E C O N O M I C S   B U L L E T I N

The Aggregated Structural-Change Model

Terence Tai-Leung Chong
The Chinese University of Hong Kong

Isabel Kit-Ming Yan
City University of hong Kong

Guoxin Liu
York Uninversity

Abstract

Most of the existing structural-change models presume that the impact of a change is instantaneous and occurs at the same time for all individuals. In this paper, we develop a new structural-change model to measure the lag length between the time when an economic crisis breaks out and the time when the impact is transmitted to various economic sectors. Our model allows different transmission lags for individuals with heterogenous characteristics. Simulation results for the performance of the estimators are reported.

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1 Introduction

Existing structural-change models usually presume no lag or identical lag in the diffusion[1] of shocks to individuals with heterogeneous economic characteristics. Although this type of abrupt change approach provides a parsimonious simplification of individual behavior, it is rather unrealistic. For instance, when there is a stock market crash, people with different positions in the stock market are likely to be affected differently. Similarly, in an oil crisis, the surge in the oil price will immediately affect the sectors that rely heavily on oil, such as the transportation and the plastic manufacturing sectors, while its impact on other sectors will appear later.

The objective of this paper is to develop a model for this kind of multiple-order diffusion process and estimate the parameters which govern the diffusion duration. The model combines the structural-change model (Kurozumi, 2005; Chong, 2001, 2003) and the aggregation model (Theil, 1954; Granger, 1980; Forni and Lippi, 1997; Chong, 2006)[2]. We assume that the break date for each individual is drawn from a distribution with unknown parameters. By aggregating the individual abrupt-change functions, we get the aggregate structural-change model.

The remaining of this paper is organized as follows: In Section 2, we develop the aggregate structural-change model and derive a procedure to estimate the pre- and post-shift parameters, as well as the parameters of the underlying distribution that governs the lag length. Section 3 presents the simulation results of the proposed estimation method. Section 4 concludes the paper.

[1] For more details on diffusion, one is referred to Jovanovic and MacDonald (1994), Lippi and Reichlin (1994) and Andolfatto and MacDonald (1998).

[2] A comprehensive literature review on the problem of aggregations over individuals is provided in Stoker (1993). More recent works in the area of aggregation include Olsen (2000) and Aoki (2002).

2 The Model

Suppose that an economic crisis occurs at time $t = 0$ in an economy with n individuals. The crisis is likely to trigger a domino effect, as different people may be affected by the crisis differently. We construct the following simple structural-break model to capture the heterogeneity in the diffusion of shocks for individual i at time t:

$$\begin{aligned} y_{it} &= (a_1 + b_1 x_{it})\, 1\{t < k_i\} + (a_2 + b_2 x_{it})\, 1\{t \geqslant k_i\} + \varepsilon_t + \eta_{it} \qquad (1) \\ (i &= 1, 2, ..., n; t = 0, 1, 2, ..., T.) \end{aligned}$$

where

$k_i \in \{0, 1, 2..., T\}$ is the lag in the diffusion of shocks triggered by a crisis occurred at time 0. The duration varies across individuals.

$1\{\cdot\}$ is an indicator function which equals one if the condition inside the bracket is true, and equals zero otherwise.

The error components ε_t and $\eta_{1t}, \eta_{2t}, ..., \eta_{nt}$ are pair-wise independent, with zero means and with variances $var(\varepsilon_t) = \sigma_\varepsilon^2$ and $var(\eta_{it}) = \sigma_\eta^2$ for all i. It is also assumed that ε_t are serially independent.

Given the structural-break functions of all the individuals in the economy, we can derive the aggregate structural-change model by performing a linear aggregation. Our analysis is based on the per capita series, which has the same time series properties as the aggregate series. Let

$$
\begin{aligned}
\overline{y}_t &= \frac{1}{n}\sum_{i=1}^{n} y_{it} \\
&= a_1 \frac{1}{n}\sum_{i=1}^{n} 1\{t < k_i\} + b_1 \frac{1}{n}\sum_{i=1}^{n} x_{it} 1\{t < k_i\} \\
&\quad + a_2 \frac{1}{n}\sum_{i=1}^{n} 1\{t \geq k_i\} + b_2 \frac{1}{n}\sum_{i=1}^{n} x_{it} 1\{t \geqslant k_i\} + \varepsilon_t + \frac{1}{n}\sum_{i=1}^{n} \eta_{it}. \qquad (2)
\end{aligned}
$$

Let

$$
\overline{x}_t = \frac{1}{n}\sum_{i=1}^{n} x_{it}.
$$

We assume that k_i are independent across i and that the regressor x_{it} is independent of k_i for all i and t.

As $n \to \infty$, we have

$$
\frac{1}{n}\sum_{i=1}^{n} 1\{t < k_i\} \to \Pr(k_i > t),
$$

$$
\frac{1}{n}\sum_{i=1}^{n} 1\{t \geqslant k_i\} \to \Pr(k_i \leqslant t),
$$

$$
\frac{1}{n}\sum_{i=1}^{n} x_{it} 1\{t < k_i\} \to \overline{x}_t \Pr(k_i > t),
$$

$$
\frac{1}{n}\sum_{i=1}^{n} x_{it} 1\{t \geqslant k_i\} \to \overline{x}_t \Pr(k_i \leqslant t),
$$

$$
\frac{1}{n}\sum_{i=1}^{n} \eta_{it} \to 0,
$$

and

$$\overline{y}_t = (a_1 + b_1\overline{x}_t)\Pr(k_i > t) + (a_2 + b_2\overline{x}_t)\Pr(k_i \leq t) + \varepsilon_t. \tag{3}$$

A salient feature of this model is that it is derived from the aggregation of individual structural change functions that allow for heterogeneous individual specific characteristics.

2.1 Estimation in Finite Sample

For finite value of T, we use a Poisson distribution[3] (with mean m_0) to model the arrival time of the impact. Thus,

$$\Pr(k_i = j) = \frac{m_0^j \exp(-m_0)}{j!}, \qquad j = 0, 1, 2, ... \tag{4}$$

where $0 \leq m_0 \leq T$ is assumed. Hence,

$$\Pr(k_i \leqslant t) = \sum_{x=0}^{t} \frac{m_0^x \exp(-m_0)}{x!} = \frac{\Gamma(t, m_0)}{\Gamma(t)}, \tag{5}$$

where

$$\Gamma(t, m) = \sum_{x=0}^{t} \frac{m^x \exp(-m)}{x!} \Gamma(t) \tag{6}$$

and

$$\Gamma(t) = (t-1)!. \tag{7}$$

Thus, we have

[3] Aoki (2002) studies the aggregation of individual binary decision rules, he shows that under certain conditions, one can obtain a Poisson distribution.

$$\overline{y}_t = (a_1 + b_1\overline{x}_t)\left(1 - \frac{\Gamma(t,m)}{\Gamma(t)}\right) + (a_2 + b_2\overline{x}_t)\frac{\Gamma(t,m)}{\Gamma(t)} + \varepsilon_t. \tag{8}$$

We estimate the model by using a two-step nonlinear LS estimation procedure. For any given m, we find the LS estimators $\widehat{a}_{1T}(m)$, $\widehat{a}_{2T}(m)$, $\widehat{b}_{1T}(m)$ and $\widehat{b}_{2T}(m)$ to minimize the total sum of squared residuals $RSS(m)$ defined as

$$\sum_{t=0}^{T}\left(\overline{y}_t - \left(\widehat{a}_{1T}(m) + \widehat{b}_{1T}(m)\,\overline{x}_t\right)\left(1 - \frac{\Gamma(t,m)}{\Gamma(t)}\right) - \left(\widehat{a}_{2T}(m) + \widehat{b}_{2T}(m)\,\overline{x}_t\right)\frac{\Gamma(t,m)}{\Gamma(t)}\right)^2. \tag{9}$$

We then search for a value of m which minimizes $RSS(m)$. The estimator for m is defined as

$$\widehat{m} = \underset{m\in(0,T)}{Arg\min} RSS(m). \tag{10}$$

After obtaining the $\widehat{m}$, the final structural estimators are

$$\widehat{a}_{iT}(\widehat{m}) \text{ and } \widehat{b}_{iT}(\widehat{m}) \text{ for } i = 1, 2. \tag{11}$$

2.2 Estimation in Large Sample

In order to allow for asymptotic analysis, we need to convert the time scale from $\{0, 1, 2, 3..., T\}$ into the $[0, 1]$ interval. The Poisson distribution does not have a continuous counterpart in the zero-one interval. Thus, we need to employ a continuous distribution in this interval. Since the time is re-scaled, the model should be rewritten as

$$\overline{y}_t = (a_1 + b_1\overline{x}_t)\Pr(l_i > h) + (a_2 + b_2\overline{x}_t)\Pr(l_i \leq h) + \varepsilon_t. \tag{12}$$

where
$h = \frac{t}{T} \in [0, 1]$;
$l_i = \frac{m_i}{T} \in [0, 1]$ is the diffusion duration of a crisis for individual i, i.e., the lag length in time before individual i is impacted by the crisis.

For simplicity, we employ the Beta distribution, which is in [0,1]. For the Beta distribution, we have

$$\Pr(l_i \leq h) = \int_0^h f(\tau; \alpha, \beta) d\tau, \tag{13}$$

where

$$f(\tau; \alpha, \beta) = \begin{cases} \frac{\Gamma(\alpha+\beta)\tau^{\alpha-1}(1-\tau)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} & \text{if} \quad 0 \leq \tau \leq 1\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{14}$$

When $h = 0$, we have $\Pr(l_i \leq 0) = 0$; when $h = 1$, we have $\Pr(l_i \leq 1) = 1$.

For any given positive α and β, the residual sum of squares is equal to

$$RSS(v) = \sum_{t=0}^{T} \left[\overline{y}_t - (a_1 + b_1 \overline{x}_t) \Pr(l_i > h) - (a_2 + b_2 \overline{x}_t) \Pr(l_i \leq h)\right]^2, \tag{15}$$

where

$$v = (a_1, a_2, b_1, b_2, \alpha, \beta). \tag{16}$$

The nonlinear LS estimators are

$$\widehat{v} = (\widehat{a}_1, \widehat{a}_2, \widehat{b}_1, \widehat{b}_2, \widehat{\alpha}, \widehat{\beta}) = \arg\min_{v \in R^4 \times R_+^2} RSS(v). \tag{17}$$

3 Simulations

In this section, we study the performance of our estimators for the following mean-shift model:

$$y_{it} = a_1 1\{t < k_i\} + a_2 1\{t \geqslant k_i\} + \varepsilon_t + \eta_{it}.$$

The pre- and post-shift parameters are $a_1 = -1$ and $a_2 = 1$ respectively. All the error terms are drawn independently from a standard normal distribution.

Experiment 1:

The objective of this experiment is to demonstrate the performance of the estimators of the pre- and post-shift regression coefficients and the Poisson mean coefficient m_0. The sample sizes of the simulations are $T = 50, 100$ and 200. The true values of m_0 are set to $0.5T$ and $0.3T$. We carry out the simulation for a sample with 100 individuals ($n = 100$) and a sample with a larger number of individuals ($n = 1000$). The number of replications is $R = 200$.

The Poisson random variable with mean m_0 is simulated by using the following inverse transformation method:

(1) Generate a uniform(0,1) random variable U.
(2) $i = 0$, $p = \exp(-m_0)$, $F = p$.
(3) If $U < F$, set $X = i$ and stop.
(4) $p = \dfrac{\lambda p}{i+1}$, $F = F + p$, $i = i + 1$.
(5) Go to step (3) until all the simulated observations are obtained.

Table 1 reports the average estimates across all replications. Note that the estimates get closer to the true values as T increases. When we compare the results with different number of individuals (n=100 and 1000), the performance of the estimators are similar.

Table 1: The Mean of the Estimators in the Poisson Case

T	50	50	100	100	200	200
m_0	25	15	50	30	100	60
$(n = 100, R = 200)$						
$\overline{\widehat{a}}_1$ (true value = -1)	-1.0108	-1.0260	-1.0060	-0.9943	-0.9998	-1.0090
$\overline{\widehat{a}}_2$ (true value = 1)	1.0281	1.0304	1.0124	1.0209	1.0060	1.0016
$\overline{\widehat{m}}$	24.87	15.16	49.95	30.28	99.88	59.80
$(n = 1000, R = 200)$						
$\overline{\widehat{a}}_1$ (true value = -1)	-1.0131	-1.0034	-1.0166	-1.0143	-1.0090	-0.9969
$\overline{\widehat{a}}_2$ (true value = 1)	1.0479	1.0188	0.9973	1.0060	0.9942	0.9968
$\overline{\widehat{m}}$	24.94	15.21	50.07	29.88	99.67	60.17

Experiment 2:

This experiment evaluates the performance of the estimators in the Beta case. The sample sizes are $T = 50, 100$ and 150. The numbers of individuals are set to $n = 150, 300$ and 1000. The number of replications is set to $R =$ 500. The parameters of Beta distribution are α and β. Let $\overline{\widehat{a}}_1$, $\overline{\widehat{a}}_2$, $\overline{\widehat{\alpha}}$ and $\overline{\widehat{\beta}}$ be the average of the estimates from the 500 replications. There are four cases of interest, namely, (i) $\alpha < 1$ and $\beta < 1$, (ii) $\alpha > 1$ and $\beta < 1$, (iii) $\alpha < 1$ and $\beta > 1$, and (iv) $\alpha > 1$ and $\beta > 1$. We exclude the cases where α or β equals 1. The results are reported in Table 2.

Table 2: The Mean of the Estimators in the Beta Case

T	50	50	50	100	100	100	150	150	150
n	150	300	1000	150	300	1000	150	300	1000
				Case 1: $\alpha = 0.4, \beta = 0.7$					
$\overline{\widehat{a}}_1$	0.983	0.986	0.990	0.981	0.988	0.982	0.979	0.986	0.988
$\overline{\widehat{a}}_2$	−1.007	−1.060	−1.011	−1.017	−1.011	−1.016	−1.014	−1.012	−1.012
$\overline{\widehat{\alpha}}$	0.397	0.379	0.394	0.384	0.387	0.386	0.390	0.384	0.392
$\overline{\widehat{\beta}}$	0.685	0.696	0.679	0.680	0.693	0.689	0.688	0.691	0.683
				Case 2: $\alpha = 1.3, \beta = 0.5$					
$\overline{\widehat{a}}_1$	0.989	0.993	0.991	0.990	0.988	0.992	0.987	0.991	0.997
$\overline{\widehat{a}}_2$	−1.015	−1.011	−1.012	−1.010	−1.014	−1.007	−1.011	−1.012	−1.002
$\overline{\widehat{\alpha}}$	1.297	1.294	1.289	1.293	1.296	1.292	1.288	1.291	1.297
$\overline{\widehat{\beta}}$	0.495	0.487	0.489	0.494	0.490	0.493	0.498	0.494	0.491
				Case 3: $\alpha = 0.5, \beta = 1.6$					
$\overline{\widehat{a}}_1$	0.981	0.984	0.989	0.993	0.984	0.994	0.985	0.987	0.995
$\overline{\widehat{a}}_2$	−1.020	−1.017	−1.012	−1.011	−1.015	−1.008	−1.011	−1.019	−1.017
$\overline{\widehat{\alpha}}$	0.487	0.490	0.492	0.484	0.492	0.489	0.488	0.485	0.486
$\overline{\widehat{\beta}}$	1.587	1.589	1.592	1.584	1.587	1.586	1.595	1.592	1.597
				Case 4: $\alpha = 1.2, \beta = 2$					
$\overline{\widehat{a}}_1$	0.989	0.986	0.987	0.983	0.995	0.982	0.987	0.994	0.992
$\overline{\widehat{a}}_2$	−1.017	−1.015	−1.013	−1.003	−1.011	−1.012	−1.012	−1.012	−1.010
$\overline{\widehat{\alpha}}$	1.189	1.192	1.190	1.192	1.194	1.192	1.189	1.190	1.189
$\overline{\widehat{\beta}}$	1.989	1.991	1.996	1.982	1.985	1.987	1.982	1.990	1.992

Table 2 shows that the estimation method provides good estimates of the pre- and post-shift regression coefficients and the distribution parameters α and β without the need of using a sample with very large number of individuals. For all specified values of T, the estimators perform well for $n = 150$, 300 and 1000. Similarly, for all specified values of n, the estimators also perform well for $T = 50$, 100 and 150.

4 Concluding Remarks

This paper contributes to the existing literature by developing a model that combines the features of an aggregation model and a structural-change model. The model is constructed by aggregating the individual abrupt-change models, with the transmission lag for each individual drawn from a distribution with an unknown mean. A salient feature of the model is that it allows for individual heterogeneity in the diffusion lag of economic crisis. The model can be estimated via the nonlinear LS method. Simulation results suggest that our estimators perform quite well in finite samples.

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