E C O N O M I C S   B U L L E T I N

A Dynamic Approach to a Consistent Value under Plurality-Efficiency

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Abstract

Inspired by Maschler and Owen (1989), we provide a dynamic approache to the multi-choice Shapley value proposed by Hsiao and Raghavan (1992,1993) under plurality-efficiency.

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1 Introduction

A multi-choice TU game, introduced by Hsiao and Raghavan (1992,1993), is a generalization of a TU game in which each player has several activity levels. There are several branches of solutions for these games that are extensions of the Shapley value in literature. In this paper, we apply the multi-choice Shapley value proposed by Hsiao and Raghavan (1992,1993), which we name the H&R Shapley value.

Maschler and Owen (1989) characterized a consistent Shapley value for hyperplane games. Subsequently, they also provided a dynamic approach that lead the players to the consistent solution, starting from an arbitrary Pareto optimal payoff vector.

Based on the result of Maschler and Owen (1989), we offer an analogue result for the H&R Shapley value. Specifically, for the H&R Shapley value, we define an x-dependent reduced games. Furthermore, we show how the H&R Shapley value can be reached dynamically from an payoff vector which is plurality-efficient in some games. The dynamic process should make some use of the consistency.

2 Definitions and Notation

Let U be the universe of players and $N \subseteq U$ be a set of players. Let $m = (m_i)_{i\in N}$ be the vector that describes the number of activity levels for each player, in which he can actively participate. For $i \in U$, we set $M_i = \{0, 1, \cdots, m_i\}$ as the action space of player i, where the action 0 means not participating, and $M_i^+ = M_i \setminus \{0\}$. For $N \subseteq U$, $N \neq \emptyset$, let $M^N = \prod_{i\in N} M_i$ be the product set of the action spaces for players in N. Denote the zero vector in $\mathbb{R}^N$ by 0_N.

A **multi-choice TU game** is a triple (N, m, v), where N is a non-empty and finite set of players, m is the vector that describes the number of activity levels for each player, and $v : M^N \to \mathbb{R}$ is a characteristic function which assigns to each action vector $x = (x_i)_{i\in N} \in M^N$ the worth that the players can obtain when each player i plays at activity level $x_i \in M_i$ with $v(0_N) = 0$. If no confusion can arise a game (N, m, v) will sometimes be denoted by its characteristic function v. Given (N, m, v) and $x \in M^N$, we write (N, x, v) for the multi-choice TU subgame obtained by restricting v to $\{y \in M^N \mid y_i \leq x_i \ \forall i \in N\}$ only. Denote the class of all multi-choice TU games by MC.

Given $(N, m, v) \in MC$, let $L^{N,m} = \{(i, j) \mid i \in N, j \in M_i^+\}$. A **solution** on MC is a map ψ assigning to each $(N, m, v) \in MC$ an element

$$\psi(N,m,v) = \big(\psi_{i,j}(N,m,v)\big)_{(i,j)\in L^{N,m}} \in \mathbb{R}^{L^{N,m}}.$$

Here $\psi_{i,j}(N,m,v)$ is the power index or the value of the player i when he takes action j to play game v. For convenience, given $(N,m,v) \in MC$ and a solution ψ on MC, we define $\psi_{i,0}(N,m,v) = 0$ for each $i \in N$.

Given $S \subseteq N$, let $|S|$ be the number of elements in S, and let $e^S(N)$ be the binary vector in $\mathbb{R}^N$ whose component $e^S_i(N)$ satisfies

$$e^S_i(N) = \begin{cases} 1 & \text{if } i \in S\ , \\ 0 & \text{otherwise}\ . \end{cases}$$

Note that if no confusion can arise $e^S(N)$ will be denoted by e^S. Given $(N,m,v) \in MC$ and $x \in M^N$, we define $S(x) = \{k \in N \mid x_k \neq 0\}$.

Let $x, y \in \mathbb{R}^N$, we say $y \leq x$ if $y_i \leq x_i$ for all $i \in N$. The analogue of unanimity games for multi-choice games are **minimal effort games** (N,m,u^x_N), where $x \in M^N$, $x \neq 0_N$, defined by for all $y \in M^N$,

$$u^x_N(y) = \begin{cases} 1 & \text{if } y \geq x\ ; \\ 0 & \text{otherwise} \end{cases}$$

Hsiao and Raghavan (1992,1993) showed that for $(N,m,v) \in MC$ it holds that $v = \sum\limits_{\substack{x\in M^N \\ x\neq 0_N}} a^x(v)\ u^x_N$, where $a^x(v) = \sum_{S\subseteq S(x)} (-1)^{|S|}\ v(x - e^S)$.

Definition 1 *Hsiao and Raghavan (1992,1993) proposed a multi-choice Shapley value, the* **H&R Shapley value**. *We denote the symmetric form of the H&R Shapley value by γ. Formally, the H&R Shapley value γ is the solution on MC which associates with each $(N,m,v) \in MC$, each $i \in N$ and each $j \in M^+_i$ the value* *

$$\gamma_{i,j}(N,m,v) = \sum_{\substack{x\in M^N \\ x_i \leq j}} \frac{a^x(v)}{|S(x)|}.$$

3 The Axioms

For $x \in \mathbb{R}^N$, we write x_S to be the restriction of x at S for each $S \subseteq N$. Let $N \subseteq U$, $i \in N$ and $x \in \mathbb{R}^N$, we introduce the substitution notation x_{-i} to stand for $x_{N\setminus\{i\}}$ and let $y = (x_{-i}, j) \in \mathbb{R}^N$ be defined by $y_{-i} = x_{-i}$

*We define the H&R Shapley value in terms of the dividends. Hsiao and Raghavan (1993) provided an alternative formula of the H&R Shapley value.

and $y_i = j$. In this paper, we will make use of the following axioms: Let ψ be a solution on MC.

• **Efficiency (EFF)**: For all $(N,m,v) \in MC$, $\sum\limits_{i\in N} \psi_{i,m_i}(N,m,v) = v(m)$.

The following axioms are analogues of the balanced contributions property due to Myerson (1980).

• **Strong balanced contributions (SBC)**: For each $(N,m,v) \in MC$ and for all $(i,k_i),(j,k_j) \in L^{N,m}, i \neq j$,

$$\begin{aligned} &\psi_{i,k_i}\big(N,(m_{-j},k_j),v\big) - \psi_{i,k_i}\big(N,(m_{-j},0),v\big) \\ =\ &\psi_{j,k_j}\big(N,(m_{-i},k_i),v\big) - \psi_{j,k_j}\big(N,(m_{-i},0),v\big), \end{aligned}$$

Upper Balanced Contributions (UBC) only requires that SBC holds if $k_i = m_i$ and $k_j = m_j$.

• **Independence of individual expansions (IIE)**[†]: For each $(N,m,v) \in MC$ and for each $(i,j) \in L^{N,m}$, $j \neq m_i$,

$$\psi_{i,j}\big(N,(m_{-i},j),v\big) = \psi_{i,j}\big(N,(m_{-i},j+1),v\big) = \cdots = \psi_{i,j}(N,m,v).$$

Weak independence of individual expansions (WIIE) only requires that IIE holds if $|S(m)| = 1$.

Theorem 1 (Hwang and Liao (2006))

1. *A solution ψ on MC satisfies EFF, IIE, and UBC if and only if $\psi = \gamma$.*
2. *A solution ψ on MC satisfies EFF, WIIE, and SBC if and only if $\psi = \gamma$.*

Interpretation : Given $(N,m,v) \in MC$. In the usual TU setting the basic assumption is that the grand coalition N forms, and consequently that $v(N)$ is the amount that has to be divided. Therefore a payoff vector is a vector $(x_i)_{i\in N}$ where x_i is the amount paid to player i, and *efficiency* dictates that $\sum_{i\in N} x_i = v(N)$, all the gains (or maybe losses) are to be divided among the participants. In the framework of multi-choice games, the basic assumption is still that the grand coalition N forms, and consequently there are many cooperative situations of N. This means that

[†]This axiom was introduced by Hwang and Liao (2006).

for each $z \in M^N$ with $z_i \neq 0$ for all $i \in N$, it is possible that $v(z)$ is the amount that has to be divided. Therefore a payoff vector is a "configuration" $(x_{i,j})_{(i,j)\in L^N}$ where $x_{i,j}$ is the amount paid to player i corresponding to his activity level j when player i participates at his activity level j. In order to reach the maximal benefit of "individuality", each individuality hopes that all other players are supposed to participate at their maximum level of effort when he participates at his activity level j in a game. Hence we will only concern that "all other players are supposed to participate at their maximum level of effort". A particular situation is that "all players participate at their maximum level of effort", which is also most interesting situation. The definition of the corresponding concept of efficiency are as the definition as follows.

Definition 2 *Given $(N, m, v) \in MC$ and $x \in \mathbb{R}^{L^{N,m}}$. x satisfies* **Plurality-efficiency (PEFF)** *in (N, m, v) if for all $(i, j) \in L^{N,m}$,*

$$x_{i,j} + \sum_{k\in N\setminus\{i\}} x_{k,m_k} = v(m_{-i}, j).$$

Furthermore, a solution ψ on MC satisfies PEFF in (N, m, v) if

$$\psi_{i,j}(N, m, v) + \sum_{k\in N\setminus\{i\}} \psi_{k,m_k}(N, m, v) = v(m_{-i}, j).$$

Note that if there exists (N, m, v) such that a solution satisfies PEFF in (N, m, v), then it satisfies EFF in (N, m, v).

For $S \subseteq N$, we denote $S^c = N \setminus S$ and 0_S the zero vector in $\mathbb{R}^S$. Given a solution ψ, $(N, m, v) \in MC$, and $\emptyset \neq S \subseteq N$, the **reduced game** $(S, m_S, v^{\psi}_{S,m})$ with respect to ψ, S and m is defined by for all $x \in M^S$,

$$v^{\psi}_{S,m}(x) = v(x, m_{S^c}) - \sum_{i\in S^c} \psi_{i,m_i}\big(N, (x, m_{S^c}), v\big)$$

• **Consistency (CON)**: For all $(N, m, v) \in MC$, for all $\emptyset \neq S \subseteq N$, for all $i \in S$ and for all $j \in M_i^+$,

$$\psi_{i,j}(S, m_S, v^{\psi}_{S,m}) = \psi_{i,j}(N, m, v).$$

Theorem 2 (Hwang and Liao (2006)) *The solutions γ satisfy CON.*

4 Dynamic Approaches

In this section, we will find a dynamic process that lead the players to solutions, starting from an arbitrary payoff vector which satisfies plurality-efficiency in a game. The dynamic process should make some use of the consistency. In order to exhibit such process, let us define an (x, γ)-dependent reduced game:

Definition 3 *Given $(N, m, v) \in MC$, $S \subseteq N$ and $x \in \mathbb{R}^{L^{N,m}}$ be a payoff vector which satisfies PEFF in (N, m, v). The* (x, γ)**-reduced game** $(S, m_S, v^x_{\gamma,S})$ *is given by for all $y \in M^S$,*

$$v^x_{\gamma,S}(y) = \begin{cases} v(m_{-i}, j) - \sum_{i \in S^c} x_{i,m_i} & y = \big((m_S)_{-i}, j\big) \text{ for all } (i,j) \in L^{S,m_S}, \\ v^\gamma_{S,m}(y) & \text{otherwise .} \end{cases}$$

Given $(N, m, v) \in MC$ with $N \geq 3$, $(i, j) \in L^{N,m}$ and solution γ, define $f_{i,j} : M^N \to \mathbb{R}$ to be

$$f_{i,j}(x) = x_{i,j} + \alpha \sum_{k \in N \setminus \{i\}} \Big(\gamma_{i,j}\big(\{i,k\}, m_{\{i,k\}}, v^x_{\gamma,\{i,k\}}\big) - x_{i,j}\Big), \qquad (1)$$

where α is a fixed postive number, which reflects the assumption that player i does not ask for full correction (when $\alpha = 1$) but only (usually) a fraction of it. By the PEFF of x and the definitions of (x, γ)-reduced game and $f_{i,j}$, it is clearly to see that $\big(f_{i,j}\big)_{(i,j) \in L^{N,m}}$ also satisfies PEFF in (N, m, v). Define $f = (f_{i,j})_{(i,j) \in L^{N,m}}$ and $x^0 = x$, $x^1 = f(x^0)$, $x^2 = f(x^1), \cdots,\ x^q = f(x^{q-1})$ for all $q \in \mathbb{N}$.

Theorem 3 *Given $(N, m, v) \in MC$ such that γ satisfies PEFF in (N, m, v). If $0 < \alpha < \frac{4}{|N|}$, then for each vector $x \in \mathbb{R}^{L^{N,m}}$ which satisfies PEFF in (N, m, v), $\{x^q\}_{q=1}^\infty$ converges geometrically to $\big(\gamma_{i,j}(N, m, v)\big)_{(i,j) \in L^{N,m}}$.*

Proof. Given $(N, m, v) \in MC$ such that γ satisfies PEFF in (N, m, v). And given $i, k \in S(m)$ and a vector $x \in \mathbb{R}^{L^{N,m}}$ which satisfies PEFF in (N, m, v). Let $S = \{i, k\}$, by PEFF, UBC and IIE of γ, and definitions of $v^\gamma_{S,m}$ and $v^x_{\gamma,S}$, we have that for all $j \in M_i^+$,

$$\gamma_{i,j}(S, m_S, v^x_{\gamma,S}) + \gamma_{k,m_k}(S, m_S, v^x_{\gamma,S}) = x_{i,j} + x_{k,m_k}.$$

and

$$\begin{aligned} & \gamma_{i,j}(S, m_S, v^x_{\gamma,S}) - \gamma_{k,m_k}(S, m_S, v^x_{\gamma,S}) \\ = \ & \gamma_{i,j}(S, ((m_S)_{-i}, j), v^x_{\gamma,S}) - \gamma_{k,m_k}(S, m_S, v^x_{\gamma,S}) \ \text{ (by IIE of } \gamma) \\ = \ & \gamma_{i,j}(S, (((m_S)_{-i}, j)_{-k}, 0), v^x_{\gamma,S}) - \gamma_{k,m_k}(S, ((m_S)_{-i}, 0), v^x_{\gamma,S}) \ \text{ (by UBC of } \gamma) \\ = \ & \gamma_{i,j}(S, (((m_S)_{-i}, j)_{-k}, 0), v^\gamma_{S,m}) - \gamma_{k,m_k}(S, ((m_S)_{-i}, 0), v^\gamma_{S,m}) \ \text{ (by definition of } v^x_{\gamma,S}) \\ = \ & \gamma_{i,j}(S, ((m_S)_{-i}, j), v^\gamma_{S,m}) - \gamma_{k,m_k}(S, m_S, v^\gamma_{S,m}) \ \text{ (by UBC of } \gamma) \\ = \ & \gamma_{i,j}(S, m_S, v^\gamma_{S,m}) - \gamma_{k,m_k}(S, m_S, v^\gamma_{S,m}) \ \text{ (by IIE of } \gamma). \end{aligned}$$

Therefore,

$$\begin{array}{rl} & 2 \cdot \Big[\gamma_{i,j}(S, m_S, v^x_{\gamma,S}) - x_{i,j}\Big] \\ = & \gamma_{i,j}(S, m_S, v^\gamma_{S,m}) - \gamma_{k,m_k}(S, m_S, v^\gamma_{S,m}) - x_{i,j} + x_{k,m_k}. \end{array} \tag{2}$$

By (1), (2) and CON of γ,

$$\begin{array}{rcl} f_{i,j}(x) & = & x_{i,j} + \frac{\alpha}{2} \cdot \Big[\sum_{k\in N\setminus\{i\}} \gamma_{i,j}(S, m_S, v^\gamma_{S,m}) - \sum_{k\in N\setminus\{i\}} x_{i,j} \\ & & \quad - \sum_{k\in N\setminus\{i\}} \gamma_{k,m_k}(S, m_S, v^\gamma_{S,m}) + \sum_{k\in N\setminus\{i\}} x_{k,m_k} \Big] \\ & = & x_{i,j} + \frac{\alpha}{2} \cdot \Big[\big(|N| - 1\big)\gamma_{i,j}(S, m_S, v^\gamma_{S,m}) - \big(|N| - 1\big)x_{i,j} \\ & & \quad - \big(v(m) - \gamma_{i,j}(S, m_S, v^\gamma_{S,m})\big) + \big(v(m) - x_{i,j}\big) \Big] \\ & = & x_{i,j} + \frac{|N|\cdot\alpha}{2} \cdot \Big[\gamma_{i,j}(N, m, v) - x_{i,j}\Big]. \end{array}$$

Hence, for all $q \in \mathbb{N}$,

$$\begin{array}{rcl} \big(1 - \frac{|N|\cdot\alpha}{2}\big)\big[\gamma_{i,j}(N, m, v) - x^q_{i,j}\big] & = & \big[\gamma_{i,j}(N, m, v) - f_{i,j}(x^q)\big] \\ & = & \big[\gamma_{i,j}(N, m, v) - x^{q+1}\big]. \end{array}$$

If $0 < \alpha < \frac{4}{|N|}$, then $-1 < \Big(1 - \frac{|N|\cdot\alpha}{2}\Big) < 1$ and $\{x^q\}_{q=1}^\infty$ converges geometrically to $\big(\gamma_{i,j}(N, m, v)\big)_{(i,j)\in L^{N,m}}$. ■

Corollary 1 *If $0 < \alpha < \frac{4}{|N|}$, then for each $(N, m, v) \in MC$ and for each efficient vector $x \in \mathbb{R}^{L^{N,m}}$, $\{(x^q)_{i,m_i}\}_{q=1}^\infty$ converges geometrically to $\gamma_{i,m_i}(N, m, v)$ for all $i \in S(m)$.*

Proof. The proof of Corollary 1 is similar to Theorem 3, we omit it. ■

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