E c o n o m i c s B u l l e t i n

A monotone comparative static result on contract incompleteness

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Abstract

The purpose of this paper is to generalize the incomplete contract model of Bajari and Tadelis (2001) into a continuous model, and to derive the condition under which the monotone comparative statics (MCS) methods can be applied. I will show that a type of single-crossing condition on the isoprobability curves of uncertainty is necessary and su cient for the availability of the MCS methods when a player has a supermodular ex post utility function. In this case, the greater the magnitude of uncertainty, the less his optimal contract allows the project to proceed smoothly.

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1 Introduction

Bajari and Tadelis (2001, henceforth, B-T) have developed an incomplete contract model, and have analyzed the types of contracts in construction industries. They showed that the difference was not based on the asymmetry between builders and owners as the principal-agent theory suggests, rather, it was based on the magnitude of uncertainty they both builders and owners are confronting with.

In B-T's model, an owner who wishes to procure a building prepares the design of the project in advance, and asks a builder to comply with it. Thus, the design can be considered as a contract. The project involves certain uncertainties that are not apparent at the beginning, for example, the condition of the ground, the owner's actual needs, and so on. One extreme way of dealing with such uncertainties is to describe every prescription for every possible event in the design, the builder then proceeds with the project as suggested. However, from a practical viewpoint, it not only is very difficult to anticipate all possible events but also to describe clauses concerning them. Thus, it is natural to include some clauses in advance, and to hold renegotiations among stakeholders when an unforeseen event takes place. B-T analyzed the number of prescriptions that the owner should have in the contract in advance, that is, how specific a contract she (the owner) should prepare *ex ante*. However, they showed that the greater the amount of uncertainty involved, the less specific is the contract that she prefers.

B-T used a discrete model, where they defined a construction project as a combination of the number of possible states T and a probability distribution $(f_t)_{t=1}^{T}$ on them. Moreover, they stated that "project T is more complex than project T'," if the following two conditions are satisfied: (i) $(f_t)_{t=1}^{T}$ dominates $(f'_t)_{t=1}^{T'}$ in the sense of first-order stochastic dominance, (ii) the states in project T are constructed by dividing some state in project T'. However, in their analysis, they assumed that the second condition is automatically implied by the first condition, and treated the number of states T as an index of complexity. Therefore, there was no discussion about the property that the probability distributions must satisfy. My main purpose is to derive such a property in a more general setting.

Since the work of Milgrom and Shannon (1994), the monotone comparative statics (MCS) methods have been widely researched recently. Among the recent works, see Athey (1998) and Athey (2002) for details about the application of the MCS methods to decision-making problems under uncertainty.

This paper is organized as follows. Section 2 presents the model. The set of possible events is assumed to be continuous, and I assume that the player's *ex post* utility function is supermodular. In section 3, I derive the condition under which the MCS methods can be applied to the model. I will also show that the optimal contracts have a monotone property with respect to the magnitude of uncertainty. Section 4 concludes the work. In the appendix, I list the theorems of Topkis (1998) that are used in my proofs.

2 The Model

In this section, I propose a general model of contract incompleteness. This model is an extension of B-T's model. The main purpose is to specify the conditions under which the monotone comparative statics methods can be applied to the model.

A player is to implement a project with uncertainty. The uncertainty will be resolved after the launch of the project, and an event will be brought to realization during the project. The event affects the implementation of the project. Let $t \in \mathbb{R}_+$ be the event that is realized *ex post*. I assume that t is *ex post* verifiable, and is drawn from a family of distribution functions indexed by $\theta \in \Theta$. Let the cumulative distribution functions be $F : \mathbb{R}_+ \times \Theta \mapsto \mathbb{R}_+$. Suppose that F is atomless and is absolutely continuous in t for all θ.

The parameter θ represents the magnitude of *ex ante* uncertainty. The following is an assumption about it.

Assumption **1.** For all $\theta'' \geq \theta'$,

$$F(t, \theta'') \leq F(t, \theta'). \quad \forall t \tag{1}$$

That is, the larger θ becomes, the more uncertain it would be *ex ante.*

The events are ordered from the most probable event $t = 0$ to infinity, based on the likelihood of its occurrence *ex ante*, irrespective of the parameter θ. This assumption is expressed by allowing the density function of the events to be monotonically nonincreasing in t.

Assumption **2.** For all $t'' \geq t'$,

$$F_t(t'', \theta) \leq F_t(t', \theta), \tag{2}$$

where the subscripts represent the derivatives of the functions.

As the *ex post* events can affect the progression of the project, the player may make a list of clauses about them *ex ante.* I assume that there is a cost involved for including clauses about *ex post* events, thus, it is impossible to have a contract that includes clauses for all events *ex ante.* I will later discuss this point in detail. In this list, the player has arrangements that describe prescriptions for each event that he has selected, moreover, he treats θ as given. For example, consider an insurance contract. The buyer has to decide which contract she should buy *before* any accidents take place, however, on the other hand, she knows how uncertain they would be, therefore, depending on her *ex ante* knowledge, she determines which clauses should or should not be included in the contract.

As I have mentioned above, it is costly to preface a contract. I represent the set of events that are selected in the contract by S. S is assumed to be an element in the family of Borel set $\mathscr{B}$ of $\mathbb{R}_+$. The cost of prefacing a contract $S \in \mathscr{B}$ is represented by $C(S) \equiv g(\mu(S))$ for the Lebesgue measure μ and some nondecreasing function g, which is assumed to be twice continuously differentiable, $g' \geq 0$ and $g'' \geq 0$.

When the uncertainty is resolved and some event t is brought about, if the contract contains a clause about t, then the project would progress according to it. On the other hand, if there is no clause about t, the stakeholders have to coordinate their actions with regard to the implementation of the project. Therefore, the *ex post* difficulty of the project depends on whether or not $t \in S$. The difficulty is represented as an *ex post* state

$\omega \in \Omega \equiv \{0, 1\}$. As ω depends on both the *ex ante* contract $S \in \mathscr{B}$ and the *ex post* event $t \in \mathbb{R}_+$, I describe it as a function $h : \mathscr{B} \times \mathbb{R}_+ \mapsto \Omega$, and

$$h(S,t) = \begin{cases} 1 & \text{if } t \in S \\ 0 & \text{if } t \notin S \end{cases}. \tag{3}$$

Finally, I define the player's *ex post* utility. Let $X \subseteq \mathbb{R}$ be a nonempty set. X is a set of the player's *ex post* alternatives. The terminal utility is determined by his choice of x, the *ex post* event t, and the *ex post* state ω. Let his utility function be $u : X \times \mathbb{R}_+ \times \Omega \mapsto \mathbb{R}_+$. The properties of u are assumed as follows.

ASSUMPTION **3.** $u(x, t, \omega)$ is supermodular in $(x, -t, \omega)$.

REMARK **1.** *This assumption implies that the order of the ex post event t not only depends on the likelihood of its occurrence but also on the player's utility function. Although this appears to be too restrictive, what is important in the discussions below is the supermodularity between x and ω. Thus, assuming that the utility does not depend on t does not change the following results.*

3 Analysis

First, I will solve the player's problem, assuming the *ex post* event and state as given. Then, taking this solution into consideration, the *ex ante* optimal contract will be derived.

Given (t, ω), the player seeks an optimal x. The problem is

$$\max_{x \in X} u(x, t, \omega) \tag{4}$$

Let its maximized value be $v(t, \omega)$. The next lemma is derived from the results of Topkis (1998).

LEMMA **1.** [1] *Let $(-T)$ be the reverse-ordered set of $\mathbb{R}_+$. Suppose either*

(i). X is finite or

(ii). X is compact and $u(x, t, \omega)$ is upper semicontinuous in x on X for every (t, ω) on $\mathbb{R}_+ \times \Omega$

is satisfied. Then, $v(t, \omega)$ exists and is supermodular in $(-t, \omega)$ on $(-T) \times \Omega$. Furthermore, there exist a greatest and a least element of

$$\arg\max_{x \in X} u(x, t, \omega),$$

and they are monotonically increasing in $(-t, \omega)$ on $(-T) \times \Omega$.

Proof. When either *(i)* or *(ii)* is satisfied, $u(x, t, \omega)$ is bounded above on X for all (t, ω). And this implies that there always exists a maximum value of $u(x, t, \omega)$ on X for all (t, ω). Therefore, by Theorem A.1, $v(t, \omega)$ is supermodular in $(-t, \omega)$ on $(-T) \times \Omega$.

Moreover, by Theorem A.2, there exist a greatest element and a least element of $\arg\max_{x \in X} u(x, t, \omega)$, and they are increasing in $(-t, \omega)$ on $(-T) \times \Omega$. □

[1] I thank an anonymous referee for suggestions to improve this lemma.

Next, given $v(t,\omega)$ acquired above, define the player's *ex ante* expected utility respective to t and ω. If he selects a set $S \in \mathscr{B}$ as an *ex ante* contract, then the *ex ante* expected utility will become

$$V(S,\theta) \equiv \int_S v(t,1)F_t(t,\theta)dt + \int_{S^C} v(t,0)F_t(t,\theta)dt.$$

He selects a contract S, which maximizes this expected utility. I can show that an interval $[0,\hat{t}]$, which has the origin as an end, maximizes the expected utility with respect to the Lebesgue measure being constant.

Lemma **2.** *For any given $\bar{\mu}$, let $S = [0,\bar{\mu}]$. Then*

$$S \in \arg \max_{\substack{S' \in \mathscr{B} \\ \mu(S')=\bar{\mu}}} V(S',\theta). \tag{5}$$

Thus, the interval $S = [0,\bar{\mu}]$, which begins at 0, maximizes the expected payoff V under the condition that its Lebesgue measure is equal to $\bar{\mu}$.

Proof. Let $t' < \bar{\mu} < t''$ and $S = [0,\bar{\mu}]$. Starting with S, define S' as follows: remove an infinitesimal interval dt at $t = t'$, and add dt at $t = t''$ to S. Then, $\mu(S) = \mu(S') = \bar{\mu}$, and its associated variation dV of V is calculated as

$$\begin{aligned} dV &= v(t'',1)F_t(t'',\theta)dt - v(t'',0)F_t(t'',\theta)dt - (v(t',1)F_t(t',\theta)dt - v(t',0)F_t(t',\theta)dt) \\ &= (v(t'',1) - v(t'',0))F_t(t'',\theta)dt - (v(t',1) - v(t',0))F_t(t',\theta)dt. \end{aligned}$$

This is nonpositive by Assumption 2 and Lemma 1. □

Lemma 2 reveals that I can concentrate on intervals $[0,\hat{t}]$ as *ex ante* contracts. Therefore, I treat either $\hat{t}$ or the probability $p = \text{Prob}(t \le \hat{t})$ as control variables instead of S. Given p and θ, define $\hat{t}(p,\theta)$ as $p = F(\hat{t}(p,\theta),\theta)$. Furthermore, identifying p as $S = [0,\hat{t}(p,\theta)]$, the functions defined on $\mathscr{B}$ will be treated as functions defined on $[0,1]$.

Rewriting the *ex ante* expected utility respective to t and ω as a function of p and θ provides

$$V(p,\theta) \equiv \int_0^{\hat{t}(p,\theta)} v(t,1)F_t(t,\theta)dt + \int_{\hat{t}(p,\theta)}^{\infty} v(t,0)F_t(t,\theta)dt. \tag{6}$$

Before checking the property of this function, I will prove the supermodularity of $\hat{t}(p,\theta)$.

Lemma **3.** *$\hat{t}(p,\theta)$ is increasing in (p,θ) on $[0,1] \times \Theta$. In addition, it is supermodular in (p,θ) on $[0,1] \times \Theta$ if and only if*

$$t'' \ge t' \implies F_t(t',\theta') \ge F_t(t'',\theta''), \quad \forall (t',\theta'),(t'',\theta'') \text{ s.t. } F(t',\theta') = F(t'',\theta'') \tag{7}$$

holds.

Moreover, suppose F is twice continuously differentiable in t, differentiable in θ, and has the cross-derivative $\partial^2 F/\partial t \partial \theta$. Then, $\hat{t}(p,\theta)$ is supermodular in (p,θ) if and only if

$$\frac{\partial}{\partial t}\left(-\frac{F_\theta(t,\theta)}{F_t(t,\theta)}\right) \ge 0. \tag{8}$$

Proof. Since F is nondecreasing in t for all θ, the inverse function $\hat{t}$ is nondecreasing in p. Furthermore, for all $\theta'' \geq \theta'$ and p, it follows that

$$p = F(\hat{t}(p,\theta'),\theta') = F(\hat{t}(p,\theta''),\theta'') \leq F(\hat{t}(p,\theta''),\theta').$$

By Assumption 1. Since F is nondecreasing in t, I obtain $\hat{t}(p,\theta'') \geq \hat{t}(p,\theta')$.

Suppose (7) holds for all (t',θ') and (t'',θ'') such that $F(t',\theta') = F(t'',\theta'')$. F is assumed to be absolutely continuous for all θ, and has its derivative F_t. Since $\hat{t}$ is the inverse function of F, there exists the derivative respective to p for all θ, and it is given by

$$\frac{\partial \hat{t}(p,\theta)}{\partial p} = \frac{1}{F_t(\hat{t}(p,\theta),\theta)}.$$

For all $\theta'' \geq \theta'$, I have $F(\hat{t}(p,\theta''),\theta'') = F(\hat{t}(p,\theta'),\theta') = p$ and $\hat{t}(p,\theta'') \geq \hat{t}(p,\theta')$. By (7), these conditions imply that

$$F_t(\hat{t}(p,\theta'),\theta') \geq F_t(\hat{t}(p,\theta''),\theta'').$$

Therefore,

$$\frac{\partial \hat{t}(p,\theta'')}{\partial p} - \frac{\partial \hat{t}(p,\theta')}{\partial p} = \frac{1}{F_t(\hat{t}(p,\theta''),\theta'')} - \frac{1}{F_t(\hat{t}(p,\theta'),\theta')} \geq 0,$$

that is, $\hat{t}$ is supermodular in (p,θ). The opposite direction is straightforward and I skip the proof.

Next, I will analyze the case in which some additional differentiability conditions hold. Suppose F is twice continuously differentiable in t, differentiable in θ, and has the cross-derivative $F_{t\theta}$. Let $p = F(\hat{t}(p,\theta),\theta)$. Differentiating both sides of this equation by p and θ, I obtain

$$\begin{aligned} 1 &= F_t(\hat{t}(p,\theta),\theta)\frac{\partial \hat{t}(p,\theta)}{\partial p}, \\ 0 &= F_t(\hat{t}(p,\theta),\theta)\frac{\partial \hat{t}(p,\theta)}{\partial \theta} + F_\theta(\hat{t}(p,\theta),\theta). \end{aligned}$$

These equalities imply that

$$\frac{\partial \hat{t}(p,\theta)}{\partial p} = \frac{1}{F_t(\hat{t}(p,\theta),\theta)} \geq 0, \tag{9}$$

$$\frac{\partial \hat{t}(p,\theta)}{\partial \theta} = -\frac{F_\theta(\hat{t}(p,\theta),\theta)}{F_t(\hat{t}(p,\theta),\theta)} \geq 0. \tag{10}$$

To prove the supermodularity of $\hat{t}$, calculating the cross-derivative of $\hat{t}(p,\theta)$ leads to the following.

$$\begin{aligned}\frac{\partial^2 \hat{t}(p,\theta)}{\partial\theta\partial p} &= \frac{\partial}{\partial\theta}\left(\frac{1}{F_t(\hat{t}(p,\theta),\theta)}\right)\\ &= -\frac{1}{F_t(\hat{t}(p,\theta),\theta)^2}\left(F_{tt}(\hat{t}(p,\theta),\theta)\frac{\partial\hat{t}(p,\theta)}{\partial\theta} + F_{t\theta}(\hat{t}(p,\theta),\theta)\right)\\ &= -\frac{1}{F_t(\hat{t}(p,\theta),\theta)^2}\\ &\quad\times\left\{F_{tt}(\hat{t}(p,\theta),\theta)\left(-\frac{F_\theta(\hat{t}(p,\theta),\theta)}{F_t(\hat{t}(p,\theta),\theta)}\right) + F_{\theta t}(\hat{t}(p,\theta),\theta)\right\}.\end{aligned}$$

The expression in the curly brackets is nonpositive for any (p,θ) if and only if (8) holds for any (t,θ). This concludes the proof. □

The supermodularity of the cost function $C(p,\theta) = g(\hat{t}(p,\theta))$ immediately follows this lemma.

I will discuss some points about the conditions in this lemma. First, to identify the equivalence between (7) and (8), it will suffice to differentiate $F_t(\hat{t}(p,\theta),\theta)$ in θ. Next, (7) can be interpreted as follows. Suppose, given a probability p, the player has to prepare a contract in which the probability of including a clause about events *ex post* is exactly p. If he attempts to prepare this contract more specifically and include some additional clauses, then the greater the magnitude of uncertainty, the less the increment of the *ex ante* probability. In another words, if he wishes to increase the *ex ante* probability by some fixed degree, then the more greater the magnitude of uncertainty *ex ante*, the larger is the number of clauses that he must include. Finally, note that (8) implies that the slope of the isoprobability curve of F is nondecreasing in t. Therefore, this condition can be regarded as a type of single-crossing condition about the isoprobability curve. However, they never cross each other on their domain.

Using the supermodularity of $\hat{t}$ derived as above, I will prove the supermodularity of V.

Lemma 4. *If $\hat{t}$ is supermodular in (p,θ) on $[0,1]\times\Theta$, then $V(p,\theta)$ is supermodular in $(-p,\theta)$ on $[0,1]\times\Theta$.*

Proof. Differentiating $V(p,\theta)$ with p provides

$$\begin{aligned}\frac{\partial V(p,\theta)}{\partial p} &= (v(\hat{t},1)F_t(\hat{t}(p,\theta),\theta) - v(\hat{t},0)F_t(\hat{t}(p,\theta),\theta))\frac{\partial\hat{t}(p,\theta)}{\partial p}\\ &= (v(\hat{t}(p,\theta),1)F_t(\hat{t}(p,\theta),\theta) - v(\hat{t}(p,\theta),0)F_t(\hat{t}(p,\theta),\theta))\left(\frac{1}{F_t(\hat{t}(p,\theta),\theta)}\right)\\ &= v(\hat{t}(p,\theta),1) - v(\hat{t}(p,\theta),0).\end{aligned}$$

For any $\theta'' \geq \theta'$, I have

$$\begin{aligned}&\frac{\partial V(p,\theta'')}{\partial p} - \frac{\partial V(p,\theta')}{\partial p}\\ &\quad= (v(\hat{t}(p,\theta''),1) - v(\hat{t}(p,\theta''),0)) - (v(\hat{t}(p,\theta'),1) - v(\hat{t}(p,\theta'),0)) \leq 0.\end{aligned}$$

The last inequality is derived from the fact that $\hat{t}$ is increasing in θ and that v is supermodular in $(-t, \omega)$. □

I shall now prove the main theorem.

THEOREM 1. *Suppose either* (7) *or* (8) *is satisfied. Then, for all* θ*, there exist a greatest and a least element of*

$$\arg\max_{p \in [0,1]} V(p, \theta) - C(p, \theta), \tag{11}$$

and they are monotonically nonincreasing in θ*.*

Proof. Note that $\arg\max_{p\in[0,1]} V(p, \theta) - C(p, \theta)$ is nonempty. By Lemmas 3 and 4, $V(p, \theta) - C(p, \theta)$ is supermodular in $(-p, \theta)$. Hence, by Theorem A.2, there exist a greatest and a least element of $\arg\max_{p\in[0,1]} V(p, \theta) - C(p, \theta)$, and they are monotonically nonincreasing in θ. □

Intuitively speaking, if the single-crossing condition on the uncertainty is satisfied, the greater the magnitude of uncertainty, the less the optimal choice of the probability with which an *ex ante* contract contains a clause about an *ex post* event. Note that the increase of θ affects the optimal choice of the probability in two ways. First, for any given *ex ante* contract, it reduces the probability with which the events listed in the contract would happen. Second, the greater the magnitude of uncertainty, the more clauses you need to improve an *ex ante* contract, and this leads to the higher marginal cost of writing the contract. As a result, the optimal choice of the probability is nonincreasing function of the magnitude of uncertainty.

4 Conclusion

In this paper, I extended B-T's model into a continuous model, and derived a condition under which the MCS methods can be applied. The intuitive interpretation of the condition is also provided. The main result indicated that if the supermodularity of *ex post* utility function is assumed, then a type of single-crossing condition of the isoprobability curves of uncertainty is necessary and sufficient.

A Appendix

In this appendix, I list the theorems of Topkis (1998) that are used in my proofs.

The main result on the maximization of supermodular functions is the following theorem.

THEOREM **A.1** (Topkis (1998))**.** *If X and T are lattices, S is a sublattice of $X \times T$, $f : X \times T \mapsto \mathbb{R}$ is supermodular in (x, t) on S, S_t is the section of S at t in T, and $g(t) = \sup_{x \in S_t} f(x, t)$ is finite on the projection $\Pi_T S$ of S on T, then $g(t)$ is supermodular on $\Pi_T S$.*

The next theorem assures the monotonicity of solutions.

THEOREM **A.2** (Topkis (1998))**.** *Suppose that X is a nonempty lattice, T is a partially ordered set, $f : X \times T \mapsto \mathbb{R}$ is supermodular in x on X for each t in T, and f is supermodular in (x, t) on $X \times T$. If either*

(i). X is finite or

(ii). X is a compact subset of $\mathbb{R}^n$ and f is upper semicontinuous in x on X for each t in T,

then each $\arg\max_{x \in X} f(x, t)$ (being either finite or a compact subset of $\mathbb{R}^n$) is a nonempty sublattice of X with a greatest element and a least element, and the greatest (least) element of $\arg\max_{x \in X} f(x, t)$ is increasing in t on T.

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