E C O N O M I C S   B U L L E T I N

Assignment submarkets with a segment core

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Abstract

The Böhm-Bawerk horse markets are assignment markets with homogeneous goods that are known to have a one-dimensional core. We show here that, although there exist two-sided assignment games with non-homogeneous products and with a segment as a core, the Böhm-Bawerk horse markets are the only ones where every submarket also has a segment as a core.

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1. Introduction: the assignment model

Let M be a finite set of buyers, M' a finite set of sellers and let us denote by m and m' their cardinalities. We may think of the formal model of assignment games as arising from a situation where each seller $j \in M'$ has an object for sale which he valuates in $c_j \in \mathbf{R}_+$ (reservation price of seller j), being $\mathbf{R}_+$ the set of non negative real numbers, while each buyer $i \in M$ wants exactly one indivisible object and places a value of $h_{ij} \in \mathbf{R}_+$ in the object offered by seller j, $h_i = (h_{ij})_{j \in M'}$. Then, if $h = (h_i)_{i \in M}$ and $c = (c_j)_{j \in M'}$, a matrix $A = A(h,c) = (a_{ij})_{(i,j) \in M \times M'}$ is defined, where $a_{ij} = \max\{h_{ij} - c_j, 0\}$ are the potential gains from the trade between i and j. An assignment market is then a triple (M, M', A).

A *matching* $\mu \subseteq M \times M'$ between M and M' is a bijection from some $M_0 \subseteq M$ to some $M'_0 \subseteq M'$ such that $|M_0| = |M'_0| = \min\{|M|, |M'|\}$. We write $(i,j) \in \mu$ as well as $j = \mu(i)$ and $i = \mu^{-1}(j)$. We denote the set of matchings between M and M' by $\mathcal{M}(M, M')$. We say a buyer $i \in M$ is not assigned by μ if $(i,j) \notin \mu$ for all $j \in M'$ (and similarly for sellers).

A matching $\mu \in \mathcal{M}(M, M')$ is *optimal* for the two-sided assignment market (M, M', A) if for all $\mu' \in \mathcal{M}(M, M')$, we have $\sum_{(i,j) \in \mu} a_{ij} \geq \sum_{(i,j) \in \mu'} a_{ij}$, and we denote the set of optimal matchings by $\mathcal{M}^*_A(M, M')$. Given $S \subseteq M$ and $T \subseteq M'$, we denote by $\mathcal{M}(S,T)$ and $\mathcal{M}^*_A(S,T)$ the set of matchings and optimal matchings of the submarket $(S, T, A_{|S \times T})$ defined by the subset S of buyers, the subset T of sellers and the restriction of A to $S \times T$. If $S = \emptyset$ or $T = \emptyset$, then the only matching is $\mu = \emptyset$ and by convention $\sum_{(i,j) \in \emptyset} a_{ij} = 0$.

The above two–sided market can be described by means of a cooperative game (Shapley and Shubik, 1972) where the player set is $M \cup M'$ and the characteristic function is defined by $w_A(S \cup T) = \sum_{(i,j) \in \mu} a_{ij}$, for any $\mu \in \mathcal{M}^*_A(S,T)$.

Shapley and Shubik (1972) prove that the core, $C(w_A)$, of the assignment game $(M \cup M', w_A)$ is nonempty and coincides with the set of *stable outcomes*. This means that given any optimal matching $\mu \in \mathcal{M}^*_A(M, M')$, a payoff vector $(u,v) \in \mathbf{R}^M_+ \times \mathbf{R}^{M'}_+$ is in the core if $u_i + v_j = a_{ij}$ for all $(i,j) \in \mu$, $u_i + v_j \geq a_{ij}$ for all $(i,j) \in M \times M'$, and the payoff to unmatched agents is zero.

Moreover, the core has a lattice structure with two special extreme core allocations: the *buyers–optimal core allocation*, $(\overline{u}, \underline{v})$, where each buyer attains her maximum core payoff, and the *sellers–optimal core allocation*, $(\underline{u}, \overline{v})$, where each seller does.

From Demange (1982) and Leonard (1983) we know that the maximum core payoff of any player coincides with his marginal contribution:

$$\overline{u}_i = w_A(N) - w_A(N \setminus \{i\}) \text{ and } \overline{v}_j = w_A(N) - w_A(N \setminus \{j\}) \tag{1}$$

for all $i \in M$ and all $j \in M'$. As a consequence, for each optimally matched pair (i,j), the minimum core payoffs are

$$\begin{array}{l} \underline{u}_i = w_A(M \cup M' \setminus \{j\}) - w_A(M \cup M' \setminus \{i,j\}) \text{ and} \\ \underline{v}_j = w_A(M \cup M' \setminus \{i\}) - w_A(M \cup M' \setminus \{i,j\}) \,. \end{array} \tag{2}$$

The present paper is devoted to the analysis of those assignment games which have a segment as a core, the segment with extreme points the buyers–optimal and the sellers–optimal core allocations, that is to say, $C(w_A) = [(\underline{u}, \overline{v}), (\overline{u}, \underline{v})]$. We also analyze which are the assignment markets such that this property is inherited by all the submarkets.

2. Assignment markets with a segment as a core

A well known example of an assignment market with a one-dimensional core is the *Böhm-Bawerk horse market* (1891), which is first studied in 1972 from the viewpoint of game theory by Shapley and Shubik (see also Núñez and Rafels, 2005). In this market, each seller has one horse for sale and each buyer wishes to buy one horse and places the same valuation in all the horses available, as they are all alike (we say goods are homogeneous). Let $0 \leq c_1 \leq c_2 \leq \cdots \leq c_{m'}$ be the reservation prices of the sellers and $h_1 \geq h_2 \geq \cdots \geq h_m \geq 0$ the valuations of the buyers. If $h_i < c_j$, no transaction is possible between these two agents but whenever $h_i \geq c_j$, agents i and j can trade and obtain a joint profit of $h_i - c_j$. Thus, the assignment matrix describing this market is defined by $a_{ij} = \max\{h_i - c_j, 0\}$.

It is known from Shapley and Shubik (1972) that the core of the Böhm–Bawerk horse market game consists of a segment, with extreme points the buyers–optimal and the sellers–optimal core allocations. These authors also point out that the assignment matrix of a Böhm-Bawerk horse market satisfies the following property: *in each* 2×2 *submatrix with nonzero entries, the sums of the diagonals are equal.* However, this property is not enough to characterize the matrices defining a Böhm–Bawerk horse market.

A 2×2 assignment matrix defines a Böhm–Bawerk horse market if and only if either two optimal matchings exist ($h_2 \geq c_2$) or there is only one optimal matching but one of the optimally matched pairs has a null outcome ($h_2 < c_2$). Thus, 2×2 matrices defining a Böhm–Bawerk horse market are, up to possible permutations of buyers or sellers,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ with } a_{11} + a_{22} = a_{12} + a_{21} \text{ or } \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix} \text{ with } a_{11} \geq a_{12} + a_{21} \,. \tag{3}$$

It is not difficult to prove that these are the only 2×2 assignment markets with a segment as a core. But this is not the case when the market has more than two agents on each side.

The assignment market with three buyers and three sellers and defined by

$$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

has a one–dimensional core $C(w_{A_1}) = [(1,1,1;0,0,0),(0,0,0;1,1,1)]$, but is not a Böhm–Bawerk horse market. To see that, notice that from $a_{11} = a_{21}$ we deduce that buyers 1 and 2 have the same valuation for the object of seller 1, but this enters in contradiction with $a_{12} \neq a_{22}$.

What is it that characterizes the Böhm-Bawerk horse markets among all assignment markets with a segment as a core? Notice that any submarket of a Böhm-Bawerk horse market is also a Böhm-Bawerk horse market and thus it has a segment as a core. On the other hand, the submarket of (M, M', A_1) defined by the submatrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ has a two-dimensional core.

Next theorem characterizes the Böhm–Bawerk horse markets in terms of its submarkets. Notice first that if one side of the market has only one agent, then trivially the market is a Böhm–Bawerk horse market.

Theorem 1 *Let $(M,\ M',\ A)$ be an assignment market with at least two agents on each side. The following statements are equivalent:*

1. *$(M,\ M',\ A)$ is a Böhm–Bawerk horse market.*
2. *Every 2×2 submarket of $(M,\ M',\ A)$ is a Böhm–Bawerk horse market.*
3. *Every submarket of $(M,\ M',\ A)$ has a segment as a core.*

PROOF: 1) $\Rightarrow$ 3) If (M, M', A) is a Böhm–Bawerk horse market, then all subgames are also Böhm–Bawerk horse markets and their core is a segment.

3) $\Rightarrow$ 2) In particular, every 2×2 submarket has a segment as a core, and this implies every 2×2 submarket is a Böhm-Bawerk horse market.

2) $\Rightarrow$ 1) Let us assume, without loss of generality, that rows and columns have been ordered in such a way that $a_{1j} \geq a_{1j+1}$ for all $j \in \{1, \ldots, m'-1\}$, $a_{i1} \geq a_{i+11}$ for all $i \in \{1, \ldots, m-1\}$ and, moreover, $a_{11} \geq a_{ij}$ for all $i \in M$ and $j \in M'$. Notice that this can always be achieved.

Under the assumption that all 2×2 submatrices define Böhm–Bawerk markets, we claim that the above ordering implies that, for all $i \in M$ and $j \in M'$, $a_{ij} \geq a_{ij'}$ for all $j' \geq j$ and $a_{ij} \geq a_{i'j}$ for all $i' \geq i$.

We prove the first inequality of the claim (the second one is proved analogously). Take $j' > j$ and consider the matrix $A' = \begin{pmatrix} a_{11} & a_{1j} \\ a_{i1} & a_{ij} \end{pmatrix}$. As this matrix defines a Böhm–Bawerk horse market, and because of the given orders in the sets of buyers and sellers, if $a_{1j} = 0$, then $a_{ij} = 0$. But on the other side, as $a_{1j} \geq a_{1j'}$, we obtain $a_{1j'} = 0$ and since matrix $\begin{pmatrix} a_{11} & a_{1j'} \\ a_{i1} & a_{ij'} \end{pmatrix}$ is a Böhm–Bawerk horse market, we deduce that $a_{ij'} = 0$ and thus $a_{ij} \geq a_{ij'}$.

If $a_{1j} > 0$ we will first see that $a_{1j} \geq a_{ij}$. As this is obvious when $a_{ij} = 0$, let us assume $a_{ij} > 0$. Then, since A' is a Böhm–Bawerk horse market, we obtain $a_{11} + a_{ij} = a_{1j} + a_{i1}$, which from $a_{11} \geq a_{i1}$ implies $a_{1j} \geq a_{ij}$.

Now take matrix $A'' = \begin{pmatrix} a_{1j} & a_{1j'} \\ a_{ij} & a_{ij'} \end{pmatrix}$. If $a_{ij'} = 0$, then trivially $a_{ij} \geq a_{ij'}$. If $a_{ij'} > 0$, since A'' is a Böhm–Bawerk horse market, $a_{1j} + a_{ij'} = a_{ij} + a_{1j'}$ which, as $a_{1j} \geq a_{1j'}$, implies $a_{ij} \geq a_{ij'}$.

Once proved the claim, we define valuations for buyers and sellers which show that A is a Böhm–Bawerk horse market.

Define $h_i = a_{i1}$ for all $i \in M$ and $c_j = a_{11} - a_{1j}$ for all $j \in M'$. Let us consider the submarket $A' = \begin{pmatrix} a_{11} & a_{1j} \\ a_{i1} & a_{ij} \end{pmatrix}$ which, by assumption, is a Böhm–Bawerk horse market. If $a_{ij} > 0$, then $A' > 0$ and $a_{11} + a_{ij} = a_{i1} + a_{1j}$, which implies

$$\max\{h_i - c_j, 0\} = \max\{a_{i1} - (a_{11} - a_{1j}), 0\} = \max\{a_{ij}, 0\} = a_{ij}\,.$$

If $a_{ij} = 0$, then $a_{11} \geq a_{1j} + a_{i1}$, which means

$$\max\{h_i - c_j, 0\} = \max\{a_{i1} - (a_{11} - a_{1j}), 0\} = 0 = a_{ij}\,. \qquad \square$$

Let us remark that, as a consequence of the above Theorem, we can easily recognize when a given matrix defines a Böhm-Bawerk horse market just by checking that every 2×2 submarket either has two optimal matchings or its unique optimal matching has a null entry (see expression (3)). Once that is done, if we reorder rows and columns as in the proof of Theorem 1, then, $h_i = a_{i1}$ for all $i \in M$ and $c_j = a_{11} - a_{1j}$ for all $j \in M'$ give a representation of the market as a Böhm-Bawerk horse market.

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