An exact consumption rule with liquidity constraints and stochastic income

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Abstract

This model provides a closed form solution to the problem of liquidity constrained consumption with stochastic income. To keep the model tractable we employ a quadratic utility function. Income follows a geometric Brownian motion. The analytical solution exhibits a smooth, non linear, relation between consumption and income along the optimizing path even when the constraint binds. This outcome confirms the assertions in the literature that even liquidity constrained consumers may satisfy the standard Euler equation. But, in our model this result emerges from the analytical solution.

1   Introduction

Closed form solutions to the problem of liquidity constrained consumption cannot, in general, be derived even without uncertainty. Thus, numerical solutions have become the standard tool for modelling consumption under constraints and uncertainty (Zeldes 1989a, 1989b; Deaton, 1991; Carroll and Kimball, 2001). Recently, Park (2006) has found a closed form solution to the constrained consumption function in continuos time, but under perfect foresight. His work is very close to Seater (1997) who was the first to provide an optimal control solution to the liquidity constraint problem under certainty. In his setup the relationship between consumption and income is *non* linear. Our model can be seen as an update of these papers. We provide a closed form solution to the problem of consumption with liquidity constraints and stochastic income in continuos time. We formally define the problem as follows. A quadratic utility function is employed to keep the model tractable. The theory of Brownian motion and its control is used to solve the consumption problem. Our closed form solution exhibits a smooth *non* linear relation between consumption and stochastic income along the entire optimizing path *even when* the constraint binds. This result confirms the assertion in literature that even liquidity constrained consumers may satisfy the standard Euler equation. But, in our model this result emerges from the analytical solution.

2   The model

Consumer acts in an imperfect capital market, where financing constraints are simple quantity restrictions on wealth. We assume that the interest rate r is constant over time. δ is the consumer's discount rate. As in Seater (1997) we confine our attention to the case $\delta = r$.[1] The dynamics of income follows a continuous-time random walk. We assume an infinite horizon. The consumer's problem is

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[1] Under the assumption $r = \delta$ it is easy to show that the results of Deaton (1991) on falling consumption in presence of liquidity constraints "arise because Deaton assumes $r < \delta$, *not* because of the liquidity constraint. The presence of liquidity constraint in his model actually causes c to fall less rapidly over time than it would without the constraint" (Seater 1997, p.131).

$$\max_{c_s} \int_t^\infty E_t \{u(c_s)\} e^{-r(s-t)} ds \tag{1}$$

subject to the constraint $\dot{a}_s = ra_s + y_s - c_s$, where c_s and y_s are consumption and income processes and a_s is wealth. The transversality condition is $\lim_{s\to\infty} a_s e^{-r(s-t)} = 0$. If utility function is quadratic, the Euler equation implies constant expected consumption over time $E_t(c_s) = c_t$. Solving the budget constraint we get

$$c_t = ra_t + r \int_t^\infty E_t \{y_s\} e^{-r(s-t)} ds \tag{2}$$

with

$$h_t = \int_t^\infty E_t \{y_s\} e^{-r(s-t)} ds \tag{3}$$

the discounted value of the expected future income.

We assume that income is distributed lognormally

$$dy_t = \mu y_t dt + \sigma y_t dz \tag{4}$$

where μ is the growth rate and σ is the volatility, with dz being a random variable with zero mean and variance dt. This is a geometric Brownian motion with the property that percentage changes in y_t are normally distributed. The expected value of income is

$$E_t \{y_s\} = y_t + \left(\mu - \frac{\sigma^2}{2}\right)(s - t) \tag{5}$$

with variance $\sigma^2(s - t)$. Substituting (5) in (3) it gives the solution

$$h_t = \frac{y_t}{r} + \frac{\mu - \frac{\sigma^2}{2}}{r^2} \tag{6}$$

and using (6) in (2) we see that current changes in income imply changes of current consumption in the same direction, with variance affecting negatively its value.

The y_t process drifts upwards but also fluctuates randomly, and this variation affects actual consumption. In practice there are restriction on the range. Following Besley (1995) it may be that

$$0 \leq y_t \leq y_{\max} \tag{7}$$

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Income cannot fall below the floor $y_{\min} = 0$ and it is bounded above because it cannot rise beyond a certain level. We require that assets be non-negative at all times

$$a_s \geq 0 \qquad \text{with } s > t \tag{8}$$

This is a set of restrictions which inhibits the consumer to anticipate the future income. Note that when $0 < y_t < y_{\max}$ the inability to borrow does not imply inability to save. Within the bounds the income can change freely, but once the barrier $y_{\max}$ has been reached, its value can only decrease randomly with probability 1. Thus, $y_{\max}$ is a reflecting barrier. We refer to $y_{\max}$ as the "liquidity constraint".

3 Constraints and uncertainty

The model admits a closed form solution. Consumption is determined by the discounted value of income (3). But, expression (3) is a function $h_t = h(y_t)$ so that to calculate its variation we must compute the differentials of such function w.r.t. income. By Ito's lemma we have

$$\frac{1}{dt} E_t (dh) = \mu y h'(y_t) + \frac{1}{2} h''(y_t) y_t^2 \sigma^2$$

where $h'(y_t)$ and $h''(y_t)$ denote the partial derivatives. From the differential form of (3)

$$r h(y_t) = y_t + \frac{1}{dt} E_t (dh)$$

substituting for the expectation we obtain the second order differential equation of $h(y_t)$

$$\frac{1}{2} h''(y_t) \sigma^2 y_t^2 + h'(y_t) \mu y_t + y_t - r h(y_t) = 0 \tag{9}$$

Note that the expression in (6) is a particular solution of $h_t = h(y_t)$. So, all solutions of (9) are a linear combination of (6) with solutions of its homogeneous part

$$h(y_t) = B_1 y_t^{\alpha_1} + B_2 y_t^{\alpha_2} \tag{10}$$

where B_1 and B_2 are two constants to be determined from the boundary conditions, and $\alpha_1 > 0$ and $\alpha_2 < 0$ are the roots of the characteristic equation

$$\frac{1}{2} \alpha^2 \sigma^2 + \left(\mu - \frac{\sigma^2}{2} \right) \alpha - r = 0$$

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The boundary condition (7) implies that $B_2 = 0$ to avoid that consumption becomes bigger and bigger as income tends to zero. The general solution of the integral (3) can, thus, be written as

$$h(y_t) = \frac{y_t}{r} + \frac{\mu - \frac{\sigma^2}{2}}{r^2} + B_1 y_t^{\alpha_1}$$

3.1 Switching between regimes

To solve this problem we look for a rule in which consumption is a function $c_t = c(y_t)$. Using the previous result we may write (2) as

$$c(y_t) = y_t + \frac{\mu - \frac{\sigma^2}{2}}{r} + rB_1 y_t^{\alpha_1} \tag{11}$$

which is the general solution of $c(y_t)$ under the assumption $a_t = 0$ and $\delta = r$.

If the constraint is absent, the constant B_1 must be set to zero because no effective liquidity constraint occurs when income tends to $+\infty$. However, we get that the variance of income σ^2 reduces the level of consumption for a given y_t even with the quadratic utility function.

If the constraint limits the random increase of income, the upper bound impinges upon consumption. In this scenario, B_1 exerts its effects on consumption even in the intermediate phases before the constraint binds. Indeed, as long as y_t lies within the range its evolution is described by (4). Once y_t reaches $y_{\max}$ its process changes since income can only decrease randomly. As a consequence, the level of consumption approaches to its maximum value $c_{\max}$ when income approaches to the threshold $y_{\max}$. Also, consumption decreases randomly when income falls down.

Any rational agent anticipates this process, and finds optimal to select a policy which accounts for constraints at the outset. To maximize his expected utility, the forward-looking consumer smooths consumption along the entire path even in the neighborhood of $c_{\max}$. Whether this condition were violated the consumption function $c(y_t)$ would have no derivative in $y_{\max}$, implying a violation of the Euler equation along the optimal *switching* path.[2] The

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[2]Saltari and Travaglini (2006) use this same principle to study the behavior of firms which are neither always constrained nor always unconstrained. They calculate the optimal investment path switching between regimes. Their analytical solution shows that the future financing constraints affect the optimal investment policy of firms at the outset.
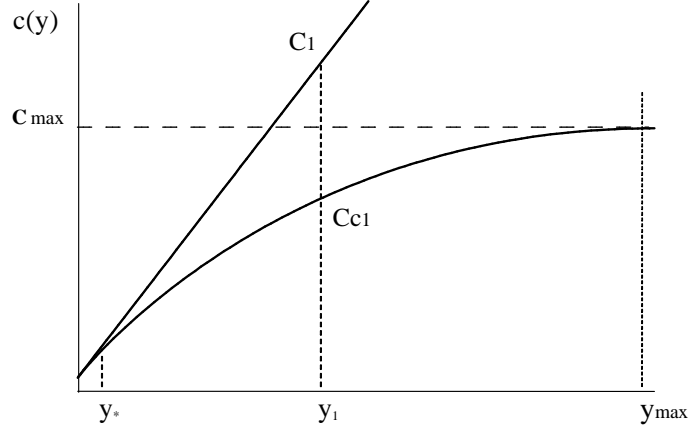
Figure 1: The consumption function with stochastic income and liquidity constraints

boundary condition corresponding to this control is

$$c'(y_{\max}) = 0 \qquad (12)$$

Expression (12) is sometimes called *smooth pasting condition* and it is sufficient to fix the constant B_1. Applying (12) to (11) we get

$$c(y_t) = y_t + \frac{\mu - \frac{\sigma^2}{2}}{r} - \left\{ \frac{1}{\alpha_1 \left(y_{\max}\right)^{\alpha_1 - 1}} \right\} y_t^{\alpha_1} \qquad (13)$$

Figure 1 shows both the linear function $y_t + \frac{\mu - \frac{\sigma^2}{2}}{r}$, and the locus representing the *non* linear functional form $c(y_t)$ which is tangent at $c_{\max}$ when the liquidity constraint binds.

Note that when $y_t < y^*$ the level of consumption rises along the straight line $y_t + \frac{\mu - \frac{\sigma^2}{2}}{r}$. Once, however, income passes this critical value the curve becomes concave smoothly since the consumer anticipates the effect of the constraint: the consumer perceives closeness to the upper bound as an exacerbation of the liquidity constraint becoming more reluctant to consume.

Two are the consequences of this behavior.

Firstly, note that the precautionary component of saving rises as the value of income is over the value y^*. But, for $y < y^*$ saving, $s_t = y_t - c_t$, is given by the expression

$$s_t = \frac{1}{r}\left(\frac{\sigma^2}{2} - \mu\right)$$

that is, saving rises with uncertainty (precautionary effect) relative to the drift of income (income effect) even when current income is low.

Secondly, when $y_t > y^*$ consumption rises more slowly than income, and saving increases as well. Figure 1 illustrates what happens when income is equal to y_1. If consumer behaves in a myopic way consumption would be C_1 on the straight line. But, the forward-looking consumer anticipates that income will never move outside $y_{\max}$, and that the closer is the liquidity constraint the higher is the probability that income will be lower in the future. This bearish expectation affects consumption, reducing its current level along the concave curve in Cc_1. The distance between this curve and the straight line provides a measure of the additional saving. Hence, the propensity to consume is much greater at low levels of income than at high level.

Finally, note that the more uncertain is income, the less is spent and the more is saved. Differentiating (13) totally with respect to σ^2 we get $c_{\sigma^2} < 0$. The greater is the amount of uncertainty the larger is the wedge between the straight line and the function $c(y_t)$, that is, the larger is saving to smooth consumption in bad times.

3.2   Consumption absent uncertainty

Assume that uncertainty is absent. Yet, when $\sigma = 0$ the effect of the upper bound remains. The income process becomes

$$\frac{dy_t}{y_t} = \mu dt$$

Let's denote the initial income as y_0, so that over time $y_t = y_0 e^{\mu t}$. Consider the following dynamics for y_t

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\begin{aligned}
y_t &= y_0 e^{\mu t} &&\text{when} &&t \leq T = \frac{\ln y_{\max} - \ln y_0}{\mu}\\
y_t &= y_{\max} &&\text{otherwise}
\end{aligned}
$$

For $0 \leq t \leq T$ income grows at the rate μ, then it remains for ever to the upper bound when $y_{\max}$ is realized. Now, $y_{\max}$ is an absorbing barrier. The

changes of consumption over time can be expressed through the changes of y_t because $c_t = c(y_t)$. Also consumption is a continuos function of time because c_t is a continuos function of y_t, and y_t is a continuos function of time. Thus, as long as $t < T$, changes in c_t can be written as

$$\frac{dc_t}{dt} = c'(y_t)\frac{dy}{dt} = c'(y_t)\mu y_t$$

For $t < T$ the differential equation is

$$c(y_t) = y_t + c'(y_t)\mu y_t$$

with solution

$$c(y_t) = \frac{y_t}{1-\mu} + By_t^{\frac{1}{\mu}} \tag{14}$$

To determine the constant B, note that as time approaches T, y_t tends to $y_{\max}$. Given, however, that consumption depends on income the function $c(y_t)$ also tends to its maximum value $c_{\max}$ as time approaches T. Thus, as $c(y_t)$ approaches the bound its variation with respect to income tends to zero. As before, this implies

$$c'(y_{\max}) = 0$$

that is $B = -\dfrac{\mu}{(1-\mu)(y_{\max})^{\frac{1}{\mu}-1}} < 0$. Finally, substituting the expression of B in (14) we obtain the explicit consumption function

$$c(y_t) = \frac{y_t}{1-\mu} - \left\{\frac{\mu}{(1-\mu)(y_{\max})^{\frac{1}{\mu}-1}}\right\} y_t^{\frac{1}{\mu}} \tag{15}$$

This is a concave function with respect to y_t. Given an initial value for income, any future increase (or decrease) in income, implies a corresponding increase (or decrease) in current consumption. A forward-looking consumer will anticipate this trend so that it is not surprising that as y_t tends to $y_{\max}$ consumption converges smoothly to $c_{\max}$, becoming tangent at this value in such a way as to satisfy the Euler equation. Finally, note that the solution (15) does not depend on σ. Hence, the concavity of the consumption function is induced by the anticipated effect of the latent constraint, and saving is driven exclusively by the optimal constrained consumption smoothing.

4 Conclusions

In this paper we have provided a closed form solution to the problem of consumption with stochastic income and liquidity constraints. To derive the analytical solution we have used a quadratic utility function and a geometric Brownian motion for income. The logic of the model is that consumers free from restrictions at the current time find optimal to make sequential decisions in order to achieve a coherent and optimal consumption plan at the outset of the planning horizon. This forward-looking behavior gives rise to a concave consumption rule with the variance of income affecting consumption in each period. Further, if income is deterministic, the consumption function remains concave. Consumption is sensitive only to what consumer believes about future financing constraints, and concavity is the consequence of the optimal constrained consumption behavior.

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