E c o n o m i c s   B u l l e t i n

A generalization of monotone comparative statics

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Abstract

In this paper, we generalize the lattice theoretical comparative statics by Li Calzi and Veinott, and Milgrom and Shannon. While their theorem is constructed on lattices, particularly on partially ordered sets, we do not require the antisymmetry on a binary relation defined on the set. On the basis of this result, we can deal with the comparative statics of constrained optimization problems, including the cases with nonlinear constraints, in a very intuitive, but considerably general fashion. Specifically, we can extend the gvalue order methods h proposed by Antoniadou and Mirman and Ruble in the context of consumer problems with linear constraints. It is also worth noting that our results on the value order can be applicable for any comparative criterion as long as it is a complete preorder on the domain of the objective function.

I am grateful to Professor Ryo Nagata, Professor Takashi Oginuma, and Associate Professor Hisatoshi Tanaka for giving me a lot of helpful comments. Needless to say, any errors are on my own.
Citation: Shirai, Koji, (2008) "A generalization of monotone comparative statics." *Economics Bulletin,* Vol. 3, No. 39 pp. 1-9
Submitted: July 26, 2008. **Accepted:** August 4, 2008.
URL: http://economicsbulletin.vanderbilt.edu/2008/volume3/EB-08C00009A.pdf

1 Introduction

Our intention in this paper is to contribute a mathematical generalization of the lattice theoretical comparative statics. Our results are obtained in the following two steps. First, we generalize the fundamental theorem of this subject. Then, by applying the extended theorem, we establish an intuitive but considerably general treatment for the comparative statics of optimization problems constrained by real-valued functions.

Since the pioneering studies of Topkis (1978), Milgrom and Roberts (1990), and Vives (1990), the theory of lattice programming has been applied in many fields in economics. Although many benefits can be derived from this theory, it is the monotone comparative statics that plays a basal role in application to economics. In particular, Milgrom and Shannon (1994) or almost the same result by Li Calzi and Veinott (1991) show the necessary and sufficient condition for the solution set of a constrained optimization problem to have global monotonicity. Their theorems are explicated on the platform of a partially ordered set[1], because a lattice is nothing but a partially ordered set satisfying some special conditions. Our first aim in this paper is to extend the theorem of monotone comparative statics to a more general environment, specifically, a preordered set with "lattice-like" properties.

Whether it is the original theorem by Milgrom and Shannon or our extended version, what is important for their application is that an "appropriate" binary relation is defined on the domain of an objective function. In many cases in economics, the objective function of an optimization problem is defined on a subset of the Euclidean space $\mathbb{R}^n$, which is partially ordered by the standard vector order. However, this is not always an appropriate binary relation for the lattice theoretical comparative statics. In such cases, one must artificially construct partial orders to meet the purpose. For the maximization problems with linear constraints, Antoniadou (1996), (2007), and Mirman and Ruble (2003) propose a rather reasonable way to define a partial order, which is called "the value order method". By applying the generalized version of Milgrom and Shannon's theorem, it is revealed that their method is essentially valid for comparative statics of optimization problems with nonlinear constraints. This is our second achievement in this paper. Moreover, our results on the value order method seem to have sufficient generality, in the sense that we do not impose any condition on the criterion for

[1] A partial order is a binary relation which satisfies reflexibility, antisymmetry, and transitivity. If antisymmetry is excluded, it is called a preorder.

the comparative statics exclusive of being a complete preorder on the domain of the objective function.

2 Comparative Statics on Preordered Sets

Let us begin with the quick review of the original theorem by Milgrom and Shannon (1994). In this paper, we concentrate on the comparative statics with respect to the changes of the feasible sets, although Milgrom and Shannon also deal with the changes of the objective functions. For the full-fledged theory of the lattice, including its other applications, see Topkis (1998). Generally, what is important for the lattice theoretical comparative statics of constrained optimization problems are that the supremum and the infimum of any two elements in the domain of the objective function exist, that is, the domain is a "lattice", and that the feasible sets are comparable with respect to some special binary relation, which is called as the "strong set order". Under these conditions, we obtain the monotone comparative statics result if and only if the objective function satisfies the property called as "quasisupermodularity". The formal definitions of these fundamental notions are all given below.

Definition 2.1: Let X be a partially ordered set endowed with a partial order $\leq_X$. We say that X is a *lattice* if both the supremum and infimum of any two elements in X exist. The supremum and infimum are defined with respect to the partial order on X. Let us write the supremum of $x, y \in X$ as $a(x, y)$ and the infimum as $t(x, y)$.

Definition 2.2: Let $S, S' \subset X$. We say that S' is higher than S with respect to the *strong set order* if we have $a(x, y) \in S'$ and $t(x, y) \in S$ for all $x \in S$ and $y \in S'$. We write this situation as $S \leq_a S'$.

We give an intuitive example to clarify the notions of a lattice and the strong set relation defined above. Let $X = [0, 2] \times [0, 2]$. Then, X is a lattice under the standard vector order. For example, the supremum of $(0.5, 1)$ and $(1, 0.5)$ is $(1, 1) \in X$, while the infimum is $(0.5, 0.5) \in X$. Suppose $A = [0, 1.5] \times [0, 1.5]$ and $B = [0.5, 2] \times [0.5, 2]$; then $A \leq_a B$, where $\leq_a$ denotes the strong set order induced by the vector order. For example, consider $(0.5, 1.5) \in A$ and $(1, 1) \in B$. Then, their infimum is $(0.5, 1) \in A$ and their supremum is $(1, 1.5) \in B$. We then define the quasisupermodularity of a function, which can be viewed as a very weak form of monotonicity with respect to the partial order defined on the domain.

Definition 2.3: Let X be a lattice and consider a function $f : X \to \mathbb{R}$. We say that f has *quasisupermodularity* if

$$f(x) \geq (>) f(t(x,y)) \Rightarrow f(a(x,y)) \geq (>) f(y)$$

for all $x, y \in X$.

With these preliminaries, we can state Milgrom and Shannon's theorem, which asserts that the quasisupermodularity of the objective function is a necessary and sufficient condition for the global monotone comparative statics. In the following, let $M(S)$ denote the solution set of the maximization problem:

$$\max_{x \in S} f(x).$$

Theorem 2.1: Let X be a lattice and $S, S' \subset X$. Suppose $S \leq_a S'$. Then $M(S) \leq_a M(S')$ if and only if the objective fuction $f : X \to \mathbb{R}$ satisfies quasisupermodularity.

Proof. See Milgrom and Shannon (1994).

One of our main tasks in this paper is to extend the above theorem on preordered sets. We can no longer use the notions stated in Definitions 1–3 directly, since those are constructed on partially ordered sets. Nevertheless, the conditions under which we have the monotone comparative statics are essentially similar to partially ordered cases. Indeed, we can reach our goal by extending the notions of a lattice, the strong set order, and quasisupermodularity to preordered sets. First, we define the "supremum" and "infimum" of two elements in preordered sets.

Definition 2.4: Suppose X is a preordered set endowed with a preorder $\preccurlyeq_X$. An element $z \in X$ is an upper bound of $x, y \in X$ if $x \preccurlyeq_X z$ and $y \preccurlyeq_X z$. With U denoting the set of all upper bounds of $x, y \in X$, z is said to be a supremum of $x, y \in X$ if $z \preccurlyeq_X u$ for all $u \in U$.

In the above definition, the prefix of the supremum is not redundant, because the supremum a pair of elements is not necessarily unique in preordered sets. We can define the notion of an infimum by analogy. Then, we define a "prelattice", which corresponds to a lattice in the partially ordered set, as follows.

Definition 2.5: Let X be a preordered set. We say X is a *prelattice* if both the set of the supremums and that of the infimums of any two elements in X are nonempty. In the following definition, the former is denote as $A_{x,y}$ and the latter is denoted by $T_{x,y}$.

It is straightforward that, if X is a partially ordered, a prelattice is a lattice. Other notions which play crucial roles in the Milgrom and Shannon's theorem are extended as follows.

Definition 2.6: Let X be a prelattice and $S, S' \subset X$. We say S' is higher than S with respect to the *w-strong set order* if $A_{x,y} \cap S' \neq \emptyset$ and $T_{x,y} \cap S \neq \emptyset$ for all $x \in S$ and $y \in S'$. This relation is denoted as $S \leq_{wa} S'$. Moreover, if $A_{x,y} \subset S'$ and $T_{x,y} \subset S$, S' is higher than S w.r.t the *s-strong set order* and is denoted by $S \leq_{sa} S'$.

Definition 2.7: Consider a prelattice X and a fuction $f : X \to \mathbb{R}$. We say that f has *w-quasisupermodularity* if the condition

$$\forall t \in T_{x,y} \ f(x) \geq (>) f(t) \Rightarrow \exists a \in A_{x,y} \ f(a) \geq (>) f(y)$$

holds for all $x, y \in X$. If the above relation holds for all $a \in A_{x,y}$, that is,

$$\forall t \in T_{x,y} \ f(x) \geq (>) f(t) \Rightarrow \forall a \in A_{x,y} \ f(a) \geq (>) f(y),$$

then we say that f has *s-quasisupermodularity*.

As is the case with the definition of a prelattice, if X is partially ordered, then the w- (s-) strong set order coincides with the strong set order, and w- (s-) quasisupermodularity coincides with quasisupermodularity. Now, we are in a position to state our main result in this section. In the reminder of this paper, all the mathematical proofs are listed in the Appendix.

Proposition 2.1: *Let X be a prelattice and $S, S' \subset X$. (a) Suppose $S \leq_{sa} S'$. Then, we have $M(S) \leq_{wa} M(S')$ if and only if f has w-quasisupermodularity. (b) Suppose $S \leq_{sa} S'$. Then, we have $M(S) \leq_{sa} M(S')$ if and only if f has s-quasisupermodularity.*

Similar to the notions stated in Definitions 5–7, it is also straightforward that when X is a partially ordered set and is a lattice, two claims in the foregoing proposition are equivalent with each other and coincide with Theorem

1, the original theorem by Milgrom and Shannon.

3 General Value Order

In this section, by applying Proposition 2.1 presented in the previous section, we establish a general treatment for the comparative statics of optimization problems constrained by real-valued functions, including the cases with nonlinear constraints. To pursue this, we follow the idea of the value order method which is proposed by Antoniadou (1996),(2007) and Mirman and Ruble (2003) and extend it.

To clarify the essense of the value order method, consider the consumer problem with the standard linear budget constraint. For simplicity, suppose the consumption set of the consumer is $\mathbb{R}^n_+$ then, it is a lattice with respect to the standard Euclidean order. However, the feasible sets, that is the budget sets under two income levels are not strong set comparable, and hence, at least under the Euclidean order, we cannot apply Theorem 2.1. To overcome this difficulty, one can artificially define the partial order on the consumption set in such a way that the budget sets under any two income levels are comparable with respect to the strong set order induced by that order. In other words, the value order method defines the binary relation appropriately for applying the lattice theoretical comparative statics.

On the other hand, thus far, the usage of the value order method is limited to the linear constraint cases. As one of the reasons for this, it is prevalent that constructing the value order as a partial order is, in general, impossible in nonlinear constraint cases. Now that we have the monotone comparative statics which is also valid for preordered sets, it is possible to extend this method to more general cases.

In the following, let X be a set, $f : X \to \mathbb{R}$ is the objective function, and $G : X \to \mathbb{R}$ is the constraint function. We consider the maximization problem

$$\max_{x \in G^{-1}((-\infty,k])} f(x)$$

for some real number k and write the set of maximizar $M(k)$. We compare the solution sets under various k with respect to a complete preorder $\preccurlyeq_c$ on X. As an example of $\preccurlyeq_c$, suppose that $X = \mathbb{R}^n$ and that one attempts to perform comparative statics with respect to the i-th component of the solution, then, $x \preccurlyeq_c y \iff x_i \leq y_i$ for each $x, y \in \mathbb{R}^n$. Now, we state the formal definition of the monotonicity of the solution set. Since, in general, $M(k)$ is not a singleton, the notions of monotonicity are not necessarily unique. Here, we adopt the most canonical definition in the related literature, Mirman and Ruble (2003), Antoniadou (2007), and Quah (2007)[2].

Definition 3.1: The solution set $M(k)$ is pathwisely monotonic if there exists $x \in M(k)$ and $y \in M(k')$ such that $x \preccurlyeq_c y$ for each $k \leq k'$. We write this as $M(k) \leq_p M(k')$.

As we intend to apply Proposition 2.1 to assure the monotonicity defined above, some appropriate preorder must be defined on set X. To define the preorder in a very intuitive fashion, we impose the following condition.

Condition A: Define the set $I_x = \{z \in X \mid x \preccurlyeq_c z \textit{ and } z \preccurlyeq_c x\}$ for each $x \in X$. If $x \preccurlyeq_c y$, then $G(I_x) \leq_a G(I_y)$, where $\leq_a$ denotes the strong set order introduced by the standard Euclidean order on $\mathbb{R}$.

Note that this condition does not require any kind of concavity or supermodularity on the constraint function $G(\cdot)$. Rather, it implies weak monotonicity of G with respect to the comparative criterion $\preccurlyeq_c$. We give the following example to prove that Condition A is not very restrictive.

Example 3.1: Suppose that $X = \mathbb{R}^n$ and $x \preccurlyeq_c y \iff x_i \leq y_i$. If $G(\cdot)$ is increasing and continuous, then Condition A is satisfied. This is shown as follows. Here, we write the supremum and infimum of x and y, a pair of elements in $\mathbb{R}^n$, with respect to the Euclidean order as $x \wedge y$ and $x \vee y$, respectively. Let $x_i < y_i$. If $G(x) \leq G(y)$, the statement is obvious. Suppose that $G(x) > G(y)$. Note that $x \vee y \in I_y$ and $x \wedge y \in I_x$. By increasingness of G, $G(x \vee y) \geq G(x) > G(y)$ and $G(x \wedge y) \leq G(y) < G(x)$. Then, since I_x and I_y are convex and G is continuous, applying the intermediate value theorem, there exists $z \in I_y$ such that $G(z) = G(x)$, and $w \in I_x$ such that $G(w) = G(y)$. This implies that if $x_i \leq y_i$, then $\min\{G(x), G(y)\}$ can be realized by some element of I_x and $\max\{G(x), G(y)\}$ can be realized by some element of I_y, which means $G(I_x) \leq_a G(I_y)$.

Under Condition A, we can define the preorder on X by directly using the "value" of the constraint $G(\cdot)$. Formally, we define it as follows.

[2] Quah (2007) gives another interesting treatment for the comparative statics. Instead of the value order, he uses the joint assumptions of concavity and supermodularity of objective functions to apply Milgrom and Shannon's theorem.

Definition 3.2: The G-value order on X is a preorder $\preccurlyeq_G$ such that

$$x \preccurlyeq_G y \iff x \preccurlyeq_c y \text{ and } G(x) \leq G(y).$$

It is obvious that the G-value order is a preorder on X and not necessarily a partial order. Now, what we have to show is that X is a prelattice under the G-value order, and the strong set comparability of the feasible sets. Fortunately, as long as Condition A holds, we have the following lemmas.

Lemma 3.1: *Under Condition A, X is a prelattice with resprect to the G-value order.*

Lemma 3.2: *Under Condition A, $G^{-1}((-\infty, k]) \leq_{sa} G^{-1}((-\infty, k'])$ for each $k \leq k'$.*

Then, by Proposition 2.1, we have the necessary and sufficient condition for the monotonicity of the solution set with respect to the w-strong set order induced by the G-value order. On the other hand, our goal here is not the monotonicity with respect to that order but the pathwisely monotonic order defined in Definition 3.1. However, we have the following lemma, which asserts that the monotonicity with respect to the w-strong set order implies the pathwise monotonicity.

Lemma 3.3: *If $M(k) \leq_{wa} M(k')$, then $M(k) \leq_p M(k')$.*

Now, we have our main proposition in this section as follows.

Proposition 3.1: *Under Condition A, if f satisfies w-quasisupermodularity with respect to the G-value order, then $M(k)$ has pathwise monotonicity in k.*

The above proposition confirms as well as enhances the flexibility of the lattice theoretical approach for comparative statics, especially, the value order method by Antoniadou and Mirman and Ruble. Indeed, we do not impose any assumptions on the domain of the objective function X. And our requirement for the comparative criterion, $\preccurlyeq_c$, is only to be a complete preorder on X. It should be noted again that Proposition 2.1 plays a fundamental role in our generalization of the value order method. Given that the original theorem by Milgrom and Shannon has been central to the lattice theoretic analyses, its generalization can be expected to serve for the generalization of various results.

Appendix

Proof of Proposition 2.1. (Sufficiency): (a) Since $S \leq_{sa} S'$, $A_{x,y} \subset S'$ and $T_{x,y} \subset S$. By hypothesis, we have $f(x) \geq f(t)$ for all $t \in T_{x,y}$; hence, there exists $a \in A_{x,y}$ satisfying $f(a) \geq f(y)$. This implies that there exists at least one t, which is contained in $M(S)$ since, if not, by the w-quasisupermodularity of f we must have $f(a) > f(y)$, which contradicts the fact that $y \in M(S')$. This asserts that $M(S) \leq_{wa} M(S')$.

(b) Make the counter hypothesis: There is an element $a' \in A_{x,y}$ such that $a' \notin M(S')$. Since $S \leq_{sa} S'$ and $a' \in S'$, thus $f(a') < f(y)$. By s-quasisupermodularity, $f(x) < f(t)$ for some $t \in T_{x,y}$ and $T_{x,y} \subset S$, contradiction.

(Necessity): (a) Suppose $S = \{x\} \cup T_{x,y}$ and $S' = \{y\} \cup A_{x,y}$. Then, $S \leq_{sa} S'$. Let $M(S) \leq_{wa} M(S')$. If $f(x) \geq f(t)$ for all $t \in T_{x,y}$, we have $f(a) \geq f(y)$ for some $a \in A_{x,y}$, whence f has w-quasisupermodularity.

(b) Let S and S' be the same as those in the previous proof, and assume $M(S) \leq_{sa} M(S')$. If $f(x) \geq f(t)$ for all $t \in T_{x,y}$, it must be satisfied $f(a) \geq f(y)$ for all $a \in A_{x,y}$ by our assumption. This implies the s-quasisupermodularity of f. *[Q.E.D.]*

Proof of Lemma 3.1. Let $x, y \in X$ and they are unordered with respect to $\preccurlyeq_G$. Without loss of generality, we can assume $x \preccurlyeq_c y$, $y \not\preccurlyeq_c x$, and $G(x) > G(y)$. By Condition A, there exists $z' \in I_y$ satisfying $G(z') = G(x)$, and $z \in I_x$ satisfying $G(z) = G(y)$. z' and z are a supremum and infimum with respect to the G-value order, respectively. *[Q.E.D.]*

Proof of Lemma 3.2. Suppose $x \in G^{-1}((-\infty, k])$ and $y \in G^{-1}((-\infty, k'])$. Let $a \in A_{x,y}$ and $t \in T_{x,y}$, which is well defined by Lemma 3.1. By the definition, $G(a) = \max\{G(x), G(y)\}$ and $G(t) = \min\{G(x), G(y)\}$. This implies $a \in G^{-1}((-\infty, k'])$ and $t \in G^{-1}((-\infty, k])$. *[Q.E.D.]*

Proof of Lemma 3.3. Suppose $x \in M(k)$ and $y \in M(k')$. By hypothesis, there exists $a \in A_{x,y}$ and $t \in T_{x,y}$ such that $a \in M(k')$ and $t \in M(k)$. It is obvious that $t \preccurlyeq_c a$. *[Q.E.D.]*

Proof of Proposition 3.1. This is obvious by Lemmas 3.1–3.3 and Proposition 2.1. *[Q.E.D.]*

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