E C O N O M I C S B U L L E T I N

Estimation of Zellner-Revankar Production Function Revisited

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Abstract

Zellner and Revankar in their paper “Generalized Production Functions” introduced a production function, which was illustrated by fitting the generalized Cobb-Douglas function to the U.S. data for Transportation Equipment Industry. For estimating the production function, they used a method in which one of the parameters (theta) is repeatedly chosen at the trial basis and other parameters are estimated so as to obtain the global optimum of the likelihood function. We show that this method of Zellner and Revankar (ZR) is caught into a local optimum trap and the estimated parameters reported by ZR are somewhat sub-optimal. Using the Differential Evolution (DE) and the Repulsive Particle Swarm (RPS) methods, we re-estimate the parameters of the ZR production function with data used by ZR and show that our estimates of parameters are better than those of ZR. We also find that the returns to scale do not vary with the size of output in the manner reported by ZR.

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Estimation of Zellner-Revankar production function revisited

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1. Introduction: Arnold Zellner and Nagesh Revankar in their well-known paper "Generalized Production Functions" [Zellner and Revankar, 1969] introduced a new production function, which was illustrated by an example specified as:

$$V\exp(\theta V) = \gamma K^{\alpha(1-\delta)} L^{\alpha\delta} \;:\; 0 < \delta < 1;\; \gamma > 0;\; \alpha > 0. \qquad \dots (1)$$

where, V, K, L stand for output, capital and labour. The parameters α, δ, $(1-\delta)$ and γ relate to the parameters of returns to scale, output elasticities with respect to labour and capital and efficiency. The parameter θ attribute to other parameters the scale variability character and thus makes the function specified above "general". In particular, for $\theta = 0$ the Zellner-Revankar production function (ZRPF) degenerates into the simple Cobb-Douglas production function. The returns to scale function obtained from the ZRPF is given as $\alpha(V) = \alpha/(1+\theta V)$ that changes with the volume of output.

2. Estimation of ZRPF: Now we present the Zellner-Revankar method of estimation of the ZRPF parameters. Let us have sample data on output, capital and labour in n observations. Introducing multiplicative random error and log-transforming we have

$$\log(V_i) + \theta V_i = \log(\gamma) + \alpha(1-\delta)\log(K_i) + \delta\log(L_i) + u_i \;:\; i = 1,2,...,\; n \qquad \dots (2)$$

where u_i's are random errors, normally and independently distributed, each with mean zero and common variance σ^2. It is also assumed that $\log(K_i)$ and $\log(L_i)$ are distributed independently of the error term, u_i, or they are fixed quantities. Then, the logarithm of the likelihood function, $\log(l)$, is:

$$\log(l) = const. - \tfrac{n}{2}\log(\sigma^2) + \log(J) - \tfrac{1}{2\sigma^2}\sum_{i=1}^{n}\{z_i(\theta) - c_0 - c_1\log(K_i) - c_2\log(L_i)\}^2 \qquad \dots (3)$$

where $z_i(\theta) = \log(V_i) + \theta V_i;\; c_0 = \log(\gamma);\; c_1 = \alpha(1-\delta),\; c_2 = \alpha\delta$ and J is the Jacobian of the transformation from u_i's to the V_i's, or

$$J = \prod_{i=1}^{n}\frac{\partial u_i}{\partial V_i} = \prod_{i=1}^{n}\left[\frac{1+\theta V_i}{V_i}\right] \qquad \dots (4)$$

Now, substituting from (4) in (3) we get

$$\log(l) = const. - \tfrac{n}{2}\log(\sigma^2) + \sum_{i=1}^{n}\log(1+\theta V_i) - \tfrac{1}{2\sigma^2}\sum_{i=1}^{n}\{z_i(\theta) - c_0 - c_1\log(K_i) - c_2\log(L_i)\}^2 \qquad \dots (5)$$

Differentiating (5) partially with respect to σ^2 and setting the derivatives equal to zero we obtain

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}\{z_i(\theta) - c_0 - c_1\log(K_i) - c_2\log(L_i)\}^2 \qquad \dots (6)$$

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as the conditional maximizing value for σ^2. When $\hat{\sigma}^2$ in (6) is substituted for σ^2 in (5), we obtain

$$\log(l^*) = const. - \frac{n}{2}\log\left[\sum_{i=1}^{n}\{z_i(\theta) - c_0 - c_1\log(K_i) - c_2\log(L_i)\}^2\right] + \sum_{i=1}^{n}\log(1+\theta V_i) \qquad \ldots (7)$$

Now, for any given value of $\theta = \theta_0$, the conditional maximizing values of c_0, c_1 and c_2 may be obtained by regression of $z_i(\theta_0)$ on the explanatory variables, $\log(K_i)$ and $\log(L_i)$ by minimizing

$$\sum_{i=1}^{n}\{z_i(\theta) - c_0 - c_1\log(K_i) - c_2\log(L_i)\}^2 \qquad \ldots (8)$$

Minimization of (8) can be done with different trial values of θ, say, $\theta_1,\ \theta_2,\ \theta_3,\ \ldots$ such that we find out the best values of $(\theta,\ c_0,\ c_1,\ c_2)$ that obtains the global optimum of the likelihood function in (7). Zellner and Revankar (Z&R) mention that this procedure of maximizing the likelihood function is similar to the procedure described by Box and Cox (1963). This procedure of estimation will be examined and revisited in this paper.

Z&R apply this procedure for estimating the optimal values of $\theta,\ c_0,\ c_1,\ c_2$, the parameters of the ZRPF, for the U.S. Transportation Equipment Industry. The data used by them have been presented in their paper (Z&R, 1969). They measure output (represented by the net value added, V), capital (K) and labour (L) per unit of establishment, that is, $V_i = (V_{a,i}/N_i);\ K_i = (K_{a,i}/N_i);\ L_i = (L_{a,i}/N_i)$. They obtain:

$$(\hat{\theta},\ \hat{c}_0,\ \hat{c}_1,\ \hat{c}_2) = (0.134,\ 3.0129,\ 0.3330,\ 1.1551) \qquad \ldots (9)$$

and, since the estimate of returns to scale parameter, $\hat{\alpha} = \hat{c}_1 + \hat{c}_2$, they also obtain for each state of the U.S. $Est.\ \alpha(V_i) = (\hat{c}_1 + \hat{c}_2)/(1+\hat{\theta}V_i) = 1.49/(1+0.134V_i)\ approx.$ According to their estimates, Indiana, Kentucky, Georgia, Ohio, Connecticut, Missouri, Kansas and Michigan exhibit decreasing returns ($\hat{\alpha}(V)$ decreasing in that order); Illinois, Pennsylvania, New Jersey, Maryland and Washington show $1 < \hat{\alpha} \le 1.1$ while other states have $\hat{\alpha} > 1.1$. Florida has the highest value of $\hat{\alpha} = 1.45$ (see Z&R, 1969).

3. The Objective of this Paper: We intend to demonstrate here that the estimates of parameters of ZRPF as reported by Z&R in their paper are somewhat sub-optimal, that is: $(\hat{\theta},\ \hat{c}_0,\ \hat{c}_1,\ \hat{c}_2) = (0.134,\ 3.0129,\ 0.3330,\ 1.1551)$ do not quite maximize the likelihood function. However, that is so due to the trial and error method used by Z&R in which a trial value of θ is chosen, and c_i's are estimated by minimization of (8). This is done repeatedly for different trial values of θ so as to maximize the likelihood function.

To show that Z&R estimates are sub-optimal, we use two methods of global optimization to minimize (8) in which $\theta,\ c_0,\ c_1,\ c_2$ are estimated together. This approach frees us from the risk of obtaining a sub-optimal set of estimated parameters of ZRPF.

4. Global Optimization: Most of the conventional methods of optimization that work very well in optimizing convex functions often perform poorly when the problem has multiple or ill-conditioned minima/maxima. They are often caught or trapped in the local minima/maxima. Versatile search methods such as those of Nelder and Mead (1964) and Box (1965) succumb to the traps of local optima.

Since the work of Holland (1975) several methods have been developed to escape from being caught in such local optima. A brief history of development of the methods of global optimization is available in Mishra (2006-a). Among these methods, the Genetic Algorithms (GA), the Simulated Annealing (SA) and the Generalized Simulated Annealing (GSA) procedures, the Particle Swarm (PS) and the Repulsive Particle Swarm (RPS) methods, and the Differential Evolution (DE) method have found numerous applications in various disciplines. A general-purpose Genetic Algorithm based optimization subroutine (PIKAIA) in FORTRAN-77 is freely downloadable from the High Altitude Observatory site (http://www.hao.ucar.edu/Public/models/pikaia/pikaia.html). The program is particularly useful (and robust) in treating multi-modal optimization problems. SIMANN, a global optimization algorithm using simulated annealing (Kirkpatrick et al., 1983) written in FORTRAN-77 by William Goffe et al. is very effective. It may be downloaded from http://www.netlib.no/netlib/opt/simann.f absolutely free of cost. Mundim (1996) provides a Fortran program for Generalized Simulated Annealing (Tsallis and Stariolo, 1995) on www.unb.br/iq/kleber/GSA/artigo2/node2.html. A Fortran program for Repulsive Particle Swarm Optimization (written by the present author) is available on http://www1.webng.com/economics/rps.txt. This program also lists the Fortran codes of (over) 90 benchmark functions of different dimensions, complexities and difficulty levels, and the RPS method that minimizes them. A Fortran program of the Differential Evolution method (written by the present author) is available on http://www1.webng.com/economics. It uses the most recent advances in the crossover scheme as recently suggested by Kenneth Price.

It may be noted, however, that the methods of global optimization are probabilistic in nature. Therefore, one cannot take their results for sure or those methods infallible. Secondly, all of them adapt themselves to the surface on which they find the global optimum. The scheme of adaptation is largely based on some guesswork since nobody knows as to the true nature of the problem (environment or surface) and the most suitable scheme of adaptation to fit the given environment. Surfaces may be varied and different for different functions. A particular type of surface may be suited to a particular method while a search in another type of surface may be a difficult proposition for it. Further, each of these methods operates with a number of parameters that may be changed at choice to make it more effective. This choice is often problem oriented and that for obvious reasons. A particular choice may be extremely effective in a few cases, but it might be ineffective (or counterproductive) in certain other cases. Additionally, there is a relation of trade-off among those parameters. These features make all these methods a subject of trial and error exercises. Nevertheless, RPS and DE find optima more frequently and accurately than the other methods of global optimization. They also have the least number of parameters to adjust.

5. Some Details on the Particle Swarm and the Differential Evolution Methods: In this study we have used two methods of global optimization: the RPS and the DE. Our choice is based on their efficiency in searching the optima of complicated functions.

The Particle Swarm (PS) method of global optimization (Eberhart and Kennedy, 1995) is an instance of a successful application of the philosophy of *bounded rationality* and decentralized decision-making to solve the global optimization problems (Simon, 1982; Bauer, 2002; Fleischer, 2005). It is observed that a swarm of birds or insects or a school of fish searches for food, protection, etc. in a very typical manner. If one of the members of the swarm sees a desirable path to go, the rest of the swarm will follow quickly. Every member of the swarm searches for the best in its locality - learns from its own experience. Additionally, each member learns from the others, typically from the best performer among them. Even human beings show a tendency to learn from their own experience, their immediate neighbours and the ideal performers. The Particle Swarm method of global optimization mimics the said behaviour (see Wikipedia: http://en.**wikipedia**.org/wiki/Particle_**swarm**_optimization). The Repulsive Particle Swarm method (see Wikipedia, http://en.wikipedia.org/wiki/RPSO) is a variant of the PS. It is particularly effective in finding out the global optimum in very complex search spaces (although it may be slower on certain types of optimization problems).

The DE method was developed by Kenneth V. Price and Rainer Storn (1995). The crucial idea behind the DE is a scheme for generating trial parameter vectors. Initially, a population of points (p in d-dimensional space) is generated and evaluated (i.e. f(p) is obtained) for their fitness. Then for each point (p_i) three different points (p_a, p_b and p_c) are randomly chosen from the population. A new point (p_z) is constructed from those three points by adding the weighted difference between two points ($w(p_b-p_c)$) to the third point (p_a). Then this new point (p_z) is subjected to a crossover with the current point (p_i) with a probability of crossover (c_r), yielding a candidate point, say p_u. This point, p_u, is evaluated and if found better than p_i then it replaces p_i else p_i remains. Thus we obtain a new vector in which all points are either better than or as good as the current points. This new vector is used for the next iteration. This process makes the differential evaluation scheme completely self-organizing (Price et al., 2005). The DE is perhaps the fastest and the most accurate method among all methods of global optimization.

6. Estimation of Zellner-Revankar Production Function by the Methods of Global Optimization: As mentioned before, in this paper we have estimated the parameters of ZRPF by two methods; the RPS and the DE. We have used our own program for estimation of the said function. The program (in FORTRAN 77) is downloadable from http://www.geocities.com/artha_indica/revankar.txt or http://ssrn.com/abstract=950731.

Table-A: Estimated Parameters of ZRPF for U.S. Transport Equipment Industry

Accuracy	Method	$\hat{c}_0$	$\hat{c}_1$	$\hat{c}_2$	$\hat{\theta}$	SSQD	(l^*)
Low Accuracy (LA)	**Zellner-Revankar**	**3.0129**	**0.3330**	**1.1551**	**0.134**	**1.2016#**	**5.4790**
	Differential Evoln	2.91527	0.352646	1.087540	0.106441	1.0689	5.5769
	R Particle Swarm	2.91476	0.350784	1.090654	0.106506	1.0691	5.5773
High Accuracy (HA)	**Zellner-Revankar**	**3.0129**	**0.3330**	**1.1551**	**0.134**	**1.2118#**	**5.4945**
	Differential Evoln	2.91161	0.350226	1.090161	0.106184	1.0665	5.5917
	R Particle Swarm	2.91587	0.350255	1.092447	0.106811	1.0692	5.5918

SSQD = Sum of Squared Deviations; # = Computed by us; l* = Log Max Likelihood

We present here (Table-A) two sets of estimates of the parameters of ZRPF: the one based on highly accurate values (correct up to 8 places after decimal) of V_i, K_i and L_i and the other when these variables are measured with values correct only up to two places after decimal (rounded off at the third place after decimal). We do not know of the accuracy level of the original computations (done by Z&R).

Table-B: 1957 U.S. Transportation Equipment Industry Value Added [Per Establishment] Zellner-Revankar Production Function Estimated by Different Methods

	V (Emp)	$V(DE)_{LA}$	$V(RPS)_{LA}$	$V(ZR)_{LA}$	$V(DE)_{HA}$	$V(RPS)_{HA}$	$V(ZR)_{HA}$
Florida	0.193	0.245	0.244	0.245	0.245	0.244	0.241
Maine	0.364	0.306	0.305	0.306	0.306	0.305	0.303
Iowa	0.477	0.478	0.477	0.478	0.477	0.476	0.477
Louisiana	0.638	0.601	0.601	0.601	0.600	0.600	0.608
Massachusetts	1.404	1.363	1.362	1.363	1.363	1.363	1.375
West Virginia	1.513	1.704	1.703	1.704	1.700	1.700	1.723
Texas	1.712	1.997	1.999	1.997	1.998	1.999	2.061
Alabama	1.855	2.378	2.384	2.378	2.381	2.384	2.502
New York	2.040	2.954	2.954	2.954	2.956	2.958	3.012
Virginia	2.052	2.088	2.090	2.088	2.093	2.095	2.140
California	2.333	2.128	2.127	2.128	2.127	2.127	2.124
Wisconsin	2.463	2.510	2.509	2.510	2.507	2.508	2.508
Illinois	2.629	2.354	2.350	2.354	2.354	2.354	2.309
Pennsylvania	2.651	2.762	2.763	2.762	2.765	2.766	2.777
New Jersey	2.701	2.087	2.086	2.087	2.084	2.084	2.063
Maryland	3.219	3.303	3.307	3.303	3.301	3.304	3.321
Washington	3.558	3.134	3.135	3.134	3.136	3.137	3.094
Indiana	3.816	4.979	4.975	4.979	4.972	4.975	4.840
Kentucky	4.031	2.281	2.280	2.281	2.281	2.280	2.188
Georgia	4.289	3.793	3.798	3.793	3.801	3.803	3.742
Ohio	4.440	5.964	5.963	5.964	5.957	5.961	5.783
Connecticut	4.485	5.616	5.622	5.616	5.614	5.619	5.535
Missouri	5.217	4.340	4.342	4.340	4.336	4.336	4.141
Kansas	6.507	5.238	5.254	5.238	5.259	5.261	5.067
Michigan	7.182	6.663	6.657	6.663	6.655	6.652	6.001

Abbreviations used: Emp = Empirical; LA = Low Accuracy; HA = High Accuracy; ZR = Estimates of Zellner-Revankar Method of Estimation; DE = Differential Evolution; RPS = Repulsive Particle Swarm

As it has been shown in Table-A, first, there is no significant difference in the values of estimated parameters (of ZRPF) due to accuracy in computation. HA and LA estimates are more or less same. Secondly, there is no significant difference between the estimated parameters obtained by DE and RPS. However, the Zellner-Revankar estimates of parameters are quite different from those obtained by the methods of global optimization (DE and RPS). The SSQD (sum of squared deviations) of ZR is larger (and l^* is smaller) than those of DE and RPS. It shows very clearly that the ZR estimates are somewhat sub-optimal. This sub-optimality of ZR estimates may thus be clearly appreciated by the results presented in Table-A and Table-B.

Two points deserve a special mention. First, the returns-to-scale parameter, $\hat{\alpha} = \hat{c}_1 + \hat{c}_2$ obtained by DE/RPS method is 1.44 approx, against 1.488 obtained by the ZR estimation. Further, the value of $\hat{\theta}$ obtained by DE/RPS is about 0.106, while it is 0.134 obtained by ZR. A consequence of all these changes is that $\alpha(V_i)$ values for different states are different from those obtained by ZR method. The estimates of $\alpha(V_i)$ are presented in Table-C.

Table-C. Estimated Variable Returns to Scale by Zellner Revankar Method of Estimation							
State	V	Est $\alpha(V)$		State	V	Est . $\alpha(V)$	
		ZR*	DE/RPS			ZR*	DE/RPS
Florida	0.193	1.45	1.41	Pennsylvania	2.651	1.10	1.12
Maine	0.364	1.42	1.39	New Jersey	2.701	1.09	1.12
Iowa	0.477	1.40	1.37	Maryland	3.219	1.04	1.07
Louisiana	0.638	1.37	1.35	Washington	3.558	1.01	1.05
Massachusetts	1.404	1.25	1.25	Indiana	3.816	0.98	1.03
West Virginia	1.513	1.24	1.24	Kentucky	4.031	0.97	1.01
Texas	1.712	1.21	1.22	Georgia	4.289	0.94	0.99
Alabama	1.855	1.19	1.20	Ohio	4.44	0.93	0.98
New York	2.04	1.17	1.18	Connecticut	4.485	0.93	0.98
Virginia	2.052	1.17	1.18	Missouri	5.217	0.88	0.93
California	2.333	1.13	1.15	Kansas	6.507	0.80	0.85
Wisconsin	2.463	1.12	1.14	Michigan	7.182	0.76	0.82
Illinois	2.629	1.10	1.13	* Source: Zellner & Revankar (1969), p. 248			

7. Concluding Remarks: Z&R's paper made two contributions: first, it generalized the production function to allow for the parameters to vary according to the scale of output and secondly it contributed a method to estimate such parameters by the maximum likelihood method. This paper has only an appreciation for the first contribution, but it has shown that the method of estimation (suggested by ZR) is neither convenient nor accurate. It gives us only a local optimum, *not the global optimum*, of the likelihood function. This observation may not sound very impressive when a simple function like Cobb-Douglas's is generalized, but it may be very important if the basic function is intrinsically nonlinear. It is understandable that at the time when the ZR paper was written, there were no effective methods to find global optima of nonlinear functions, especially those with numerous local optima. Now that very effective methods of global optimization have been found, it would be appropriate to estimate the parameters of ZRPF by those advance methods. Our present paper has made a modest attempt to that effect. Using such global optimization methods, we have estimated other nonlinear production functions [Sato's two-level CES and LINEX functions; Mishra, 2006(b)] as well. We have found that the performance of these methods is much better than that of the classical methods of estimation of nonlinear functions [Mishra, 2006(a)].

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