E C O N O M I C S   B U L L E T I N

Optimal monetary policy with skill shock

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Abstract

da Costa and Werning (2005) prove that the Friedman rule of setting nominal interest rate to zero is locally optimal in a monetary model where each consumer receives a privately observed skill shock and the government uses incentive compatible non-linear income taxation. In this paper, we show that when goods and money are strictly separable from labor in the consumer's utility function, the Friedman rule is the globally optimal monetary policy. Positive nominal interest rate does not improve social welfare.

AcknowledgementsI thank an anonymous referee, Pete Klenow and Akimichi Takemura for their helpful comments.
Citation: Hiraguchi, Ryoji, (2007) "Optimal monetary policy with skill shock." *Economics Bulletin,* Vol. 5, No. 15 pp. 1-11
Submitted: June 10, 2007. **Accepted:** September 18, 2007.
URL: http://economicsbulletin.vanderbilt.edu/2007/volume5/EB-07N10001A.pdf

1. Introduction

A major goal of monetary policy analysis is to derive the optimal monetary policy rule. Many authors, including Chari and Kehoe (1998), show that the Friedman rule to set the nominal interest rate as zero is the optimal policy rule in most of the representative agent models. However, many details remain to be clarified in a heterogeneous agent model.

Recently da Costa and Werning (2005) (henceforth CW) have developed a monetary model in which each agent has different skill and it is private information. They show that when the government can use incentive-compatible non-linear income taxation, the Friedman rule is locally optimal. From their results, Kocherlakota (2005b) and Chari and Kehoe (2006) argue that if the government have enough fiscal policy instruments, monetary authority should follow the Friedman rule.

However, the analysis by CW has one problem. They do not show that the Friedman rule is globally optimal. CW use the Lagrangian in the welfare maximization problem of the government. In their proof of the optimality, the sign of the multiplier on the incentive constraint plays a crucial role. However, they derived it only when the nominal interest rate i is zero. CW admit that nothing is known on the sign of the multiplier when i is positive and they just assume that the sign does not change as i increases from zero. [1] Hence their proof is valid only when the nominal interest rate is around zero.

This paper studies the money-in-the-utility-function model in which the disutility from labor and the utility from consumptions and money holdings are strictly separable. The paper extends their results and shows that the Friedman rule is globally optimal. The paper proves that for any incentive-compatible policy with positive nominal interest rate, there exists an incentive-compatible, feasible and welfare-improving policy with zero nominal interest rate. Redistributions by expansionary monetary policy do not improve social welfare if nonlinear taxation is available.

The paper is organized as follows. Section 2 describes the model. Section 3 shows the optimality of the Friedman rule. Section 4 concludes the paper. All proofs of the lemmas are in the Appendix.

2. The Model

In this section, we set-up the model and state our assumptions. Our model follows closely those in CW, Chari and Kehoe (1998), Ljungqvist and Sargent (2000) and Kocherlakota (2005a).

2.1. Preferences

Time is discrete and denoted by $t = 0, 1, 2, ... \infty$. There is a continuum of agents with measure one. Each agent is identical ex ante and the instantaneous utility function is

$$u(c, m) - h(l), \tag{1}$$

[1] CW describes the problem on the multiplier in page 9.

where $c \geq 0$ is consumption, $m \geq 0$ is money in real term and $l \geq 0$ is work time. The disutility h from labor supply and the utility u from consumption and money are strictly separable. The function u is concave, twice continuously differentiable, strictly increasing in c, weakly increasing in m and $\lim_{c \to +0} u_c(c,m) = +\infty$. On the other hand, h is strictly increasing, strictly convex and twice continuously differentiable.

There exists the satiation level of money $\phi(c)$ which is defined over $c \geq 0$ such that $\phi(0) = 0$, $\phi'(c) \geq 0$ and $\phi(c) > 0$ if $c > 0$. The marginal utility on money satisfies

$$\begin{aligned} u_m(c,m) &> 0 \text{ if } m < \phi(c), \\ u_m(c,m) &= 0 \text{ if } m \geq \phi(c). \end{aligned}$$

Hence the utility is unchanged when the real money holding exceeds the satiation level.

The ex ante homogeneous agent receives an idiosyncratic shock on her skill at the beginning of period 0. The shock is described as a parameter θ, which has the distribution function $F(\theta)$ and the density function $f(\theta) > 0$. The distribution function F has a compact support $\Theta = [\underline{\theta}, \bar{\theta}]$ with $0 < \underline{\theta} < \bar{\theta} < \infty$. Hence $\theta > 0$ for all $\theta \in \Theta$.

The skill parameter θ is private information to each agent. It is independently distributed across agents and is fixed across time. In what follows we call an agent with skill θ as *agent* θ. If agent θ supplies labor by l, she produces effective labor according to the function $y = \theta l$. Production uses a linear technology and one unit of effective labor produces one unit of single good. Effective labor y is observable, but actual labor l is not observable to the government.

The preferences of agent θ over consumption c_t real money balances m_t and effective labor y_t are expressed as $(1-\beta)\sum_{t=0}^{\infty}\beta^t \{u(c_t, m_t) - h(y_t/\theta)\}$, where β is the discount factor. Similar preferences are in Werning (2007). Since $\theta > 0$, $h(y_t/\theta)$ is well-defined.

2.2. Agent's problem

This subsection describes the problem of agents. After receiving the skill shock, each agent reports her skill parameter to the government according to the reporting strategy $\sigma(\theta) : \Theta \to \Theta$. We describe the determination of the reporting strategy later.

The government then assigns the income–labor allocation $\{x_t(\theta'), y_t(\theta')\}_{t=0}^{\infty}$ to an agent who reports θ' as her skill. Here $x_t(\cdot)$ is income and $y_t(\cdot)$ is effective labor in period $t \geq 0$.

Let $V^*(\theta', \theta)$ denote the value function of agent θ with the reporting strategy θ'. Given the allocation, agent θ who reports $\sigma(\theta)$ as her skill solves

$$V^*(\sigma(\theta), \theta) = \max_{\{c_t, M_{t+1}, B_{t+1}, y_t\}_{t=0}^{\infty}} \left[(1-\beta)\sum_{t=0}^{\infty}\beta^t \left\{u(c_t, m_t) - h\left(\frac{y_t}{\theta}\right)\right\}\right], \tag{2}$$

subject to the budget constraint

$$c_t + \frac{M_{t+1}}{p_t} + \frac{B_{t+1}}{R_t} \leq B_t + \frac{M_t}{p_t} + x_t(\sigma(\theta)) \quad t \geq 0, \tag{3}$$

and the effective labor assignment

$$y_t = y_t\left(\sigma\left(\theta\right)\right), \tag{4}$$

where M_{t+1} is money held between times t and $t+1$, p_t is the price level, B_t is the real value of government bond holdings that mature at the beginning of time t, and R_t is the real rate of return on the bonds. The real money balances m_t are written as $m_t = M_{t+1}/p_t$.

Let $i_t = R_t/(p_t/p_{t+1}) - 1$ denote the nominal interest rate. (Note that p_t/p_{t+1} is the rate of return on money.) From the arbitrage condition $R_t \geq p_t/p_{t+1}$, i_t is nonnegative. Following Chari and Kehoe (1998), here we assume that the government chooses the nominal interest rate i_t as the monetary policy. We impose the initial condition on debt $M_0/p_0 + B_0 = 0$ and the following transversality conditions on money and bonds for every agent:

$$\lim_{t\to\infty}\left(\prod_{i=0}^{t} R_i^{-1}\right) B_{t+1} = \lim_{t\to\infty}\left(\prod_{i=0}^{t} R_i^{-1}\right)\frac{M_{t+1}}{p_t} = 0. \tag{5}$$

Using (5), we can consolidate the sequence of budget constraints:

$$\sum_{t=0}^{\infty} q_t \left\{c_t + r_t m_t - x_t\left(\sigma\left(\theta\right)\right)\right\} \leq 0, \tag{6}$$

where $r_t = i_t/\left(1+i_t\right)$, $q_0 = 1$ and $q_t = \prod_{i=0}^{t-1} R_i^{-1}$ if $t > 0$. In the present value budget constraint (6), q_t is the price of good in period t and $r_t m_t$ is the opportunity cost of holding money m_t. In what follows, we call r_t as the nominal interest rate, instead of i_t.

For all θ, the reporting strategy $\sigma\left(\theta\right)$ is determined by $\sigma\left(\theta\right) \in \operatorname{argmax}_{\theta'\in\Theta} V^*\left(\theta', \theta\right)$. The strategy σ maximizes each agent's utility given the allocation $\left\{x_t\left(\theta\right), y_t\left(\theta\right)\right\}_{t=0}^{\infty}$. Obviously the $\sigma\left(\cdot\right)$ depends on the government policy.

2.3. Government Policy

In this subsection we describe fiscal and monetary policy of the government. First we define the policy.

Definition 1 *Government policy is the allocation* $\left\{x_t\left(\theta\right), y_t\left(\theta\right), r_t\right\}_{t=0}^{\infty}$ *in which* $\left\{x_t\left(\theta\right), y_t\left(\theta\right)\right\}$ *is the income–labor assignment for an agent who reports* θ *as her type and* r_t *is the nominal interest rate in period* t.

Following CW, we focus on the stationary equilibrium with price $q_t = \beta^t$ and the time-independent policy $\left\{x\left(\theta\right), y\left(\theta\right), r\right\}_{t=0}^{\infty}$. Agent θ with the reporting strategy θ' maximizes her utility $(1-\beta)\sum_{t=0}^{\infty}\beta^t\{u(c_t, m_t, \theta) - h(y(\theta')/\theta)\}$ subject to the budget constraint $\sum_{t=0}^{\infty}\beta^t\{c_t + rm_t - x(\theta')\} \leq 0$. Hence the equilibrium allocation is actually time-independent. [2] Let

$$
\begin{aligned}
\left[c\left(x,r\right),m\left(x,r\right)\right] &= \underset{(c,m):c+rm=x}{\operatorname{argmax}} \; u\left(c,m\right), && (7)\\
v\left(x,r\right) &= \max_{(c,m):c+rm=x} u\left(c,m\right) = u[c\left(x,r\right),m\left(x,r\right)], && (8)\\
V\left(x,y,r,\theta\right) &= v\left(x,r\right) - h\left(y/\theta\right). && (9)
\end{aligned}
$$

Here v and V are the indirect utility functions and $c(\cdot)$ $(m(\cdot))$ is the consumption (money) demand which is a function of income x and the nominal interest rate r. Note that r is the opportunity cost for holding money. Clearly V is increasing in x and decreasing in r. If $r = 0$, money holding is costless and agents are satiated with money. Thus $c(x,0) = x$ and $m(x,0) = \phi(x)$. If $r > 0$, on the other hand, agents are not satiated with money and $m(x,r) < \phi(c(x,r))$ since the first order conditions imply $u_m = ru_c > 0$.

We can see that the value function $V^*(\theta',\theta)$ in (2) is simply expressed by the indirect utility function V, which is known. Since agent θ with reporting strategy θ' receives $V(x(\theta'),y(\theta'),r,\theta)$ in each period as her utility,

$$V^*\left(\theta',\theta\right) = (1-\beta)\sum_{t=0}^{\infty}\{\beta^t V(x(\theta'),y(\theta'),r,\theta)\} = V(x(\theta'),y(\theta'),r,\theta). \qquad (10)$$

Let us express the incentive compatibility constraint by V. The government policy $\{x(\theta),y(\theta),r\}_{t=0}^{\infty}$ is incentive-compatible if and only if

$$V^*\left(\theta,\theta\right) = \max_{\theta'\in\Theta} V^*\left(\theta',\theta\right) \qquad \text{for all } \theta\in\Theta, \qquad (11)$$

If the policy satisfies (11), each agent follows the truth-telling strategy $\sigma(\theta) = \theta$. It follows from (10) that the incentive constraint (11) holds if and only if

$$V\left(x\left(\theta\right),y\left(\theta\right),r,\theta\right) \geq V\left(x(\theta'),y(\theta'),r,\theta\right) \qquad \text{for all } \theta,\theta'\in\Theta. \qquad (12)$$

From the revelation principle, we can concentrate our attentions on incentive-compatible policies. If a policy $\{x(\theta),y(\theta),r\}_{t=0}^{\infty}$ is incentive-compatible, the value function of agent θ is $V^*(\theta,\theta) = V(x(\theta),y(\theta),r,\theta)$.

The government maximizes the expected utility of the ex ante homogeneous agent $E[V^*(\theta,\theta)]$ by choosing (i) the nominal interest rate (monetary policy) r and (ii) incentive-compatible and feasible income taxation (fiscal policy). Here we assume the government expenditure is time-independent constant $G > 0$ and is given. If the policy is incentive-compatible, the consumption of agent θ in period t is $c_t = c(x(\theta),r)$ and the resource constraint becomes

$$\int_{\underline{\theta}}^{\bar{\theta}} \left(y(\theta) - c(x(\theta),r)\right) dF(\theta) \geq G. \qquad (13)$$

The resource constraint is also time-independent. Note that, as argued in Albanesi and Sleet (2006), we can interpret $f(\theta)$ as the ex-post density of agent θ from the law of large numbers. Formally, we define the problem of the government as follows.

[2] Lagrangian L is $L = (1-\beta)\sum_{t=0}^{\infty}\beta^t\{u(c_t,m_t,\theta) - h\left(y(\theta')/\theta\right) + \lambda\left(x(\theta') - c_t - rm_t\right)\}$, where λ is the multiplier on the budget constraint.

Definition 2 *The optimal policy of the government is the incentive-compatible, feasible and stationary policy* $\{x(\theta), y(\theta), r\}_{t=0}^{\infty}$ *which maximizes each agent's expected utility (social welfare):*

$$E[V^*(\theta, \theta)] = \int_{\underline{\theta}}^{\bar{\theta}} V(x(\theta), y(\theta), r, \theta)\, dF(\theta).$$

The government policy is incentive-compatible if (12) *holds. If the policy is incentive-compatible, it is feasible if* (13) *holds.*

The indirect utility function $V(x, y, r, \theta)$ satisfies the single crossing property since

$$-\frac{\partial}{\partial \theta}\left(\frac{V_y}{V_x}\right) = \frac{1}{v_x}\frac{\partial}{\partial \theta}\left(\frac{1}{\theta} h'\left(\frac{y}{\theta}\right)\right) < 0.$$

Therefore the incentive-compatible policy satisfies the monotonicity condition $x'(\theta) \geq 0$ for all θ (see Mirrlees (1976)).

3. Global optimality of The Friedman rule

In this section, we show that the Friedman rule is the globally optimal monetary policy. We prove that for any incentive-compatible and feasible policy $A_r = \{x(\theta), y(\theta), r\}_{t=0}^{\infty}$ with $r > 0$, there exists an incentive-compatible, feasible and *welfare-improving* policy with zero nominal interest rate. The proof consists of two steps below.

Step 1. Find a zero interest rate policy $\bar{A}_0$ so that agents get the same utilities from $\bar{A}_0$ and A_r, but $\bar{A}_0$ uses less resources than A_r.

Step 2. Construct a new zero interest rate policy A_0^* from $\bar{A}_0$ by redistributing the extra resources generated at Step 1 so that every agent increase her utility.

3.1 Resource-saving government policy

In this subsection, we first construct an incentive-compatible and feasible policy $\bar{A}_0$ in which the nominal interest rate is zero and all agents achieve the same utility as $A_r = \{x(\theta), y(\theta), r\}_{t=0}^{\infty}$. Then we show that it is resource-saving compared with the original policy A_r.

First, using the indirect utility function v in (8), we let $\bar{x}(\theta)$ be such that

$$v(\bar{x}(\theta), 0) = v(x(\theta), r). \tag{14}$$

Equation (14) states that the utility (from consumption and money) when labor income is $x(\theta)$ and the nominal interest rate is r and the utility when income is $\bar{x}(\theta)$ and the interest rate is zero are the same. Using $\bar{x}(\theta)$, we let $\bar{A}_0 = \{\bar{x}(\theta), y(\theta), 0\}_{t=0}^{\infty}$ denote a new policy with the Friedman rule. The following lemma shows that $\bar{A}_0$ is well-defined.

Lemma 1 *For all* θ, $\bar{x}(\theta)$ *always exists.*

Equation (14) means that, for all θ and θ', $\bar{A}_0$ satisfies

$$V(\bar{x}(\theta'), y(\theta'), 0, \theta) = v(x(\theta'), r) - h(y(\theta')/\theta) = V(x(\theta'), y(\theta'), r, \theta), \tag{15}$$

Equation (15) states that, regardless of the reporting strategy, each agent gets the same utility from A_r and $\bar{A}_0$. Note that the effective labor $y(\theta')$ is the same in A_r and $\bar{A}_0$. Substituting (15) into the incentive constraint (12) yields

$$V\left(\bar{x}\left(\theta\right), y\left(\theta\right), 0, \theta\right) \geq V(\bar{x}(\theta'), y(\theta'), 0, \theta) \text{ for all } \theta \text{ and } \theta'. \tag{16}$$

Hence $\bar{A}_0$ is incentive-compatible. The following lemma shows that agents consume less under $\bar{A}_0$ than under A_r. In other words, $\bar{A}_0$ uses less resources than A_r.

Lemma 2 *Suppose $r > 0$. The new income level $\bar{x}(\theta)$ satisfies*

$$c(x(\theta), r) \geq c(\bar{x}(\theta), 0) = \bar{x}(\theta) \text{ for all } \theta \in \Theta, \tag{17}$$

where $c\left(\cdot\right)$ is the consumption demand function. The strict inequality holds if $x(\theta) > 0$.

The proof of the lemma works as follows. Equation (15) states that each agent achieves the same utility level from two policies $\bar{A}_0$ and A_r. Since she is benefited by the monetary satiation in $\bar{A}_0$, the satiation saves consumption.

The consumption reduction by agent θ is equal to $c(x(\theta), r) - \bar{x}(\theta)$. Hence the total amount of the extra resources K by adopting $\bar{A}_0$ is

$$K = \int_{\underline{\theta}}^{\bar{\theta}} \{c(x(\theta), r) - \bar{x}(\theta)\} dF(\theta). \tag{18}$$

Lemma 2 implies $K \geq 0$. The next lemma shows that A_r is not the optimal if all agents receive zero income.

Lemma 3 *A policy $A_r = \{x(\theta), y(\theta), r\}_{t=0}^{\infty}$ such that $r > 0$ and $x(\theta) = 0$ for all θ is not the optimal. There is a welfare-improving policy with $r = 0$.*

Now we can limit our attention to a case where $x\left(\theta\right) > 0$ for some θ. From the monotonicity condition $x'(\theta) \geq 0$, there exists $\eta \in [\underline{\theta}, \bar{\theta})$ such that $x(\theta) > 0$ for all $\theta \geq \eta$. It follows from (17) and (18) that $K \geq \int_{\eta}^{\bar{\theta}} \{c(x(\theta), r) - \bar{x}(\theta)\} dF(\theta) > 0$. Hence $K > 0$.

3.2 Reallocation of the extra resources

This subsection shows that we can reallocate the extra resources $K > 0$ without violating the incentive constraint and that the reallocation increases the social welfare. Given the policy $\bar{A}_0$ and a constant $\Delta v > 0$, define $w\left(\theta, \Delta v\right)$ by a unique solution of

$$\Delta v = v\left(\bar{x}(\theta) + w, 0\right) - v\left(\bar{x}(\theta), 0\right), \quad \theta \in \Theta. \tag{19}$$

where $\bar{x}(\cdot)$ is defined in (14). Equation (19) states that if the current government policy is $\bar{A}_0$ and agent θ receives an additional income $w(\theta, \Delta v)$, her utility increases by Δv. Clearly $w(\theta, \cdot)$ is positive, increasing, continuous and $w(\theta, 0) = 0$. Hence for sufficiently small utility increase $\Delta v^* > 0$, we have

$$\int_{\underline{\theta}}^{\bar{\theta}} w(\theta, \Delta v^*)\, dF(\theta) \leq K. \tag{20}$$

Equation (20) states that if the government increase the income of agent θ by $w(\theta, \Delta v^*)$, the policy changes does not use up the extra resources K.

Now define a new zero interest rate policy by using $\bar{x}(\theta)$:

$$A_0^* = \{x^*(\theta), y(\theta), 0\}_{t=0}^{\infty}, \qquad x^*(\theta) = \bar{x}(\theta) + w(\theta, \Delta v^*). \tag{21}$$

Let us compare A_0^* with $\bar{A}_0$. Agent θ gets additional income $w(\theta, \Delta v^*)$ in A_0^*, but the effective labor assignment $y(\theta)$ is the same. Hence every agent increases her utility by Δv^* under A_0^* as long as she adopts the truth telling strategy. The next lemma shows that this seemingly better policy is actually incentive-compatible and feasible.

Lemma 4 *The policy $A_0^* = \{x^*(\theta), y(\theta), 0\}_{t=0}^{\infty}$ is incentive-compatible and feasible.*

Since the extra income $w(\theta, \Delta v^*)$ raises the utility of agent θ by Δv^*, her utility from the policy A_0^* is written as

$$\begin{aligned} V(x^*(\theta), y(\theta), 0, \theta) &= v(x^*(\theta), 0) - h(y(\theta)/\theta) \\ &= \Delta v^* + v(\bar{x}(\theta), 0) - h(y(\theta)/\theta). \end{aligned} \tag{22}$$

Substituting (14) into (22) yields

$$V(x^*(\theta), y(\theta), 0, \theta) = \Delta v^* + V(x(\theta), y(\theta), r, \theta). \tag{23}$$

Equation (23) states that if we compare the new policy $A_0^* = \{x^*(\theta), y(\theta), 0\}_{t=0}^{\infty}$ with the original positive nominal interest policy $A_r = \{x(\theta), y(\theta), r\}_{t=0}^{\infty}$, A_0^* gives the additional utility to each agent. Finally we get the following proposition.

Proposition 1 *The Friedman rule is the optimal monetary policy.*

Proof. Since every agent raises her utility by Δv^* under A_0^*, the social welfare also increases by Δv^*. Therefore, the positive nominal interest rate policy A_r cannot be socially optimal. Thus the optimal nominal interest rate is zero. ■

4. Conclusion

This paper considers the money-in-the-utlity-function model in which agents are heterogeneous in their skills and the government can use nonlinear income taxation. It shows that the Friedman rule maximizes social welfare when the disutility from labor and the utility from consumptions are strictly separable. The model described here could lead to a better understanding of monetary policy analysis in a heterogeneous agent model.

APPENDIX A: Proof of Lemma 1

If $x(\theta) = 0$ for all θ, effective labor $y(\theta)$ is also the same across agents. Otherwise the policy violates the incentive compatibility. Hence for some constant $y \geq 0$, $y(\theta) = y$. From feasibility, $y \geq G > 0$. Note that if the income is zero, nominal interest rate does not affect the utility. Hence $v(0, r) = v(0, 0)$. The social welfare $W(A_r)$ from the policy $A_r = \{x(\theta), y(\theta), r\}_{t=0}^{\infty} = \{0, y, r\}_{t=0}^{\infty}$ is given by

$$W(A_r) = E_\theta \left[v(0, r) - h\left(\frac{y}{\theta}\right) \right] = v(0, 0) - \int_{\underline{\theta}}^{\bar{\theta}} h\left(\frac{y}{\theta}\right) dF(\theta).$$

Now consider a new policy $A_0' = \{\Delta x, y + \Delta x, 0\}_{t=0}^{\infty}$ with labor income $\Delta x > 0$. Under A_0', each agent receive positive income Δx, but they have to work more than under A_r. Since the policy is type-independent, A_0' is incentive-compatible. It is also feasible because

$$\int_{\underline{\theta}}^{\bar{\theta}} \{(y + \Delta x) - c(\Delta x, 0)\} dF(\theta) = \int_{\underline{\theta}}^{\bar{\theta}} (y + \Delta x - \Delta x)\, dF(\theta) = y \geq G.$$

Let us show that for a small $\Delta x > 0$, A_0' is welfare-improving. Since $\lim_{c \to 0} u_c(c, m) = +\infty$, $\lim_{x \to 0} \partial v(x, 0)/\partial x = \lim_{c \to 0} du(c, \phi(c))/dc = +\infty$. If $r = 0$, the marginal utility of the indirect utility function v goes to $+\infty$ as labor income approaches to zero. Hence for small Δx, $[v(\Delta x, 0) - v(0, 0)]/\Delta x$ can be arbitrary large. Furthermore, $h'(y)$ is finite. Thus

$$\frac{v(\Delta x, 0) - v(0, 0)}{\Delta x} > \frac{1}{\Delta x} \int_{\underline{\theta}}^{\bar{\theta}} \left[h\left(\frac{y + \Delta x}{\theta}\right) - h\left(\frac{y}{\theta}\right) \right] dF(\theta), \tag{24}$$

for a sufficiently small Δx. It follows from (24) that the social welfare from the policy A_0', $W(A_0')$ satisfies

$$\begin{aligned} W(A_0') &= v(\Delta x, 0) - E_\theta \left[h\left(\frac{y + \Delta x}{\theta}\right) \right] \\ &> v(0, 0) - E_\theta \left[h\left(\frac{y}{\theta}\right) \right] \\ &= W(A_r). \end{aligned}$$

Hence A_r is not the optimal policy. ■

APPENDIX B: Proof of Lemma 2

First suppose $x(\theta) > 0$. The function $v(x, 0)$ is continuous in x and $v(0, 0) < v(x(\theta), r) < v(x(\theta), 0)$. Hence there exists $\bar{x}$ such that $v(\bar{x}, 0) = v(x(\theta), r)$. Next, if $x(\theta) = 0$, then obviously $\bar{x}(\theta) = 0$. ■

APPENDIX C: Proof of Lemma 3

First suppose $x(\theta) > 0$. If $r > 0$, monetary satiation does not occur and $m\left(x(\theta), r\right) < \phi\left(c\left(x(\theta), r\right)\right)$. Hence

$$\begin{aligned} v(\bar{x}(\theta), 0) &= v(x(\theta), r) \\ &= u\left[c\left(x\left(\theta\right), r\right), m\left(x\left(\theta\right), r\right)\right] \\ &< u\left[c\left(x\left(\theta\right), r\right), \phi\left(c\left(x\left(\theta\right), r\right)\right)\right] = v\left(c\left(x\left(\theta\right), r\right), 0\right). \end{aligned}$$

Since $v\left(\cdot, 0\right)$ is the increasing function, $\bar{x}(\theta) < c(x(\theta), r)$.

On the other hand, if $x(\theta) = 0$, clearly $c(x(\theta), r) = \bar{x}(\theta) = 0$. ■

APPENDIX D: Proof of Lemma 4

The proof consists of two parts, incentive compatibility and feasibility.

1 (Incentive compatibility). Since $\bar{A}_0 = \{\bar{x}(\theta), y(\theta), 0\}_{t=0}^{\infty}$ is incentive-compatible,

$$\begin{aligned} V\left(x^*(\theta'), y(\theta'), 0, \theta\right) &= \Delta v^* + v\left(\bar{x}(\theta'), 0\right) - h\left(y(\theta')/\theta\right) \\ &\leq \Delta v^* + v\left(\bar{x}\left(\theta\right), 0\right) - h\left(y(\theta)/\theta\right) \\ &= V\left(x^*(\theta), y(\theta), 0, \theta\right), \end{aligned}$$

for all θ and θ'. Thus $A_0^* = \{x^*\left(\theta\right), y(\theta), 0\}_{t=0}^{\infty}$ is also incentive-compatible.

2 (Feasibility) Equation (18) and (20) imply that

$$\int_{\underline{\theta}}^{\bar{\theta}} w\left(\theta, \Delta v^*\right) dF(\theta) \leq K = \int_{\underline{\theta}}^{\bar{\theta}} \{c(x(\theta), r) - \bar{x}(\theta)\} dF(\theta). \tag{25}$$

Since $x^*(\theta) = w\left(\theta, \Delta v^*\right) + \bar{x}\left(\theta\right)$ by definition, (25) implies

$$\int_{\underline{\theta}}^{\bar{\theta}} x^*(\theta) dF(\theta) \leq \int_{\underline{\theta}}^{\bar{\theta}} c\left(x\left(\theta\right), r\right) dF(\theta).$$

Hence

$$\int_{\underline{\theta}}^{\bar{\theta}} \left[y\left(\theta\right) - x^*(\theta)\right] dF(\theta) \geq \int_{\underline{\theta}}^{\bar{\theta}} y\left(\theta\right) - c\left(x\left(\theta\right), r\right) dF(\theta) \geq G.$$

Therefore the policy A_0^* does not violate the resource constraint. ■

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