E C O N O M I C S   B U L L E T I N

The time to shut down

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Abstract

At each time, a firm facing uncertainty over future market conditions have to make a decision whether they should continue to produce or stop the process? As the traditional principle, the firm will go out of production when the price of the typical unit does not cover the average variable cost that it must incur to produce the typical unit. In reality the firm can suffer losses today; however it can get more gains tomorrow that is enough to make up the losses. It means that this rule seems not be suitable absolutely in an uncertainty environment. And it leads to a rule that the firm only stop producing if average variable costs of unit exceed the price of unit by a positive amount. This paper expects to find this exceeding amount and when a firm will stop producing. Under uncertainty, the price of unit and the average variables cost are assumed to follow a continuous time stochastic process. We wish to apply the optimal stopping time approach in order to solve it.

We are grateful the members of MPSE, University of Toulouse 1, especially Professor Jean-Paul Decamps and Mr. Stephan Villeneuve (Produit Derive) for their valuable lectures. We are particularly indebted to Professor Christian Gollier, for his experienced guidance throughout the process of designing and writing our paper. We greatly appreciate his readiness and interest to discuss our progress. Finally, we want to express our gratitude to our friends, especially Bertrand Djembissi Kamsu (University of Toulouse 1, France) for his advices and support. All remaining errors are our own.
Citation: Pham, Thu Phuong and Anh Tuan Bui, (2007) "The time to shut down." *Economics Bulletin*, Vol. 7, No. 14 pp. 1-14
Submitted: September 3, 2007. **Accepted:** September 3, 2007.
URL: http://economicsbulletin.vanderbilt.edu/2007/volume7/EB-07G30005A.pdf

1. Introduction

There are many firms carrying out production and sale of their goods and services in the markets. At each time, a firm facing uncertainty over future market conditions have to make a decision whether they should continue to produce or not? The principle of microeconomics said that if total revenue (TR) is less than variable costs of production, the firm will shut down its production. In other words, the firm will go out of production when the price of the typical unit does not cover the average variable cost. In reality the firm can suffer losses today; however it can get more gains tomorrow that is enough to make up the losses. It means that this rule seems not be suitable absolutely in an uncertainty environment. And it leads to a rule that the firm only stop producing if average variable costs of unit exceed the price of unit **by a positive amount**. This paper expects to find this exceeding amount and when a firm will stop producing. Under uncertainty, the price of unit and the average variables cost are assumed to follow a continuous time stochastic process. The problem has a few characteristics similar to the problem "The waiting to invest". Thus we wish to apply the optimal stopping time approach in order to solve it.

Although McDonald and Seigel (1985) considered the investment problem and the value of the firm when there is an option to shut down, the shut-down problem -as the matter of fact- have not been mentioned as a separate issue under uncertainty. In their paper, they showed how option pricing techniques can be used to study the investment problem of the firm which has the option to shut down production if variable production costs exceed revenues. In our paper, we wish to detach the shut-down problem from the investment. Our contribution in this paper is to give shut-down problem in the simplest circumstance and perform an obvious result in the special case of the average variable cost. Besides, we also achieve a new solution of a new differential equation presented in section III of this paper.

The firm in the model is risk-neutral. This paper is organized as follows. Section I presents a general model where the price of unit follows a geometric Brownian motion and the average variable costs also follow a continuous time stochastic process. Section I begins with simple computation the value of the firm and the optimal stopping time with only the price of unit follows a geometric Brownian motion; the average variable costs is constant. Despite the essential different problems, the shut-down problem in this section is slightly similar to one part of the problem in the paper "Entry and Exit decision under uncertainty" by Dixit, Avinash (1986). We used the technical calculi in the literature which we will refer in details later. Section III extends to the average variable costs under the process indicated in section I. We apply popular variants of Brownian process to the average variable cost in this section. Section IV analyzes the obtained results and their reasonableness. Section V gives some conclusions of this work.

2. The shut-down problem

We are considering a firm which produces a constant quantity of commodity, says q, at every time period. The firm has to pay money to buy inputs per unit at each time period t, called C_t. It is clear that C_t is average variable costs of production. Before its production, the firm paid for expenditure of building material and technological foundation which is seen as sunk costs because the firm is not able to recover them under any circumstance. Thus, it is not necessary to take into account the fixed costs in shut-down decision. The revenue that the firm obtains for the sales of a unit of its output is exactly the price of unit, called P_t. As mentioned above, P_t and C_t are stochastic.

Assuming that P_t follows a geometric Brownian motion of the form:

$$dP_t = P_t(\mu_p dt + \sigma_p dB_t^1) \qquad (1)$$

where dB_t^1 is the increment of a Wiener process.

C_t follows a dynamic stochastic process of the form:

$$dC_t = C_t(\mu_c dt + \sigma_c(\rho dB_t^1 + \sqrt{1-\rho^2} dB_t^2)) \qquad (2)$$

where dB_t^1 and dB_t^2 are standard increment of Brownian motion and ρ is coefficient of correlation between dB_t^1 and ($\rho \, \mathrm{d} B_t^1 + \sqrt{1-\rho^2} \, \mathrm{d} B_t^2$).

B_t^1 and B_t^2 are independent of each other.

μ_p and μ_c are called the drift parameters in Brownian motion;

σ_c and σ_p are the volatility parameters.

Assuming C_t followed by equation (2) is to aim to simplify computation in next steps. This is an only special case of geometric Brownian motion, in effect. We show this in Appendix.

2.1.The objective function

Returning to the problem, it is that the firm will stop producing if the price of unit is less than the average variable costs by a positive amount as above saying. The firm receives q*(P_t - C_t) at each time t. If the firm decides not to produce, profit is zero. The stopping time problem consist of finding the number z* such that at each time, if $C_t / P_t \geq$ z* the firm will shut down and otherwise it continues to produce. The decision is made to maximize the time zero expected profits. If C_t / P_t touches the boundary z* at time t*, the firm decide to come to a stop at that time. And the value of the firm at time t can be written as

$$E[\int_0^t e^{-rt} q(P_t - C_t) dt] \qquad (3)$$

where r is the given appropriate discount rate. It is risk less interest rate, normally. We also give assumptions that the variable unit production cost, C_t , and the output price P_t are known at time zero with certainty. In other word, P_0 and C_0 are given. The firm will choose a boundary to stop producing in order to maximize (3). Solving this problem, we derive an optimal decision rule and the value of the firm.

2.2.The difference between "The investment timing problem" and "the shut-down problem"

At first sight, two problems seem to be identical, but they are not so. In the investment timing problem (R.McDonald and D.Siegel 1986), the project's costs is only installed *once* at time t – the time to invest, during the life of the project. Thus the expected present value of the payoff is E[$e^{-rt}(P_t - C_t)$] and the investment timing problem want to maximize it. With regarding to the shut-down problem, *at each time t*, the costs always is installed and the revenue always achieves continuously until the stopping time t*. Therefore, the time zero expected value of the firm is not E [$e^{-rt}(P_t - C_t)$], which is only the expected present value of the firm at one time t. In fact, the firm wishes to maximize the total of the time zero expected of the payoffs it obtained until it shuts down. In mathematical aspect, integral signal is added to the objective function in the shut-down problem like the formula (3). This suggests that the shut-down problem can be solved by the approach similar to the investment timing problem if the integral signal is relaxed. We are going to solve it by applying the approach in the following parts.

3. The price of unit is stochastic and the cost of unit is constant

We begin with the simple case of the price of commodity P_t following (1) but the cost of its is constant C. The problem becomes:

$$V(P)= \underset{t,P_0,C}{Max}\ E[\int_0^t e^{-rt} q(P_t - C)dt \mid P_0 = P] \tag{4}$$

Where $P_t = P_0 \exp[(\mu_p - \frac{\sigma_p^2}{2})t + \sigma_p B_t^1]$ because P_t is a geometric Brownian motion[1].

By arbitrage, over an interval dt, the total expected return from production must be equal to $rVdt$. This expected value of the firm includes two components, an expected value $E_0 dV$, and a flow of revenue $(P-C)dt$. By Ito's Lemma,

$$E_0 dV = \frac{\sigma_p^2}{2} P^2 V''(P)dt + \mu_p PV'(P)dt$$

So in the continuation region $V(P)$ must satisfy the following differential equation:

$$\frac{\sigma_p^2}{2} P^2 V''(P) + \mu_p PV'(P) - rV + P - C = 0 \tag{5}$$

We can see the equation (5) includes 2 parts: one part is homogeneous and the other part is nonhomogeneous. Dixit (1989) found the general solution of the equation like above one. That is to take linear combination of the solutions of the homogeneous part, and add on any particular solution of the full equation. The homogeneous part are easy to obtain the solution, and a simple substitution showed $\left(\frac{P}{r-\mu_p} - \frac{C}{r} \right)$ satisfied the equation.

Assuming that $r \neq \mu_p$, try a solution of the equation (5) with the form of

$$V(P) = P^{\alpha} + \frac{P}{r-\mu_p} - \frac{C}{r} \tag{6}$$

Substitute (6) into (5), we get the equation below:

$$h(\alpha) = \frac{1}{2}\sigma_p^2 \alpha(\alpha - 1) + \mu_p \alpha - r = 0 \tag{7}$$

If the volatility $\sigma_p \neq 0$, this is a quadratic equation of α with $\Delta = (\mu_p - \frac{\sigma_p^2}{2})^2 + 2\sigma_p^2 r > 0$ [2] guaranteeing that it has two roots α_1 and α_2. Since h(0)=-r <0 and $h''(\alpha) = \frac{\sigma_p^2}{2} > 0$, one root must be greater than 0 and one root is negative. Written out explicitly:

[1] See Tomas Bjork, Arbitrage theory in continuous time(1998), p.55

[2] Since r>0

$$\alpha_1 = \frac{-(\mu_p - \frac{\sigma_p^2}{2}) + \sqrt{\Delta}}{\sigma_p^2} > 0 \text{ (7) and } \alpha_2 = \frac{-(\mu_p - \frac{\sigma_p^2}{2}) - \sqrt{\Delta}}{\sigma_p^2} < 0$$

Thus we can write the general solution of (5) as $V(P) = AP_t^{\alpha_1} + BP_t^{\alpha_2} + \frac{P}{r-\mu_p} - \frac{C}{r}$. Dealing with (4), we can see that when P_t is very small, $V(P)$ becomes nearly worthless. Thus, we need B=0 and the solution is: $V(P) = AP_t^{\alpha_1} + \frac{P}{r-\mu_p} - \frac{C}{r}$ (8).

In the stopping region, the firm decides to stop producing and it does not earn the flow of revenue. Thus we have the value matching condition:

$$V(P) = 0 \tag{9}$$

and smooth-pasting condition:

$$V'(P) = 0 \tag{10}$$

Substituting the solution (8) in equations (9) and (10), we obtain

$$P_t^* = \frac{\alpha_1 C(\mu_p - r)}{r(1-\alpha_1)} \text{ where } \alpha_1 \neq 1 \tag{11}$$

As a result, if the growth rate of price μ_p is not equal to the discount rate, it is optimal for the firm to stop producing as soon as $P_t \leq P_t^* = \frac{\alpha_1 C(\mu_p - r)}{r(1-\alpha_1)}$ where α_1 is given by (7).

3. **Both the price of unit and the cost of unit are stochastic**

We still consider the same problem, except that the price of commodity is also the stochastic process (2). The issue in section II is only a simple case of that in section III. The problem is formulated as:

$$\text{V(P,C)} = \underset{t, P_0, C_0}{Max} \text{ E}[\int_0^t e^{-rt} q(P_t - C_t)dt \mid P_0 = P; C_0 = C] \tag{12}$$

We are having a function with two stochastic variables. Our objective now is to alter the function in equation (12) to equivalent function with a variable following a Brownian process. Based on that, we hope it is possible to solve the problem in this section by the similar approach in above section.

4.1 In order to apply easily the used method in the part 2 of section II, we transform equation (12) from two-stochastic variables to one stochastic variable.

Denote $Z_t = \frac{C_t}{P_t}$ proves that Z_t is also a geometric Brownian motion latter. Now, we would like to rewrite equation (12) according to Z_t:

$$V(Z)=\underset{t,P_0,C_0}{Max}\ E[\int_0^t e^{-rt}qP_t(1-Z_t)dt \mid Z_0=\frac{C}{P}] \quad (13)$$

Since P_t and C_t are under geometric Brownian motion, they can be written as an exponential function of time below[3]:

$$P_t = P_0 \exp[(\mu_p - \frac{\sigma_p^2}{2})t + \sigma_p B_t^1] \quad (14)$$

$$C_t = C_0 \exp\left[\left(\mu_c - \frac{\sigma_c^2}{2}\right)t + \sigma_c\left(\rho B_t^1 + \sqrt{1-\rho^2}B_t^2\right)\right] \quad (15)$$

Substitution equation (14) into equation (13), we get

$$V(Z)=\underset{t,P_0,C_0}{Max}\ P_0 qE\left[\int_0^t e^{-(r-\mu_p)t}\exp\left(\sigma_p B_t^1 - \frac{\sigma_p^2}{2}t\right)(1-Z_t)\Big|Z_0 = Z\right] \quad (16)$$

We see that equation (16) is nearly similar to the problem in section 2, except for the term $\exp\left(\sigma_p B_t^1 - \frac{\sigma_p^2}{2}t\right)$. It is lucky that we can apply Girsanov's theorem to transform it. Below is its application.

We know that $\{B_t^1\}_{0\le t<\infty}$ is a standard Brownian motion under the probability measure P and let $\{\Im_t\}_{0\le t<\infty}$ be the associated Brownian filtration. Under P, the process $\Lambda(t)= \exp\left(\sigma_p B_t^1 - \frac{\sigma_p^2}{2}t\right)$ is a martingale with respect to $\{\Im_t\}_{0\le t<\infty}$. The formula $Q(F)=E_P(\Lambda(t)1_F)$ defines a new probability measure Q. The distribution of the stochastic process $\{B_t^1\}_{0\le t<\infty}$ under the new probability is described as

$$W_t^1 = B_t^1 - \sigma_p t \quad (17)$$

The Girsanov's theorem said that under the new probability Q, the stochastic process $\{W_t^1\}_{0\le t<\infty}$ is also a standard Wiener process.

Let $W_t^2 = B_t^2$ (18).

Since B_t^2 and B_t^1 are the independent standard Brownian motion processes under the probability measure P, W_t^2 given by equation (18) is also a standard Wiener process under the new probability Q.

[3] See Tomas Bjork, Arbitrage theory in continuous time(1998), p.55

Now, the problem (16) under the probability measure P turns out

$$V(Z) = \underset{t, P_0, C_0}{Max}\ P_0 E_Q \left[\int_0^t e^{-(r-\mu_p)t} (1 - Z_t) dt \middle| Z_0 = Z \right] \tag{19}$$

under the new probability measure Q.

It is clear that if Z_t in equation (19) also follows a geometric Brownian motion then the solution of (19) appears to nearly alike to that in section 2. Fortunately it is true and the proof of this is relegated to the Appendix.

We can write Z_t as a stochastic process:

$$dZ_t = Z_t \left[(\mu_c - \mu_p) dt + \sigma dW_t \right] \tag{20}$$

where

$$\sigma = \sqrt{\sigma_c^2 + \sigma_p^2 - 2\rho\sigma_c\sigma_p} \tag{21}$$

$$W_t = \frac{1}{\sigma} \left[(\rho\sigma_c - \sigma_p) W_t^1 + \sigma_c \sqrt{1-\rho^2} W_t^2 \right] \tag{22}$$

It is easy to realize that W_t is a Wiener process with normally distributed which has zero mean and standard variance t because W_t is a linear function of two Brownian processes W_t^1 and W_t^2.

Thus we say Z_t is a geometric Brownian motion. In the following part, we use the same method in section 2 but the characteristic of the boundary at which the firm shuts down production is not the same.

4.2.Solution of the transformed problem

The problem we have been considering involves choosing a boundary Z* to the equation (19). The argument establishes that the firm will stop production when the ratio $Z = \frac{C}{P}$ reaches the boundary z*.

McDonald & Siegel, 1986 showed that the boundary is homogeneous of degree zero in P and C as well as independent of calendar time. Thus the correct rule is to shut down when the ratio C/P reaches a fixed boundary. The paper of McDonald & Siegel (1986) also proved that there are not multiple boundaries. Thus, the stopping region here is [z*, ∞) only.

Similar to the section 2, the firm operates under the condition of absence of arbitrage opportunity. Over an interval dt, the total expected return from producing must be equal to $(r - \mu_p)Vdt$. This expected value has two components: $E_0 dV$ and $(1 - Z)dt$. By Ito's Lemma,

$E_0 dV = \frac{\sigma^2}{2} Z^2 V''(Z)dt + (\mu_c - \mu_p)ZV'(Z)dt$. Thus in the continuation region, $V(Z)$ must satisfy the following differential equation:

$$\frac{\sigma^2}{2} z^2 V''(z) + (\mu_c - \mu_p) z V'(z) - \left(r - \mu_p\right) V(z) + (1 - Z) = 0 \qquad (23)$$

We can easily realize that a solution of the homogeneous part of the equation (23) is Z^{β} ,which both McDonald, Siegel (1985) and Dixit (1989) referred to. Thus, what this paper presents here is to find out a particular solution of the nonhomogeneous part in equation (23). A clear substitution $\left(\frac{Z}{\mu_c - r} + \frac{1}{r - \mu_p} \right)$ looks quite good in the equation.

Thus, trying a solution of the equation (23) with the form

$$V(z) = Z^{\beta} + \frac{Z}{\mu_c - r} + \frac{1}{r - \mu_p} \qquad (24)$$

Substituting (24) into (23), we get the quadratic equation of β below:

$$g(\beta) = \frac{1}{2}\sigma^2 \beta(\beta - 1) + (\mu_c - \mu_p)\beta - (r - \mu_p) = 0 \qquad (25)$$

If the volatility $\sigma \neq 0$, this is a quadratic equation of β with $\Delta_1 = (\mu_c - \mu_p - \frac{\sigma^2}{2})^2 + 2\sigma^2 (r - \mu_p) > 0$ [4] guaranteeing that it has two roots β_1 and β_2. Since $g(0) = -\left(r - \mu_p\right) < 0$ and $g''(\beta) = \frac{\sigma^2}{2} > 0$, one root must be greater than 0 and one root is negative. Written out explicitly:

$$\beta_1 = \frac{-(\mu_c - \mu_p - \frac{\sigma^2}{2}) + \sqrt{\Delta_1}}{\sigma^2} > 0 \ (26) \text{ and } \beta_2 = \frac{-(\mu_c - \mu_p - \frac{\sigma^2}{2}) - \sqrt{\Delta_1}}{\sigma^2} < 0$$

Thus we can write the general solution of (23) as V(Z)= $A_1 Z^{\beta_1} + B_1 Z^{\beta_2} \frac{Z}{\mu_c - r} + \frac{1}{r - \mu_p}$. Dealing with (19), we can see that when Z_t is very small, V(Z_t) becomes nearly worthless. Thus, we need B_1=0 and the solution is: V(Z_t)= $A_1 Z_t^{\beta_1} + \frac{Z}{\mu_c - r} + \frac{1}{r - \mu_p}$ (27).

On the other hand, in the stopping region, the firm stops producing so that the revenue it gets is equal to zero. In other word, the value matching and smooth pasting follows respectively:

$$V(Z) = 0 \qquad (28)$$

$$V'(Z) = 0 \qquad (29)$$

[4] In this section, we will only consider the case of r> μ_p and $r > \mu_c$.

Substituting the solution (27) in equations (28) and (29), we obtain

$$Z^* = \frac{\beta_1(\mu_c - r)}{(\beta_1 - 1)(\mu_p - r)} \text{ where } \beta_1 \neq 1^5 \tag{30}$$

As a result, if the growth rate of price μ_p is less than the discount rate, it is optimal for the firm to stop producing as soon as $Z \geq Z^* = \frac{\beta 1(\mu_c - r)}{(\beta_1 - 1)(\mu_p - r)}$ where $\beta_1$is given by (26).

5. Analytical Results

We derived the results in two above sections but we ignored the relations between the discount rate and the expected growth rate of the output price and that of the costs. These will be considered in this section with their economic interpretation.

In section II, we used the assumption: the expected growth rate of the output price is different from the discount rate: $\mu_p \neq$ r. Now, we deal with the case of $\mu_p = r$; A solution of the similar form to (6) is impossible to find; In other word, the equation (5) has no root (only considering the form of root is similar to solution (6)). And we cannot figure out the solution of the equation (5). This says that the firm will never stop producing when the expected growth rate of the output price is not different from the discount rate.

Continued, we think about the case that the expected growth rate of the output price is less than the discount rate $\mu_p < r \Rightarrow h(1) = \mu_p - r < 0 \Rightarrow \alpha_1 < 1 \Rightarrow P^* > 0$. In the opposite case, $\mu_p > r$ we also obtain the same result $P^* > 0$. It means that if the growth rate of the output price is different the discount rate, there always exists to a boundary such that the firms will shut down when the output price reaches it.

[5] It means the growth rate of the variable costs μ_c is not equal to the discount rate r.

6. Conclusion

The paper has studied the best time to stop producing for a firm, that is called the shut-down problem and shown the expected maximum value of the firm as well as the stopping time at which the firm makes a blackout decision. This problem is figured out thoroughly with the assumption of the constant variable cost. This paper presents the issue's outcome exactly for each relationship between the growth rate of the output price and the discount rate. However, the solution of the situation where the variable cost follows (2) have just found under condition the growth rate of the outcome price is less than the discount rate. Otherwise, the answer is out of the paper's reach.

This paper still has another limitation. As noted, all analysis based on the assumption of the special geometric Brownian motion form for C, where the increment of a Wiener process is represented linearly by two independent standard increment of Brownian motions. Of course, this assumption is relaxed in section II because C is constant. Yet, it is mandatory to keep reasonable calculi after applying Girsanov's theorem. The question raises the realizability of the variable cost, as described. Two assumptions of linearity and independence are obvious constraints. The low existence of the variable cost function as noted is synonymous with the low worth of the model.

The analysis also imposes restriction on the constancy of quantity over time. This thing maybe only occurs in special industry such as weapon, but only in short-term too. In general, quantities of commodities are affected by many factors, especially by demand for commodities. Under uncertainty, the assumption of constancy of quantity seems less persuasive in long run. Suggest that the quantity of commodity can follows a continuous time stochastic process. The problem will become very complicated and the answer is also beyond the paper's scope.

APPENDIX

1. **The continuous time stochastic process C_t represented by equation (2) is seen as a special case of geometric Brownian motion.**

Dealing with C_t as a geometric Brownian motion of the form:

$$C_t = C_t(\mu_c \,\mathrm{dt} + \sigma_c \,\mathrm{d}Z_t) \qquad (2')$$

where $\mathrm{d}Z_t$ is a standard increment of Brownian motion. Let ρ be the coefficient of correlation between the two Wiener processes dZ_t and dB_t^1.

However, the solution of the problem with C_t like equation (2') is very complicated and is impossible for me at this time. Thus we choose one special case to solve it where Z_t is transformed as a linear function of two ***independent*** Wiener processes $\mathrm{d}B_t^1$ and $\mathrm{d}B_t^2$ as follows:

$$dZ_t = \alpha\; dB_t^1 + \beta\; dB_t^2 \qquad (2'')$$

We find α and β in equation (2'').

Since dB_t^1 and dZ_t are normally distributed with mean zero and variance dt, Cov(dB_t^1, dZ_t) = ρ dt.

On the other hand, we see that:

Cov(dB_t^1,dZ_t)[6] = Cov(dB_t^1, $\alpha\; dB_t^1 + \beta\,\mathrm{d}B_t^2$)

= Cov(dB_t^1, $\alpha\; dB_t^1$)+Cov(dB_t^1, $\beta\,\mathrm{d}B_t^2$)

=Cov(dB_t^1, $\alpha\; dB_t^1$)[7]

= E[8](d$B_t^1 . \alpha\; dB_t^1$)-E(dB_t^1)E($\alpha\; dB_t^1$)

= α Var(dB_t^1)[9]

=α dt.

Thus, $\alpha = \rho$.

Now, we consider below: Var(dZ_t) = Var($\alpha\,\mathrm{d}B_t^1 + \beta\,\mathrm{d}B_t^2$)

= ρ^2Var(dB_t^1)+β^2Var(dB_t^2)

= dt

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[6] Cov(x_i, x_j): Covariance of x_i and x_j

[7] Since B_t^1 and B_t^2 are independent.

[8] E(x): Expectation of x.

[9] Var(x): Variance of x

and we know Var(d B_t^1) = Var(d B_t^2) = dt ➔ $\beta = \sqrt{1-\rho^2}$

Thus, Z_t can be rewritten as follows:

$$dZ_t = \rho\, dB_t^1 + \sqrt{1-\rho^2}\, dB_t^2$$

2. Proof: Z_t is also a geometric Brownian motion.

Substitution both equation (14) and equation (15) into the expanded Z_t, we get Z_t as:

$$Z_t = \frac{C_t}{P_t} = \frac{C_0}{P_0}\exp\left[\left(\mu_c - \mu_p - \frac{\sigma_c^2}{2} + \frac{\sigma_p^2}{2}\right)t + \left(\rho\sigma_c - \sigma_p\right)B_t^1 + \sigma_c\sqrt{1-\rho^2}B_t^2\right] \quad \text{(i.)}$$

Replace B_t^1 and B_t^2 inferred from equation (17) and equation (18) respectively in equation (i), we see:

$$Z_t = \frac{C_t}{P_t} = \frac{C_0}{P_0}\exp\left[\left(\mu_c - \mu_p - \frac{\sigma_c^2}{2} - \frac{\sigma_p^2}{2}\right)t + \rho\sigma_c\sigma_p t + \left(\rho\sigma_c - \sigma_p\right)W_t^1 + \sigma_c\sqrt{1-\rho^2}W_t^2\right] \text{(ii.)}$$

The equation (ii) is reduced as:

$$Z_t = \frac{C_0}{P_0}\exp\left[\left(\mu_c - \mu_p - \frac{\sigma^2}{2}\right)t + \sigma W_t\right] \quad \text{(iii.)}$$

where

$$\sigma = \sqrt{\sigma_c^2 + \sigma_p^2 - 2\rho\sigma_c\sigma_p}$$

$$W_t = \frac{1}{\sigma}\left[\left(\rho\sigma_c - \sigma_p\right)W_t^1 + \sigma_c\sqrt{1-\rho^2}W_t^2\right]$$

It is easy to realize that W_t is a Wiener process with normally distributed which has zero mean and standard variance t because W_t is a linear function of two Brownian processes W_t^1 and W_t^2.

From equation (iii), we can write

$$dZ_t = Z_t\left[\left(\mu_c - \mu_p\right)dt + \sigma dW_t\right]$$

where σ and W_t are given equation (21) and equation (22) respectively in the text.

Thus we say Z_t is a geometric Brownian motion.

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