E C O N O M I C S B U L L E T I N

Internal Dividend, External Loss and Value

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Abstract

The equal allocation of nonseparable costs (EANSC) can be expressed as the sum of both "internal dividends" and "external losses" for a given transferable utility (TU) game.

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1 Introduction

Let N be a player set. Harsanyi (1963) argued that when a coalition S in N forms, then the two complementary coalitions S and $N \setminus S$ form and they make conflicting threats each other in order to get their maximal profits. Also Harsanyi introduced the notion of "internal dividend". He argued that when a coalition S forms, then every member of S is influenced by S. Hence, every member of S can receive an internal dividend from S. The final payoff of a given player i will be the sum of the internal dividends to i from all coalitions of which he is a member. The Shapley value (1953) exactly refers to such a formula.

In this note, we consider a more complicated situation. Namely, when a coalition S forms, then every member of S is influenced not only by S but also by its complement $N \setminus S$. So, when coalition S forms, every member of S can receive an internal dividend from S. Also he can receive a specific amount from $N \setminus S$, which we name "external loss". The final payoff of a given player i will be the sum of both the internal dividends to i from all coalitions of which he is a member and the external losses to i from all coalitions of which he is not a member. Interesting, the equal allocation of nonseparable costs (EANSC) refers to such a formula. *

2 Preliminaries

Let U be the universe of players. A coalition is a non-empty finite subset of U. Let N be a coalition and let $I\!R$ be the set of real numbers, the cardinality of N is denoted by $|N|$.

A *transferable utility* (TU) game is a pair (N, v), where N is a coalition and $v : 2^N \to I\!R$ is a *characteristic function* satisfying $v(\emptyset) = 0$. Let G denote the set of all TU games. We call S a subcoalition if S is a subset of N. (S, v) denotes a *subgame* of (N, v) obtained by restricting v to subsets of S only.

Recall some facts for the *Shapley value*. The Shapley value ϕ is the function on G that assigns to each TU game (N, v) a vector $\phi(N, v)$ in $I\!R^N$ given by

$$\phi_i(N, v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(|S \setminus \{i\}|!)(|N \setminus S|!)}{|N|!}(v(S) - v(S \setminus \{i\})).$$

*The EANSC emerged originally in the cost-sharing literature. Later, Straffin and Heaney (1981), and Moulin (1985) investigated it on the class of TU games. Specially, Moulin (1985) introduced a reduced game in the context of *quasi-linear cost allocation problems* to characterize EANSC.

Definition 1 *A dividend function* on G is a function d assigns to each TU game $(N, v) \in G$ with a configuration $(d_T(N, v))_{T \subseteq N}$ satisfying the following condition:

$$v(S) = \sum_{T \subseteq S} |T| d_T(N, v) \quad \text{(Efficiency)}, \tag{1}$$

for all $S \subseteq N$, where $d_\emptyset(N, v) = 0$.

It is well-known that for $T \subseteq N$, $d_T(N, v)$ can be interpreted to be "internal dividend" allocated by coalition T to its members, and $d_T(S, v) = d_T(N, v)$ for $T \subseteq S \subseteq N$. For convenience, we use notation $d_T = d_T(S, v) = d_T(N, v)$ for $T \subseteq S \subseteq N$.[†] A remarkable result for the Shapley value is as follows.

Theorem 1 *There exists a unique dividend function on G. Moreover, the Shapley value can be expressed as the sum of internal dividends for a given TU game. That is, let $(N, v) \in G$ and $i \in N$, the Shapley value*

$$\phi_i(N, v) = \sum_{\substack{T \subseteq N \\ i \in T}} d_T.$$

3 Main Result

In this section, we show that EANSC can be expressed as the sum of both "internal dividends" and "external losses" for a given TU game. First, we introduce the definition of EANSC and a *dividend-loss function.* It is known that the EANSC φ of TU games can be given the following simple game theoretic formulation:

$$\begin{aligned} \varphi_i(N, v) &= v(N) - v(N \setminus \{i\}) + \tfrac{1}{|N|}[v(N) - \textstyle\sum_{k \in N} v(N) - v(N \setminus \{k\})] \\ &= \tfrac{1}{|N|}\{v(N) - (|N| - 1)v(N \setminus \{i\}) + \textstyle\sum_{k \in N \setminus \{i\}} v(N \setminus \{k\})\}. \end{aligned}$$

[†]Let (N, u_T^N) be the *unanimity* game given by, for each $T \subseteq N$,

$$u_T^N(S) = \begin{cases} 1 & , \text{ if } T \subseteq S \\ 0 & , \text{ otherwise.} \end{cases}$$

It is well-known that each TU game (N, v) can be expressed as a linear combination of unanimity games and this decomposition exists uniquely. That is, $v = \sum_{T \subseteq N} c_T(N, v) u_T^N = \sum_{T \subseteq N} |T| d_T u_T^N$.

Definition 2 *A dividend-loss function* on G is a function D assigns to each TU game $(N, v) \in G$ with a pair configuration $(D_T^+(N, v), D_T^-(N, v))_{T \subseteq N}$ satisfying the following two conditions:

$$v(S) = \sum_{T \subseteq S} D_T^+(N, v) + D_T^-(N, v) \quad \text{(Efficiency)}, \tag{2}$$

for all $S \subseteq N$*, and*

$$D_T^+(N, v) + D_T^-(N, v) = |T| \ D_T^+(N, v) - |N \setminus T| \ D_T^-(N, v) \ \text{(Balancedness)}, \tag{3}$$

for all $T \subseteq N$*, where* $D_\emptyset^+(N, v) = D_\emptyset^-(N, v) = 0$.

The condition (2) can be referred to the efficiency property. As to condition (3), we image that when a coalition T forms, then

1. T possesses both the internal dividend "$D_T^+(N, v)$" and the external loss "$D_T^-(N, v)$", hence, the sum is "$D_T^+(N, v) + D_T^-(N, v)$"
2. T allocates "$D_T^+(N, v)$" to its every member as internal dividend
3. T allocates "$-D_T^-(N, v)$" to every member in $N \setminus T$ as external loss.

These mean that "$|T| \ D_T^+(N, v)$" is the sum of internal dividends and "$-|N \setminus T| \ D_T^-(N, v)$" is the sum of external losses. Hence, the left part of the equality in condition (3) can be interpreted as the amount of "supply" when a coalition T forms; and the right part is referred to the amount of "demand" when a coalition T forms. The condition (3) means that supply is equal to demand. Note that $D_T^+(S, v) \neq D_T^+(N, v)$ and $D_T^-(S, v) \neq D_T^-(N, v)$ for $T \subseteq S \subseteq N$ in general.

Theorem 2 *There exists a unique dividend-loss function on* G*. Moreover, the EANSC can be expressed as the sum of both internal dividends and external losses for a given TU game. That is, let* $(N, v) \in G$ *and* $i \in N$*, the EANSC*

$$\varphi_i(N, v) = \sum_{\substack{T \subseteq N \\ i \in T}} D_T^+(N, v) - \sum_{\substack{T \subseteq N \\ i \notin T}} D_T^-(N, v).$$

Proof: Let $(N, v) \in G$ and $T \subseteq N$. Put $D_T^+(N, v) = \frac{|N| - |T| + 1}{|N|} |T| d_T$ and $D_T^-(N, v) = \frac{|T| - 1}{|N|} |T| d_T$, then it is easy to verify that there exists a unique dividend-loss function on G, we omit it.

To verify the expression, let $T \subseteq N$, by substituting $v(T)$ with $\sum_{K \subseteq T} |K| d_K$ to the formulation of the EANSC of (N, v), we obtain that

$$\begin{aligned}\varphi_i(N,v) &= \tfrac{1}{|N|}\{v(N) - (|N|-1)v(N \setminus \{i\}) + \sum_{k \in N\setminus\{i\}} v(N \setminus \{k\})\} \\ &= \tfrac{1}{|N|}\{\sum_{T \subseteq N} |T| d_T - (|N|-1) \sum_{T \subseteq N\setminus\{i\}} |T| d_T + \sum_{k \in N\setminus\{i\}} \sum_{T \subseteq N\setminus\{k\}} |T| d_T\}.\end{aligned}$$

Repeat to calculate the above expression, we see that

$$\begin{aligned}\varphi_i(N,v) &= \tfrac{1}{|N|}\{\sum_{\substack{T \subseteq N \\ i \in T}} |T| d_T + \sum_{\substack{T \subseteq N \\ i \notin T}} |T| d_T - (|N|-1) \sum_{T \subseteq N\setminus\{i\}} |T| d_T + \sum_{\substack{T \subseteq N \\ i \in T}} (|N| - |T|)|T| d_T \\ &\quad + \sum_{\substack{T \subseteq N \\ i \notin T}} (|N| - |T| - 1)|T| d_T\} \\ &= \tfrac{1}{|N|}\{\sum_{\substack{T \subseteq N \\ i \in T}} (|N| - |T| + 1)|T| d_T + \sum_{\substack{T \subseteq N \\ i \notin T}} [1 - (|N|-1) + (|N| - |T| - 1)]|T| d_T\} \\ &= \tfrac{1}{|N|}\{\sum_{\substack{T \subseteq N \\ i \in T}} [|N| + (1 - |T|)]|T| d_T + \sum_{\substack{T \subseteq N \\ i \notin T}} (1 - |T|)|T| d_T\} \\ &= \tfrac{1}{|N|}\{\sum_{T \subseteq N\setminus\{i\}} [(|N| - |T|)(|T|+1) d_{T \cup \{i\}} - (|T|-1)|T| d_T]\} \\ &= \sum_{\substack{T \subseteq N \\ i \in T}} D_T^{+}(N,v) - \sum_{\substack{T \subseteq N \\ i \notin T}} D_T^{-}(N,v). \qquad \textit{Q.E.D.}\end{aligned}$$

References

[1] Harsanyi, J. C. (1963) A simplified bargaining model for n-person cooperative game. *International Economic Review*, 4, pp.194-220.

[2] Moulin, H. (1985) The separability axiom and equal-sharing methods. *Journal of Economic Theory*, 36, pp.120-148.

[3] Shapley, L. S. (1953) Additive and non-additive set functions. Ph.D. Thesis, Princeton University.

[4] Straffin, P. D. and Heaney, J. P. (1981) Game theory and the Tennessee Valley Authority. *International Journal of Game Theory*, 10, pp.35-43.