E C O N O M I C S B U L L E T I N

On the strategic non-complementarity of complements

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Abstract

This paper examines the equilibrium provision of a public good if the private monetary contributions of identical agents are (im)pure complements. To reconcile complementarity in contributions with the apparent substitutability of monetary payments, we assume a setup with multiple inputs into a complementary production function. This paper proves the uniqueness and symmetry of the equilibrium for any impure complementarity if each agent is permitted to contribute to any input; in the equilibrium, contributions are strategic substitutes. Only pure complementarity exhibits multiple equilibria, where contributions are either strategic substitutes or strategic complements.

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1. Introduction

This paper revisits the strategic effects of complementarities between private contributions to a public good. If money is legal tender, payments are, by definition, pure substitutes, so there must be a transformation preventing the contributions of different agents from being pure substitutes. One interpretation of the transformation is a production function exhibiting complementarity between multiple inputs.

With multiple inputs, one can examine a class of games where the strategy set of each agent entails the vectors of contributions to all inputs. This setting has a tradition in the literature on in-kind transfers to weaker- and weakest-link public goods (Sandler, Vicary 2001; Vicary, Sandler 2002). In-kind subsidies to non-domestic inputs are indeed observed, in particular for strong complementarities (e.g., smallpox has been eradicated by financing organised by the WHO, see Barrett 2003).

Extending the strategy space of an agent implies that his or her contribution to an input has two different functions: it serves as i) an (im)pure *complement* to contributions to *other* inputs, and as ii) a pure *substitute* to contributions to the *same* input. This has interesting strategic consequences. In this paper, I compare these effects with Cornes (1993) who extended Bergström's *et al.* (1992) seminal proof of the uniqueness of the Nash equilibrium in a symmetric population on impurely complementary aggregations. In the class of the constant elasticity of substitution (CES) aggregations, Cornes proved that the equilibrium is not only unique, but also identical irrespective of complementarity. This paper follows up in demonstrating that uniqueness rests predominantly on the impureness of complementarity.

The paper proceeds as follows. Section 2 defines the standard setup with a normal private good and a normal public good, imposing the CES aggregation of inputs into the public good. Section 3 examines impure complements (weaker-link aggregations): It proves the unique Nash equilibrium but disproves the invariance of the equilibrium to the type of aggregation. This equilibrium is compared with the equilibrium in Cornes (1993), where contributions are restricted into a single input. Section 4 examines the pure complements (the weakest-link aggregation): It yields a multiple equilibria, where contributions are either strategic substitutes or strategic complements. In addition, it explores two effects of in-kind transfers: on one hand, some Pareto-dominant equilibria disappear, and, on the other hand, a new distributional conflict emerges. Section 5 concludes this paper.

2. Assumptions

We have $n \in \mathcal{N}$ identical individuals endowed with sufficiently large money income $y > 0$, and $m \in \mathcal{N}$ inputs into a public good. The individual $i \in \{1, \ldots, n\}$ contributes to an input $j \in \{1, \ldots, m\}$ an amount $g_j^i \in \mathcal{R}^+$, where the unit cost of the input is $p_j > 0$. The total amount of input j is $g_j \equiv \sum_i g_j^i$. Inputs are aggregated by a symmetric production function with a constant elasticity of substitution:

$$G(g_1, \ldots, g_m) = \left[\sum_{j=1}^{m} \frac{(g_j)^\psi}{m} \right]^{1/\psi}$$

Like in Cornes (1993), we restrict attention to complementarities, i.e. $\psi < 1$. Pure substitutes corresponding to $\psi = 1$ are strategically trivial, and better-shot aggregations with $\psi > 1$ are strategically completely different. Impure complements are $-\infty < \psi < 1$, and pure complements $\psi = -\infty$. The individual's budget constraint is $c^i + \sum_j g_j^i \leq y$, where $c^i \geq 0$ is private consumption, and the utility function is $u^i = U^i(c_i, G)$. Both private and public goods are normal goods. Hence, once we introduce the marginal rate of substitution between private and public goods as S^i, we have

$$S^i \equiv \frac{\partial U^i(c^i, G)/\partial c^i}{\partial U^i(c^i, G)/\partial G}, \qquad \frac{\partial S_i}{\partial c^i} < 0, \frac{\partial S_i}{\partial G} > 0. \tag{1}$$

Let the marginal product of the input j be G_j, $G_j \equiv \partial G(\cdot)/\partial g_j$, and the marginal rate of transformation between inputs j and k be $T_{j,k}$. Then:

$$T_{j,k}(g_1, \ldots, g_m) \equiv \frac{G_j}{G_k} = \left(\frac{g_k}{g_j} \right)^{1-\psi} \tag{2}$$

For impure complements, the marginal rate of transformation is a continuous function over the amounts of inputs, $T_{j,k} \in \mathcal{R}^+$. For pure complements, the situation is different: $T_{j,k} = 0$ if $g_j > g_k$, $T_{j,k} = \infty$ if $g_j < g_k$, or $T_{j,k}$ is undefined.

3. Impure complements

Since for impure complements the marginal rate of transformation $T_{j,k}$ is continuous in $g_j \times g_k$, we may expect that the usual interior condition of optimal production holds with equality. This is captured in Lemma 1.

Lemma 1 *For impure complements, $T_{j,k} = p_j/p_k$ holds $\forall j, k \in \{1, \ldots, m\}$.*

Proof By contradiction. First, we eliminate $G = 0$ on the basis of the normality of the public good. Now, it is easy to find that $\forall j : g_j > 0$. If not, and $g_k = 0$ and $g_j > 0$ exist, then this must be a corner implying $T_{j,k} \geq p_j/p_k$ (all individuals have re-optimised from k-th toward j-th input). Yet, imposing g_k into Equation 2 (2) yields $T_{j,k} = (g_k/g_j)^{1-\psi} = 0$. In total, $T_{j,k} \geq p_j/p_k > 0 = T_{j,k}$, which is a contradiction. □

Lemma 1 helps to identify that the unique equilibrium is symmetric in both private and public consumption, and the only arbitrary variable is the vector of divisions of costs across individuals.

Proposition 1 *For impure complements, the Nash equilibrium is symmetric and unique, and $S^i = G^i/p_j, \forall i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}$.*

Proof For all individuals, the normality of a private good implies $c^i > 0$. The normality of the public good yields $G > 0$, hence there must be an individual i who contributes to at least one input. From Lemma 1, the contributor is in an equilibrium that is indifferent between the inputs he or she pays. This implies that for the optimising contributor, the marginal rate of substitution between public and private consumption equals the marginal rates of transformation between all inputs and the private good, $S^i = G_1/p_1 = \cdots = G_m/p_m$.

The next step is to also prove this condition for any non-contributor l. If not and $S^l < G_j/p_j$, then the non-contributor l improves individual welfare by increasing g_j^l. If $S^l > G_j/p_j$, then re-optimisation is not feasible ($g_j^l = 0$). However, by symmetry, the non-contributor consumes more private goods than the contributor, $c^l > c^i$, and by (1), $S^l < S^i$. This contradicts $S^l > G_j/p_j = S^i$. Hence, the symmetry holds in equilibrium, $S^i = S^k$, and $c^1 = \cdots = c^n$.

Next we prove uniqueness. Suppose, on the contrary, there are two symmetric equilibria $0 < G' < G$, where $\kappa \equiv G'/G < 1$. Since the CES function is homogenous of degree one and the marginal rates of transformation are identical in both equilibria, we need $g_j(G') = \kappa g_j(G)$ to satisfy Lemma 1. The marginal product of an input j accordingly rewrites $G_j(G') = \kappa^{(\psi-1)/\psi} G_j(G) > G_j(G)$.

Since $G' < G$, we have $c' > c$. By Equation 2 (1), the marginal rate of the substitution of a representative individual decreases in c, thus $S(G') < S(G)$. If G is equilibrium, then $S(G) = G_j(G)/p_j$, and $S(G') < S(G) = G_j(G)/p_j < G_j(G')/p_j$. This contradicts the necessary condition for equilibrium, $S(G') = G_j(G')/p_j$. □

Notice briefly that unlike in Cornes (1993, p. 262), the unique Nash equilibrium is not invariant to ψ. In the optimum for a representative individual, $S = G_j/p_j$. To have the Nash equilibrium constant in ψ, we need both S and G_j to be constant in ψ. However, the marginal product in the equilibrium apparently depends on ψ. Consider, for example, a two-input case. For $\psi = 0$ (geometric mean), $G_1 = \sqrt{p_1/p_2}/2$. For $\psi = 1/2$, $G_1 = (1 + p_1/p_2)/4$.

4. Pure complements

The usual reason to disregard pure complements $G = \min\{g_1, \ldots, g_m\}$ as uninteresting is that in the classic symmetric case, the Pareto-efficient equilibrium is unique and the Pareto-dominant overtakes all other equilibria. The Pareto-inferior equilibria can be justified only by a coordination failure that is nevertheless likely to be avoided by pre-play communication. Also, experiments in coordination games point to a low incidence of coordination failure (Devetag, Ortmann 2007).

With the possibility to pay for all inputs, one should revisit the case of pure complements. Discrete marginal products, $G_j \in \{0, 1\}$, indicate that the marginal cost of the public good is not continuous. For instance, to boost G, one may firstly need to pay a single missing input, then two inputs and so on up to all m inputs. This step-wise character may give rise to complex strategic properties.

We demonstrate the non-uniqueness of Nash equilibria for pure complements. As a corollary, we classify the equilibria by the strategic effect of contributions. C-type equilibria exhibit strategic complementarity, and involve the classic possibility of coordination failure. These are also known as equilibria with a full specialisation (cf. Vicary, Sandler 2002). S-type equilibria exhibit strategic substitutability, and do not involve any coordination failure.

Proposition 2 *For pure complements, the set of Nash equilibria is not necessarily a singleton.*

Proof Consider an economy with two people and two inputs, $n = m = 2$, prices $p_1 \geq p_2$, and a quasilinear utility, $u = c + H(G)$, where $H(\cdot)$ is an increasing and concave function, $H_G(G) \equiv \partial H(G)/\partial G$, and by the Inada condition, $\lim_{G \to 0^+} H_G(G) = +\infty$. It is now useful to define the optimal provision depending on the inputs paid on the margin. The marginal benefit of the public good is $\partial u(c, G)/\partial G = H_G(G)$; the marginal cost is p_1 only if the extra input 1 must be paid on the margin, p_2 only if the extra input 2 must be paid, and $p_1 + p_2$ if both inputs must be paid on the margin.

The optimal amounts satisfying the marginal conditions are implicitly defined as $H_G(c) = p_2$, $H_G(e) = p_1$ and $H_G(b) = p_1 + p_2$, where $c \geq e > b$ denote paying cheap (c), expensive (e), or both (b) inputs. Now, it is easy to characterise the full set of Nash equilibria:

S-type In this equilibria, $G = g_1^1 + g_1^2 = g_2^1 + g_2^2 = b$. Here, each individual who expects payments of another individual is willing to provide additional contributions *exactly* up to the level $G = b$. Hence, contributions are strategic substitutes. Any further increase in $G > b$ would be suboptimal, since it yields marginal cost $p_1 + p_2 = H_G(b) > H_G(G)$.

C-type In this equilibria, $b < G \leq e$. Either $(g_1^1, g_2^1, g_1^2, g_2^2) = (0, G, G, 0)$, or $(g_1^1, g_2^1, g_1^2, g_2^2) = (G, 0, 0, G)$. Here, each individual who expects a sufficiently large payment of the other player is willing to *match*the contribution of the other player by paying the missing input only. Hence, contributions are strategic complements. This holds up to level e, since for $G > e$, the player who pays the more expensive good bears the prohibitive marginal cost $p_1 = H_G(e) > H_G(G)$. □

Finally, note that one cannot apply the Pareto-dominance on equilibria of different types; generally, not even on equilibria of the same type. Corollary 1 shows that a ranking is feasible only within subsets of C-type equilibria.

Corollary 1 *Not all equilibria can be ranked by Pareto-dominance.*

Proof In the example of the proof above, first consider Pareto-dominance in the subsets of equilibria. In S-type equilibria, the total utility is constant and the payoffs differ only in distributional consequences. From the Pareto perspective, all equilibria are identical. In C-type equilibria, the higher the provision, the higher the total utility. However, an increase in G associated with a switch in specialisation may harm the player who switched from a cheap to an expensive input. A ranking by Pareto-dominance is feasible only on subsets where specialisation is fixed (i.e. Player 1 consistently pays only input 1, or only input 2).

To compare equilibria across types, notice that C-type equilibria provide a higher total utility. It can be easily inferred that the total utility is maximised for $e \leq G \leq c$. If specialisation if fixed, then the higher the provision, the larger the set of S-type equilibria that are Pareto dominated by this C-type equilibrium. Nevertheless, and most importantly, even the highest C-type equilibrium is not necessarily Pareto-dominated in all S-type equilibria. Consider the most asymmetric S-type equilibrium, $(g_1^1, g_2^1, g_1^2, g_2^2) = (0, 0, b, b)$, where Player 1 is a genuine free rider. To be generous to the highest C-type equilibrium, suppose now that Player 1 pays only the cheap input 2, $(g_1^1, g_2^1, g_1^2, g_2^2) = (0, e, e, 0)$. For Player 2, the highest C-type equilibrium is clearly better,

$$u^2(0, e, e, 0) = (y - p_1 e) + H(e) > (y - p_1 b - p_2 b) + H(b) = u^2(0, 0, b, b).$$

The reason is that $H(e) - H(b) > p_1(e - b)$, i.e. there is always a surplus over the cost of extra expensive input. However, for Player 1, as long as the surplus is small, $H(e) - H(b) < p_2 b$, then

$$u^1(0, e, e, 0) = (y - p_2 e) + H(e) < y + H(b) = u^1(0, 0, b, b).$$

This occurs if the surplus over the cost of the extra cheap input is not sufficiently large to compensate for the loss of advantage of free riding. In other words, Player 2 is now harmed by having to cover an extra cheap input 2 at amount b that was previously provided to them for free. $\square$

5. Conclusion

This paper directly extends Cornes (1993) by permitting agents to contribute to all inputs into a complementary production function of a public good. It shows that impure complements are effectively strategic substitutes; in a symmetric simultaneous setup, only pure complements deliver strategic complementarity. This result contributes to the recent literature identifying strategic complementarities in a public-good provision (Kessing 2007), and the literature on non-cooperative transfers with complementary aggregations (Gregor, Gregorova 2007).

The introduction of transfers into a purely complementary (weakest-link) aggregation bring three effects: i) equilibria are of both strategic types, ii) the minimal provision is defined by a willingness to pay both inputs, hence coordination failure is partly alleviated, and iii) Pareto-dominance can be generally applied only within subsets of equilibria, hence standard coordination failure only partly explains the multiplicity in equilibria.

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