E C O N O M I C S   B U L L E T I N

Indeterminacy and market instability

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Abstract

This note shows that indeterminacy arising from an economy exhibiting production with social constant returns to scale may be related to the instability of the consumption goods market equilibrium. Furthermore, trade does not contribute to indeterminacy; indeterminacy arises becasue each country's equilibrium path is already indeterminate before trade.

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1. Introduction

Seminal works by Benhabib and Farmer (1994, 1996) and Benhabib and Nishimura (1998) on equilibrium path indeterminacy have attracted considerable attention in macroeconomic dynamics analysis. Studies along this direction have strived to elucidate why and how indeterminacy can arise even for parameterization consistent with the data. However, existing works in the literature mainly employ the autarkic framework as the modelling workhorse. The first exception is the two-country Heckscher-Ohlin framework of Nishimura and Shimomura (2002). The authors show that the equilibrium path in the world market is possible when private and social factor intensity rankings are the reverse of the other and intertemporal elasticity of substitution in consumption is small enough. Consequently, the indeterminacy of the realized path implies that the long-run Heckscher-Ohlin prediction of trade is uncertain.

Our objective is to highlight a counter-intuitive result relating to market instability and indeterminacy by employing an autarkic variant of Nishimura and Shimomura (2002). We believe that this result may have potential implications on other works in indeterminacy especially those employing a similar production setup. In particular, we show that indeterminacy may entail a downward sloping supply schedule of consumption goods, which we view as evidence against the plausibility of indeterminacy in Nishimura and Shimomura. Moreover, it is possible for the consumption goods market to be unstable when the demand schedule cuts the downward sloping supply schedule from above.

The rest of the paper is as follows. Section 2 outlines the autarkic variant of the Nishimura and Shimomura model. Discussions about the stability properties of the steady state as well as our main result are presented in Section 3. Section 4 concludes.

2. The Model

The outline of the model is as follows. The economy is populated by an infinitely-lived pseudo-planner who solves the following dynamic problem:

$$\underset{\mathrm{C}}{\mathrm{Max}} \int_0^\infty \frac{C^{1-\eta}}{1-\eta} e^{-\rho t} dt \tag{1}$$

subject to

$$\hat{I} = L_I^{a_I} K_I^{b_I} \overline{L}_I^{\alpha_I} \overline{K}_I^{\beta_I} \quad \text{and} \quad \hat{C} = L_C^{a_C} K_C^{b_C} \overline{L}_C^{\alpha_C} \overline{K}_C^{\beta_C} \tag{2}$$

$$K = K_I + K_C \quad \text{and} \quad L = L_I + L_C \tag{3}$$

$$\dot{K} = \hat{I} + p\hat{C} - \delta K - pC \tag{4}$$

$$K_0 \quad \text{given}$$

where C and I denote the level of demand for consumption and investment, $\rho \in (0, \infty)$ is the subjective rate of time discount, $1/\eta \in (1, \infty)$ is the intertemporal elasticity of substitution in consumption, and K_0 is the initial capital stock. Production in each sector follows Cobb-Douglas technology, where the production frontier for each good is described by Equation (2) with consumption and investment output denoted by $\hat{C}$ and $\hat{I}$ respectively. Labor and capital are represented by $\overline{L_i}$ and $\overline{K_i}$, and in each sector, technology exhibits constant returns to scale from the social perspective, i.e. $a_i + \alpha_i +$

$b_i + \beta_i = 1$, but decreasing returns to scale from the private perspective, i.e. $a_i + b_i < 1$. Equation (3) describes the resource constraints for capital and labor and Equation (4) is capital accumulation process where $\delta \in (0,1)$ is the capital depreciation factor. Using the market clearing conditions, that is $C = \hat{C}$ and $I = \hat{I}$, the current value Hamiltonian is

$$\begin{aligned}\mathrm{H} = &\frac{C^{1-\eta}}{1-\eta} + \lambda(I - \delta K) + P_I(L_I^{a_I} K_I^{b_I} \overline{L}_I^{\alpha_I} \overline{K}_I^{\beta_I} - I) \\ &+ P_C(L_C^{a_C} K_C^{b_C} \overline{L}_C^{\alpha_C} \overline{K}_C^{\beta_C} - C) + \overline{r}(K - K_I - K_C) + \overline{w}(L - L_I - L_C)\end{aligned}$$

where P_I, P_C, $\overline{r}$ and $\overline{w}$ are the shadow price of investment, consumption, capital and labor respectively. Let p be the price of consumption in terms of investment and normalize $r = \overline{r}/P_I$ and $w = \overline{w}/P_I$. The necessary conditions are

$$C^{-\eta} = \lambda p \tag{5}$$

$$w = a_I L_I^{a_I - 1} K_I^{b_I} \overline{L}_I^{\alpha_I} \overline{K}_I^{\beta_I} = p a_C L_C^{a_C - 1} K_C^{b_C} \overline{L}_C^{\alpha_C} \overline{K}_C^{\beta_C} \tag{6}$$

$$r = b_I L_I^{a_I} K_I^{b_I - 1} \overline{L}_I^{\alpha_I} \overline{K}_I^{\beta_I} = p b_C L_C^{a_C} K_C^{b_C - 1} \overline{L}_C^{\alpha_C} \overline{K}_C^{\beta_C} \tag{7}$$

$$\dot{\lambda} = \lambda(\rho + \delta - r) \tag{8}$$

$$\lim_{t\to\infty} K(t)\lambda(t)e^{-\rho t} = 0 \tag{9}$$

Equation (5) states that the marginal utility and the marginal cost of consumption are equalized under optimality, while (6) and (7) state that factor returns are equalized across sectors due to frictionless mobility. Equation (8) is the familiar intertemporal arbitrage condition and (9) is the transversality condition. Denote $\theta_i = a_i + \alpha_i$ as the labor share in sector i from the social perspective.

Lemma 1. *From (6) and (7), we have*

$$\frac{pr'(p)}{r(p)} = \frac{\theta_I}{\theta_I - \theta_C} \tag{10}$$

$$\frac{pw'(p)}{w(p)} = \frac{1 - \theta_I}{\theta_C - \theta_I} \tag{11}$$

where $\theta_C - \theta_I$ measures the social factor intensity rankings.
Proof. See Nishimura and Shimomura (2002). □

Equations (10) and (11) can be rationalized as the Stolper-Samuelson theorem. When labor is socially more intensive in the consumption sector, one has $\theta_C - \theta_I > 0$. Therefore, as (10) and (11) indicate, an increase in the relative price of consumption will increase the returns to labor and decrease the returns to capital.

Lemma 2. *Goods market clearing implies*

$$I = \frac{b_C w(p) L - a_C r(p) K}{\Delta} \tag{12}$$

$$C = \frac{a_I r(p)K - b_I w(p)L}{p\Delta} \tag{13}$$

where $\Delta = a_I b_C - a_C b_I$.
Proof. See Nishimura and Shimomura (2002). □

Using (12), the capital accumulation process can be expressed as

$$\dot{K} = \frac{b_C w(p)L - a_C r(p)K}{\Delta} - \delta K \tag{14}$$

The transitional path of the economy is determined by (8) and (14). In the steady state, $\dot{\lambda} = 0$ implies $\rho + \delta - r(p) = 0$, from which the steady state relative price, p^e, is determined. Before taking linear approximations around the steady state, note that K and λ are one-to-one and onto functions in p, which can be determined by substituting consumption demand from (5) into (13) and totally differentiating the resulting expression:

$$\frac{dp}{dK} = -\frac{\Sigma a_I(\rho + \delta)(\lambda p)^{1/\eta}}{\Delta} \tag{15}$$

$$\frac{dp}{d\lambda} = -\frac{p\Sigma}{\eta\lambda} \tag{16}$$

where

$$\frac{1}{\Sigma} = \frac{1}{\eta} + \frac{\theta_C a_I b_C(\rho + \delta) + b_I(1 - \theta_C)(\rho a_C + \delta a_I b_C + \delta a_C(1 - b_I))}{(\theta_I - \theta_C)\Delta(\rho + \delta(1 - b_I))} \tag{17}$$

As it is usual in the literature, we take a first order approximation of the dynamic equations. The dynamic property of the economy is then determined by the signs of the determinant and trace of the Jacobian matrix derived from linearly approximating the dynamic system. In particular, for indeterminacy to take place, the determinant must be positive and the trace must be negative, so that there are two stable (but possibly complex) eigenvalues and hence two stable solutions that solve the dynamic equations. Since any linear combination of the two (stable) solutions is also a (stable) solution,[1] there are now infinitely many possible stable solutions giving rise to the notion of indeterminacy of the equilibrium path. However, if the determinant is negative, then there must be one unstable and one stable eigenvalue, implying that the stable solution is unique. This is the case that corresponds to saddle-path stability.

3. Discussion

The determinant and trace of the Jacobian matrix are:

$$Det(J) = -\left(\frac{\rho a_C + \delta a_I b_C + \delta a_C(1 - b_I)}{\Delta}\right)\left(\frac{\Sigma\theta_I(\rho + \delta)}{\eta(\theta_I - \theta_C)}\right) \tag{18}$$

[1] This is the well-known principle of superposition in differential equations.

$$
\begin{aligned}
Trace(J) = &-\frac{\Sigma a_I(\rho+\delta)}{\Delta^2}\left(\frac{(1-\theta_C)(1-\theta_I)(\delta\Delta + a_C(\rho+\delta)) + \theta_C\theta_I b_C(\rho+\delta)}{(\theta_C-\theta_I)(\rho+(1-b_I)\delta)}\right) \\
&-\frac{\rho a_I + \delta a_I b_C + \delta a_C(1-b_I)}{\Delta} \\
&+\frac{\Sigma\theta_I(\rho+\delta)}{\eta(\theta_I-\theta_C)}
\end{aligned} \tag{19}
$$

We now examine the nature of the equilibrium under three cases. First, we consider the case where private and social factor intensities have the same ranking. Next, we examine the dynamic properties of the economy under two cases that result in factor intensity reversal between private and social factor intensities.

Case 1. *Let* $\theta_I - \theta_C < 0$ *and* $\Delta < 0$.

These restrictions imply that $1/\Sigma > 0$. Therefore, the determinant is negative, indicating that the system only has one stable eigenvalue. Consequently, the steady state is saddle-path stable.

Case 2. *Let* $\theta_I - \theta_C < 0$ *and* $\Delta > 0$.

Clearly, the sign of the determinant depends on the sign of $1/\Sigma$. In particular, if

$$
\eta \in \left(0, \frac{(\theta_C-\theta_I)\Delta(\rho+\delta(1-b_I))}{\theta_C a_I b_C(\rho+\delta) + b_I(1-\theta_C)(\rho a_C + \delta a_I b_C + \delta a_C(1-b_I))}\right) \tag{20}
$$

is holds, then $\Sigma > 0$ emerges, resulting in a positive determinant. This is the necessary condition for indeterminacy, in the sense that a positive determinant is necessary to rule out saddle-path stability. Furthermore, since the determinant is positive and the trace is negative, both eigenvalues are stable. Hence, any equilibrium path as differentiated by the choice of λ_0 given K_0 will always be convergent.

Note that indeterminacy arises in this system when two conditions are met: i) $\theta_I - \theta_C < 0$ and $\Delta > 0$, ii) $1/\Sigma > 0$. Although it is difficult to reconcile these conditions intuitively, some observations related to the consumption market equilibrium can be made. If condition i) is satisfied, then consumption supply becomes negatively sloping. This can be seen by differentiating (13) to obtain

$$
\frac{dC^S}{dp} = \frac{a_I r'(p)K - b_I w'(p)L}{p\Delta} + \frac{a_I r(p)}{p\Delta}\frac{dK}{dp} - \frac{C}{p} < 0 \tag{21}
$$

using the fact that $r'(p) < 0$, $w'(p) > 0$, $dC/dK > 0$ and $dC/dL < 0$ given $\theta_I - \theta_C < 0$ and $\Delta > 0$. These conditions imply that the dual relationship between the Stolper-Samuelson and the Rybzcynski effects is broken. Suppose it is not. Then with a labor intensive consumption good, two things must be true: the Stolper-Samuelson theorem asserts that wage should rise given an appreciation in the relative price of consumption, i.e. $w'(p) > 0$; the Rybczynski theorem asserts that consumption goods supply should increase alongside an increase in the relative supply of labor, i.e. $dC/dL > 0$. However, while $w'(p) > 0$ holds, we have $dC/dL < 0$ instead. Therefore, it must be true that the dual relationship is broken. The phenomenon is first discovered by Benhabib and Nishimura (1998).

Now, what is interesting is that with a negatively sloping consumption supply schedule, the consumption market equilibrium is unstable if the demand schedule cuts the supply schedule from above. This can indeed occur with condition ii), which states that η must be small enough so that $1/\Sigma > 0$. For instance, for η sufficiently close to 0 by which the second condition is satisfied, the slope of the consumption demand schedule, which is

$$\frac{dC^D}{dp} = -\frac{1}{\eta}\lambda^{\frac{1}{\eta}}p^{-\frac{1}{\eta}-1} \tag{22}$$

can always be made steeper than the slope of the consumption supply schedule with an appropriate choice of η that is small.

Nishimura and Shimomura demonstrate that indeterminacy may arise in the Heckscher-Ohlin world in the presence of sector-specific externalities whenever $\theta_I - \theta_C < 0$, $\Delta > 0$ and (20) hold. However, these are exactly the same set of restrictions giving rise to indeterminacy in the autarkic economy. Therefore, whenever indeterminacy holds under autarky, indeterminacy will hold in the Heckscher-Ohlin model once two of these autarkic economies are opened to trade.

The intuition behind this outcome is straightforward. Incomplete specialization in each country under trade implies that the countries are functioning together as one integrated world economy. In the trade literature, this feature is known as the integrated world equilibrium (IWE). With the IWE, one may pool the endowments of all countries and treat all the countries as a single entity with an enlarged endowment set. Clearly, as long as indeterminacy occurs in any one of the countries under autarky, indeterminacy will occur for the IWE as well. That is, as long as indeterminacy occurs in any one of the countries under autarky, it must occur for all countries under autarky since each country is only differentiated by its factor endowments. Since indeterminacy exists in each country under autarky to begin with, it is not surprising that IWE preserves indeterminacy under trade à là Heckscher-Ohlin with incomplete specialization.

Case 3. *Let $\theta_I - \theta_C > 0$ and $\Delta < 0$.*

Case 3 states that the consumption good is capital intensive from the social perspective but labor intensive from the private perspective. Interestingly, an unstable steady state may arise under this scenario. The determinant of the Jacobian remains positive just as in Case 2. As mentioned in the above, this implies that there are either two eigenvalues with positive real parts or two eigenvalues with negative real parts, the outcome of which lies in the sign of the trace. A cursory observation of the trace in (19) reveals that it may be positive. For instance, the trace is necessarily positive if $\delta\Delta + a_C(\rho + \delta) > 0$, which is easily satisfied for a large enough ρ or a_C, and a small enough Δ.

Note that Case 3 is not examined in Nishimura and Shimomura (2002). A point that can be made from examining Case 3 is that factor intensity reversal may break the saddle-point property of the steady state. However, the steady state is not necessarily a sink. From our analysis, it is a sink if the investment good is intensive in the factor that the investment good is used to accumulate. It would be important to further investigate whether this statement is generalizable in a multisector economy.

The downward sloping supply schedule for the consumption good remains. The condition $\theta_I - \theta_C > 0$ implies $r'(p) > 0$ and $w'(p) < 0$. Therefore, the term $a_I r'(p)K - b_I w'(p)L$ in (21) is positive. However, given that $\Delta < 0$, we have $\left(a_I r'(p)K - b_I w'(p)L\right)/p\Delta < 0$.

Since $dp/dK > 0$, it is clear that $dC^S/dp < 0$, implying that the supply schedule for the consumption good is downward sloping yet again. As before, with an appropriate choice of η, the market clearing condition for the consumption good becomes unstable.

4. Conclusion

Using the autarkic variant of Nishimura and Shimomura, we show that non-saddle-path stability is related to an unstable consumption goods market equilibrium, where the supply curve of consumption goods is downward sloping and is intersected by the demand curve of consumption goods from above. In addition, we point out that trade on its very own in the Heckscher-Ohlin framework does not contribute to indeterminacy in Nishimura and Shimomura. Instead, indeterminacy arises in the Heckscher-Ohlin framework because each country's equilibrium path is already indeterminate before trade.

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