\varepsilon(\boldsymbol{x},\boldsymbol{p})$ is continuous in $\boldsymbol{x}$ and $\boldsymbol{p}$ and in the target value t.

(ii) $\rho_t^\varepsilon(\boldsymbol{x},\boldsymbol{p})$ is monotonic in outcomes and probabilities; any improvement decreases ρ_t^ε. It is increasing in the target value t.

(iii) $\rho_t^\varepsilon(\boldsymbol{x},\boldsymbol{p})$ is nonnegative. If there are no risky outcomes ($\min x_i \ge t$), then $\rho_t^\varepsilon(\boldsymbol{x},\boldsymbol{p})=0$.

(iv) $\rho_t^\varepsilon(\boldsymbol{x},\boldsymbol{p})$ is symmetric in outcomes.

(v) $\rho_t^\varepsilon(\boldsymbol{x},\boldsymbol{p})$ is linearly homogeneous in the target values and outcomes, i.e. $\rho_{\lambda t}^\varepsilon(\lambda\,\boldsymbol{x},\boldsymbol{p})=\lambda\,\rho_t^\varepsilon(\boldsymbol{x},\boldsymbol{p})$ for $\lambda>0$.

(vi) Since shortfalls are not changed by the addition of the same amount to the target value *and* all outcomes, $\rho_t^\varepsilon(\boldsymbol{x},\boldsymbol{p})$ is invariant w.r.t. these changes:

$$\rho_{t+\alpha}^\varepsilon(\boldsymbol{x}+\boldsymbol{\alpha},\boldsymbol{p})=\rho_t^\varepsilon(\boldsymbol{x},\boldsymbol{p}) \text{ where } \boldsymbol{x}+\boldsymbol{\alpha}=(x_1+\alpha,\ldots,x_n+\alpha).$$

(vii) $\rho_t^\varepsilon(\boldsymbol{x},\boldsymbol{p})$ is convex [concave] in $\boldsymbol{x}$ if and only if $\varepsilon>1\left[0<\varepsilon<1\right]$. Then

$$\rho_t^\varepsilon\left(\lambda\boldsymbol{x}+(1-\lambda)\boldsymbol{y},p\right)\le\lambda\,\rho_t^\varepsilon(\boldsymbol{x},\boldsymbol{p})+(1-\lambda)\,\rho_t^\varepsilon(\boldsymbol{y},\boldsymbol{p})\le\max\left\{\rho_t^\varepsilon(\boldsymbol{x},\boldsymbol{p}),\rho_t^\varepsilon(\boldsymbol{y},\boldsymbol{p})\right\}$$

$$\left[\min\left\{\rho_t^\varepsilon(\boldsymbol{x},\boldsymbol{p}),\rho_t^\varepsilon(\boldsymbol{y},\boldsymbol{p})\right\}\le\lambda\,\rho_t^\varepsilon(\boldsymbol{x},\boldsymbol{p})+(1-\lambda)\,\rho_t^\varepsilon(\boldsymbol{y},\boldsymbol{p})\le\rho_t^\varepsilon\left(\lambda\boldsymbol{x}+(1-\lambda)\boldsymbol{y},\boldsymbol{p}\right)\right]$$

for $0<\lambda<1$. The decision maker is risk averse [risk loving].

(viii) $\rho_t^\varepsilon(\boldsymbol{x},\boldsymbol{p})$ reflects risk neutrality for $\varepsilon=1$.

The proof of these claims is obvious.

5. Conclusion

The paper has presented an axiomatization of a two-parameter family of downside risk measures and a discussion of their properties. The choice of the target value depends on the

[5] See Menezes, Geiss and Tressler (1980) and Keenan and Snow (1982) for a definition of increasing downside risk aversion in a different framework.

decision maker's preferences and on the details of the problem at hand. The selection of the parameter ε seems to be more difficult. Here it is possible to use a procedure proposed by and employed in Laughhunn, Payne, and Crum (1980). The procedure was designed for Fishburn's $\alpha - t$ measures. Since both types of measures are ordinally equivalent the method can also be applied in the above framework in order to determine the downside risk aversion ε. It is based on the ranking of distributions.

Finally it should be stressed again that downside risk measures can be employed for decision making under risk: one can minimize downside risk on the condition that a given outcome value has to be attained (see e.g. Grootveld and Hallerbach (1999)) or apply a value-risk model using a downside risk measure (see e.g. Miller (1996)). For these cases the present paper has presented an axiomatization of an important family of measures. Though this framework is different from the expected utility approach there is some relationship between both approaches (see Fishburn (1977) again and Holthusen (1989)).

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Appendix

Proof of Proposition 1

(i) Because of Axiom 1 we can confine ourselves in the following to $(\boldsymbol{x},\boldsymbol{p}) \in X$ such that $x_i \le 0$ for $i = 1,...,n$.

(ii) Choose $a < 0$ and define $X_a := \{(\boldsymbol{x},\boldsymbol{p}) \in X \,|\, a \le x_i \le 0 \text{ for } i = 1,...,n\}$. Now we prove that R_0 is a ε-order mean on X_a by using the Theorem stated on page 242, Aczel (1966).

We suppose that R_0 satisfies Axiom 2-7 and define

$$F(x,y;r,s) := R_0\left(\left(x,\frac{r}{r+s}\right),\left(y,\frac{s}{r+s}\right)\right)$$

for $a \le x \le 0,\ a \le y \le 0,\ r \ge 0,\ s \ge 0,\ r+s > 0$.

Then *F* satisfies the assumptions of the above theorem:

1. Reflexivity: Use Axiom 5
2. Internality:

$$\begin{aligned} F(a,0;1,0) &= R_0\big((a,1),(0,0)\big) \\ &= R_0\big(\big(R_0((a,p),(a,1-p)),1\big),(0,0)\big) \text{ by Axiom 5 for } 0 \le p \le 1 \\ &= R_0\big((a,p),(a,1-p),(0,0)\big) \text{ by Axiom 7} \\ &= R_0\big((a,p),(a,1-p)\big) \text{ by Axiom 6} \\ &= a \text{ by Axiom 5.} \end{aligned}$$

Using symmetry (Axiom 2) we prove analogously

$$F(a,0;0,1) := R_0\big((a,0),(0,1)\big) = 0.$$

The rest follows from monotonicity (Axiom 4).

Then $a = F(a,0;1,0) < F(a,0;r,s) < F(a,0;0,1) = 0$ for $r > 0,\ s > 0$.

3. Homogeneity in the weights:

$F(a,b;ru,su) = F(a,b;r,s)$ for $r \ge 0, s \ge 0, r+s > 0,$ and $u > 0$ by the definition of *F*.

4. Bisymmetry:

$$F\big(F(x,y;r,s),F(x',y';r',s');r+s,r'+s'\big)$$
$$= F\big(F(x,x',r,r'),F(y,y';s,s');r+r',s+s'\big)$$

It follows directly from repeated applications of Axiom 2 (Symmetry) and Axiom 7 (Substitution).

5. Increasingness in the (2nd) weight:

Use Axiom 2 (Symmetry) and Axiom 4.

6. Increasing in the (2nd) variable:

Use Axiom 2 (Symmetry) and Axiom 3.

The Theorem on page 242 implies that there is a continuous and strictly increasing function $f:[a,0]\to\mathbb{R}$ such that

$$R_0\left((x,p),(y,1-p)\right)=f^{-1}\left(p\,f(x)+(1-p)\,f(y)\right). \qquad (*)$$

Then R_0 is a continuous (!) weighted quasi-linear mean. It is clear that *f* depends on *a*. Since on the other hand R_0 is defined on *X* (not only on X_a) there is a continuous and strictly increasing function $f:[-\infty,0]\to\mathbb{R}$ such that (*) is satisfied.

(iii) By (ii) we know that

$$R_0\left((u,p),(\upsilon,1-p)\right)=-g^{-1}\left(pg(-u)+(1-p)g(-\upsilon)\right)$$

where $g(z):=f(-z)$ for $z\ge 0$. Using Axiom 8 (linear homogeneity) we get

$$R_0\left((\lambda u,p),(\lambda\upsilon,1-p)\right)=\lambda R_0\left((u,p),(\upsilon,1-p)\right) \text{ for } u\ge 0,\ \upsilon\ge 0,\ 0\le p\le 1 \text{ and } \lambda>0.$$

Therefore

$$\frac{1}{\lambda}\,g^{-1}\left(p\,g(\lambda u)+(1-p)\,g(\lambda\upsilon)\right)=g^{-1}\left(p\,g(u)+(1-p)\,g(\upsilon)\right).$$

Theorem 2 in Aczel (1966), p. 290 yields

$$g(\lambda z)=a(\lambda)\,g(z)+b(\lambda) \text{ for } z\ge 0 \text{ and } \lambda>0 \text{ and given } p.$$

Then Theorem 2.7.3 in Eichhorn (1978) implies that there are $\beta\ne 0,\varepsilon\ne 0$ and $\gamma\in\mathbb{R}$ such that

$$g(z)=\beta z^{\varepsilon}+\gamma,\ a(\lambda)=\lambda^{\varepsilon}, \text{ and } b(\lambda)=\gamma\left(1-\lambda^{\varepsilon}\right)$$

or $g(z)=\beta\ln z+\gamma$, $a(\lambda)=1$, and $b(\lambda)=\gamma\ln\lambda$ for $z>0$ and $\lambda>0$.

We obtain

$$R_0\left((x_1,p_1),(x_2,p_2)\right)=\begin{cases}-\left(\sum_{i=1}^{2}p_i\left(-x_i\right)^{\varepsilon}\right)^{1/\varepsilon} & \text{for } \varepsilon\ne 0\\ -\prod_{i=1}^{n}\left(-x_i\right)^{p_i} & \text{for } \varepsilon=0\end{cases}$$

as long as $x_i>0$ for $i=1,2$ and given p_1,p_2.

Continuity of R_0 requires that the functional structure derived also holds if $x_1=0$ or $x_2=0$. Furthermore, the parameter ε must be independent of p_1 and p_2. Now it turns out for $\varepsilon\le 0$ that $R_0\to 0$ if $x_1\to 0$ or $x_2\to 0$. Then Monotonicity (Axiom 3) is violated.

Thus we get

$$R_0\left((x_1,p_1),(x_2,p_2)\right)=-\left(\sum_{x_i<0}p_i\left(-x_i\right)^{\varepsilon}\right)^{1/\varepsilon} \quad \text{for } \varepsilon>0.$$

(iv) This result can be extended to *X* by using Axiom 7 (Substitution) and the Axiom 1 (Range). □