A note on Phillips-Perron-type statistics for cointegration testing

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Abstract

We introduce two trace statistics for the null hypothesis of no cointegration that nonparametrically correct for serial correlation in the spirit of Phillips-Perron. The limiting distributions are free of nuisance parameters. One of them coincides with the asymptotic distribution of Johansen's trace statistic. Hence, this statistic is applicable without further tabulation of critical values.

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1 Introduction

The nonparametric corrections for serial correlation of the Dickey-Fuller [DF] statistics by Phillips (1987) and Phillips and Perron (1988) are widely used in practice. Here we address the question how these unit root tests can be generalized in order to obtain trace statistics for the null hypothesis of no cointegration with limiting distributions free of nuisance parameters.

Let x_t be a vector of $I(1)$ variables of length n with $\Delta x_t = u_t$, where $E(u_t) = 0$. Consider the regression

$$x_t = \widehat{A} x_{t-1} + \widehat{u}_t \,, \quad t = 1, \ldots, T,$$

estimated by ordinary least squares (OLS), or in differences ($\Delta x_t = x_t - x_{t-1}$)

$$\Delta x_t = \widehat{\Pi} x_{t-1} + \widehat{u}_t \,, \quad \widehat{\Pi} = \widehat{A} - I_n \,, \tag{1}$$

with I_n denoting the identity matrix.

In the univariate case ($n = 1$), the DF unit root test may be based on the normalized estimator, $T(\widehat{A} - 1)$, or on the studentized statistic t_A testing for $A = 1$,

$$t_A = \frac{\sqrt{\sum x_{t-1}^2}}{\sqrt{T^{-1} \sum \widehat{u}_t^2}} \left(\widehat{A} - 1 \right) ,$$

where all sums run from $t = 1$ through T. In practice, one may augment (1) with lagged differences to account for serial correlation of u_t. Phillips (1987) and Phillips and Perron (1988) considered a different way to handle short-run memory and advocated nonparametric corrections of the test statistics:

$$Z(\widehat{A}) := T(\widehat{A} - 1) - \frac{\widehat{\Omega} - \widehat{\Gamma}(0)}{2T^{-2} \sum x_{t-1}^2}, \tag{2}$$

$$\begin{aligned} Z(t_A) \;&:=\; \frac{\sqrt{T^{-1} \sum \widehat{u}_t^2}}{\sqrt{\widehat{\Omega}}} \, t_A - \frac{\widehat{\Omega} - \widehat{\Gamma}(0)}{2\sqrt{\widehat{\Omega}} \sqrt{T^{-2} \sum x_{t-1}^2}} \\ &= \frac{\sqrt{\sum x_{t-1}^2}}{\sqrt{\widehat{\Omega}}} (\widehat{A} - 1) - \frac{\widehat{\Omega} - \widehat{\Gamma}(0)}{2\sqrt{\widehat{\Omega}} \sqrt{T^{-2} \sum x_{t-1}^2}}, \end{aligned} \tag{3}$$

where $\widehat{\Gamma}(0)$ and $\widehat{\Omega}$ are consistent estimators of the variance and long-run variance of u_t, respectively (see Phillips and Durlauf (1986) or Phillips (1987) for a discussion).

In this note we consider tests for the null hypothesis of no cointegration formally testing for $A = I_n$ or $\Pi = 0$ in (1). We propose two nonparametric corrections of the OLS estimator in the spirit of Phillips-Perron and prove that their limit distributions are free of nuisance parameters. In particular one of them coincides with the distribution discovered by Johansen (1988).

2 Trace tests

2.1 Johansen's test

Johansen's (1988) trace test for no cointegration can be considered as an extension of the DF statistic t_A. It relies on the eigenvalues λ_j of the matrix

$$M_x := S_{11}^{-1} S_{10} S_{00}^{-1} S_{10}'$$

with

$$S_{11} = T^{-1} \sum_{t=1}^{T} x_{t-1}\, x_{t-1}' \,, \quad S_{10} = T^{-1} \sum_{t=1}^{T} x_{t-1}\, \Delta x_t' \,, \quad S_{00} = T^{-1} \sum_{t=1}^{T} \Delta x_t\, \Delta x_t' \,.$$

Under the null hypothesis of no cointegration the eigenvalues converge to zero, which yields the following approximation of the likelihood ratio test statistic in terms of the trace of M_x:

$$\text{LR} \;:=\; -T \sum_{j=1}^{n} \log(1-\lambda_j) \approx T \sum_{j=1}^{n} \lambda_j = T \, \text{tr}\,[M_x] \,.$$

In case of $n = 1$, one obtains $T \, \text{tr}\,[M_x] \approx t_A^2$, where the approximation relies on $S_{00} \approx T^{-1} \sum \widehat{u}_t^2$.

To account for serial correlation of u_t, Johansen (1988) assumed a VAR model and considered lag augmentation in (1). Alternatively, we introduce two nonparametric corrections of the OLS estimator along the lines in Phillips (1987) and Phillips and Perron (1988).

2.2 Assumptions and notation

We assume that the I(0) process u_t is stationary and ergodic, although this is stronger than necessary and maintained only for convenience. A set of more general assumptions is presented for instance in Phillips and Durlauf (1986). Consequently,

$$T^{-1}\sum_{t=1}^{T-h} u_t u_{t+h}' \;\overset{p}{\to}\; \mathrm{E}\left(u_t u_{t+h}'\right) =: \Gamma(h),$$

where $\overset{p}{\to}$ stands for convergence in probability. It will be convenient to work with the following matrices

$$\Lambda := \sum_{h=1}^{\infty}\Gamma(h), \quad \Omega := \sum_{h=-\infty}^{\infty}\Gamma(h) = \Gamma(0)+\Lambda+\Lambda'.$$

We omit technical details discussed in the literature and assume instead (where joint convergence also applies):

$$T^{-2}\sum_{t=1}^{T} x_t\, x_t' \;\overset{d}{\to}\; \int_0^1 B(r)\, B(r)' dr \;=\; \int B\, B', \tag{4}$$

$$T^{-1}\sum_{t=1}^{T} x_{t-1}\, u_t' \;\overset{d}{\to}\; \int_0^1 B(r)\, dB(r)' + \Lambda \;=\; \int B\, dB' + \Lambda, \tag{5}$$

as $T \to \infty$ where $\overset{d}{\to}$ denotes convergence in distribution. The Brownian motion B is defined in terms of a standard Wiener process W of length n, $B = \Omega^{1/2}W$. Finally, the I(1) vector x_t alone is assumed to be not cointegrated, $\Omega > 0$ (positive definite).

2.3 Result

Define

$$Z(M_x) := T S_{11}^{-1}(S_{10}-\widehat{\Lambda})\widehat{\Omega}^{-1}(S_{10}-\widehat{\Lambda})', \tag{6}$$

and

$$Z(\widehat{A}) := T(\widehat{A}-I_n) - T\widehat{\Lambda}' S_{11}^{-1}, \tag{7}$$

where $\widehat{\Lambda}$ and $\widehat{\Omega}$ are again consistent estimators.

Proposition *Under the above assumptions it holds*

$$tr\,[Z(M_x)] \;\overset{d}{\to}\; tr\left[\left(\int WW'\right)^{-1}\int W dW'\left(\int W dW'\right)'\right],$$

$$tr\left[Z(\widehat{A})\right] \;\overset{d}{\to}\; tr\left[\left(\int W\, dW'\right)'\left(\int W\, W'\right)^{-1}\right]$$

as $T \to \infty$.

Proof With (4) and (5) the proof is elementary. By the continuous mapping theorem it holds:

$$\begin{aligned} Z(M_x) &\xrightarrow{d} \left(\int B\,B'\right)^{-1}\int BdB'\,\Omega^{-1}\left(\int BdB'\right)' \\ &= \Omega^{-1/2}\left(\int WW'\right)^{-1}\int WdW'\left(\int WdW'\right)'\Omega^{1/2}\,, \end{aligned}$$

which establishes the first result by the trace properties. The second result is established the same way. ■

Notice that the distribution of $\mathrm{tr}\,[Z(M_x)]$ coincides with the one given in Johansen (1988, Theorem 3).

3 Discussion

Remark 1 Consider the univariate case ($n = 1$) where $\Omega = \Gamma(0) + 2\,\Lambda$. Using $\widehat{\Lambda} = (\widehat{\Omega} - \widehat{\Gamma}(0))/2$, it is straightforward to verify that $Z(\widehat{A})$ reduces to the expression in (2), while $Z(M_x) = [Z(t_A)]^2$. Hence, our proposals from (6) and (7) are indeed direct extensions of the Phillips-Perron statistics.

Remark 2 In a univariate context Phillips and Perron (1988) observed that $Z(\widehat{A})$ is more powerful than $Z(t_A)$ in empirically relevant cases. Nevertheless, we recommend the use of $\mathrm{tr}\,[Z(M_x)]$ instead of $\mathrm{tr}[Z(\widehat{A})]$, simply because percentiles of the limiting distribution of the latter are not tabulated, while critical values for $\mathrm{tr}\,[Z(M_x)]$ are readily available, see e.g. Johansen (1995, Table 15.1) or Osterwald-Lenum (1992, Table 0).

Remark 3 The test statistic $\mathrm{tr}[Z(M_x)]$ can also be seen as a modification of the Wald statistic W testing for $\Pi = 0$ with $\widehat{\Pi} = S_{10}'\,S_{11}^{-1}$:

$$\begin{aligned} W &:= \mathrm{vec}(\widehat{\Pi})'\left[\left(\sum x_{t-1}\,x_{t-1}'\right)^{-1}\otimes T^{-1}\sum \widehat{u}_t\,\widehat{u}_t'\right]^{-1}\mathrm{vec}(\widehat{\Pi}) \\ &= \mathrm{tr}\left[\left(T^{-1}\sum \widehat{u}_t\,\widehat{u}_t'\right)^{-1}\widehat{\Pi}\sum x_{t-1}\,x_{t-1}'\widehat{\Pi}'\right] \\ &= T\,\mathrm{tr}\left[\left(T^{-1}\sum \widehat{u}_t\,\widehat{u}_t'\right)^{-1}\widehat{\Pi}\,S_{11}\widehat{\Pi}'\right]. \end{aligned}$$

Replacing the variance estimator $T^{-1}\sum \widehat{u}_t\,\widehat{u}_t'$ by $\widehat{\Omega}$ and adding two appropriate terms results in $\mathrm{tr}[Z(M_x)]$:

$$\mathrm{tr}[Z(M_x)] \;=\; T\,\mathrm{tr}\left[\widehat{\Omega}^{-1}\widehat{\Pi}S_{11}\widehat{\Pi}'\right] - 2\,T\,\mathrm{tr}\left[\widehat{\Omega}^{-1}\widehat{\Pi}\widehat{\Lambda}\right] + T\,\mathrm{tr}\left[\widehat{\Omega}^{-1}\,\widehat{\Lambda}'\,S_{11}^{-1}\,\Lambda\right].$$

Hence, $\mathrm{tr}[Z(M_x)]$ has a Wald-type representation and interpretation. A similarly modified Wald statistic has been discussed by Phillips and Durlauf (1986), however in terms of the alternative estimator

$$\widetilde{\Pi} = \frac{S_{10}' + S_{10}}{2}\,S_{11}^{-1}$$

instead of $\widehat{\Pi}$.

REMARK 4 Phillips and Ouliaris (1990) suggested a trace test which is similar in spirit to our proposal,

$$\mathrm{tr}\left[\widehat{\Omega}\,T\,S_{11}^{-1}\right] \;\stackrel{d}{\to}\; \mathrm{tr}\left[\left(\int WW'\right)^{-1}\right],$$

where critical values of this limit are tabulated in their paper. A comparison of the competing procedures via Monte Carlo experiments is beyond the scope of this note.

REMARK 5 So far we have neglected determinisitc components. If all variables are demeaned before computing S_{ij} and $\mathrm{tr}\,[Z(M_x)]$, then the resulting limiting distribution is given in terms of demeaned Wiener processes. For critical values, see e.g. Osterwald-Lenum (1992, Table 1.1*).

References

Johansen, S. (1988) “Statistical Analysis of Cointegration Vectors” *Journal of Economic Dynamics and Control* **12**, 231-254.

Johansen, S. (1995) *Likelihood-Based Inference in Cointegrated Vector Autoregressive Models*, Oxford University Press: Oxford.

Osterwald-Lenum, M. (1992) “A Note with Fractiles of the Asymptotic Distribution of the Maximum Likelihood Cointegration Rank Test Statistic: Four Cases” *Oxford Bulletin of Economics and Statistics* **54**, 461-472.

Phillips, P.C.B. (1987) “Time Series Regression with a Unit Root” *Econometrica* **55**, 277-301.

Phillips, P.C.B., and S.N. Durlauf (1986) “Multiple Time Series Regressions with Integrated Processes” *Review of Economic Studies* **53**, 473-495.

Phillips, P.C.B, and S. Ouliaris (1990) “Asymptotic Properties of Residual Based Tests for Cointegration” *Econometrica* **58**, 165-193

Phillips, P.C.B, and P. Perron (1988) “Testing for a Unit Root in Time Series Regressions” *Biometrika* **75**, 335-346.