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Inequality of Well-Being and Isoelastic Equivalence Scales

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Abstract

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Inequality of Well-Being and Isoelastic Equivalence Scales*

Udo Ebert † Patrick Moyes ‡

February 11, 2011

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1. Introduction and Motivation

The measurement of economic well-being requires among other things that the households' incomes are adjusted in order to accommodate differences in needs. Equivalence scales are designed to accomplish this adjustment by taking into account those household characteristics deemed to affect its needs. [1] Given a reference household type – generally a single adult – the procedure consists in deflating the household's original income by a scale factor which reflects

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1 Adjustments for differences in needs by means of equivalence scales may be considered too specific an approach and an alternative procedure has been proposed by Atkinson and Bourguignon (1987) (see also Bourguignon (1989), Jenkins and Lambert (1993), Bazen and Moyes (2003), Ebert (2010)). While this approach has mainly focused on the derivation of quasi-orderings like the sequential Lorenz dominance criterion for making comparisons of living standards across heterogenous populations, it is equally possible to use multidimensional (cardinal) indices (see e.g. Maasoumi (1999), Ebert (1995), Gravel, Moyes, and Tarroux (2009) among others).

the household's members' needs. Typical characteristics include region, location, age of adults and children but by far the most common factor that affects the household's members' well-being is family size. The equivalence scales currently used in empirical work are extremely varied both in the way in which they are derived and in their values. Different procedures, such as expert opinions, interviews with consumers or econometric studies, are used for determining the values of the equivalence scales (see e.g. Coulter, Cowell, and Jenkins (1992a)). This results in contrasted figures which in turn significantly affect the normative conclusions to be drawn (see e.g. Whiteford (1985), Buhmann, Rainwater, Schmaus, and Smeeding (1988), Coulter, Cowell, and Jenkins (1992b), Figini (1998)).

Among the different formulae proposed in the literature, equivalence scales with a constant elasticity with respect to family size play a prominent role (see for instance Atkinson, Rainwater, and Smeeding (1995)). According to such equivalence scales, a proportional increase in family size results in a proportional – not necessarily of the same magnitude – increase in the scale. [2] There are a number of reasons that explain why isoelastic equivalence scales meet such success in empirical work. A first reason – stressed for instance by Buhmann *et al.* (1988) – is that equivalence scales with constant elasticity approximate reasonably well the scales currently used in empirical work. More important is the fact that these scales permit one to control for the impact of family size on adjusted income through a single parameter that measures the elasticity of the scale. This in turn may have nice implications for applied work where one is interested in the changes in the extent of poverty, inequality or welfare implied by modifications of the distribution of household size. For instance, Coulter *et al.* (1992b) showed that the cardinal value of a poverty measure can be sensitive to the choice of the size elasticity parameter. Lanjouw and Ravallion (1995) addressed the question of whether large households are poorer than smaller ones and argued that the answer depends critically on the extent of dispersion in family sizes and the size elasticity of the equivalence scale. [3]

In this paper we are interested in the measurement of inequality using the distributions of the households' equivalent incomes. Typically, the equivalent income incorporates two main elements of the household's well-being: *neediness*, which is determined by the household's socio-demographic characteristics, and household *income*. Therefore, the equivalent income can be considered a bidimensional (cardinal) index and inequalities in well-being can be traced back to differences in neediness and household income. If one thinks of inequality as a multi-dimensional concept, then one is likely to require that this index reacts in an appropriate way to particular changes in the (unidimensional) distributions of needs and income as well as to the way the (joint) distribution of these two elements vary. For instance, modifications of the way neediness and income are distributed among households that leave unchanged their relative marginal distributions might be required to have no impact on inequality of well-being. It is a standard requirement for a (unidimensional) inequality index that equiproportionate changes in incomes leave inequality unchanged. Similarly, one might want that those changes in the distribution of demographic variables that do not modify the distribution of needs across households have no impact on overall inequality. Assimilating neediness with household size, this would quite naturally translate into the requirement that proportional increases in neediness do not affect the (relative) inequality of well-being.

[2] This makes only sense if the equivalence scales are independent of household income, an assumption that will be maintained throughout the paper. This assumption is clearly debatable and there is indeed empirical evidence that it is violated in practice (see e.g. Donaldson and Pendakur (2004), Koulovatianos, Schröder, and Schmidt (2005a,b)).

[3] Under certain conditions, it is indeed possible to establish the existence of a single critical value of the size elasticity for which the poverty ranking of household-size groups switches (see Lanjouw and Ravallion (1995) for details).

The implications for the structure of multidimensional inequality indices of such changes in the distributions of the households' needs and incomes is something that – to the best of our knowledge – has not been explored until now. We will show that the invariance conditions according to which inequality in well-being is immune to proportional changes in household income and size actually impose a lot of structure on the multidimensional inequality index. In particular, only isoelastic scales guarantee that inequality is invariant with respect to demographic changes: a replication of the population that leaves unchanged the relative distribution of the households' types does not modify inequality which is a natural requirement. Admittedly, for this requirement to make sense, a consensus must be reached concerning the way the index of neediness is derived from household size, something we assume throughout. To see what this condition means, suppose that the society consists of two types of households: those composed of a single individual and those comprising two individuals, where individuals are all alike. Then, the condition requires that, if – other things being equal – all household sizes double, then inequality of well-being remains the same. However, equivalence scales with a constant elasticity are very special and they are expected to imply particularly strong restrictions on the households' patterns of preference. It is convenient to distinguish between the preferences of the individuals who constitute the household and the principles the household's members rely on when deciding how to distribute resources among themselves. In this paper, we make the simplifying assumptions (i) that the household maximises the sum of utilities of its members so that the household is utilitarian, [4] and (ii) that all individuals within the household have the same preferences, which is consistent with the former assumption of identical individuals. Considering an economy with two commodities – a private good and a public (to the household's members) good – we show that the only preferences that are consistent with isoelastic equivalence scales are Cobb-Douglas.

We introduce in Section 2 the framework for our analysis. Section 3 is concerned with inequality and we show there that invariance of the (multidimensional) inequality index to neutral changes in income and neediness – coupled with standard properties of inequality measures – imply that the equivalence scales must be isoelastic. We investigate in Section 4 the implications of such patterns for the household's members' preferences under the assumption that the household maximises the sum of utilities of its members. Finally, Section 5 concludes the paper, while the proofs of our results are collected in Section 6.

2. General Framework and Notation

A *household* is a finite group of (not necessarily identical) individuals and we identify it with its size n. Typically, household size refers to the number of persons in the household and it therefore takes its values in the set of positive integers. For technical reasons we assume throughout that household size is a continuous variable ranging from unity to infinity so that $n \in [1, +\infty)$. We find it convenient to think of n as an *index of neediness* that depends on the household's composition and that is increasing with its size. This way of proceeding allows us to take into account the heterogeneity of the households which typically involves individuals with differing needs (think of adults and children both of different ages). It is reasonable to assume that a child contributes less to the household's expenses than an adult and she should therefore count less than an adult. A practical way of acknowledging this is to let $n(m_A, m_C) = m_A + \lambda m_C$ measure household size, where m_A and m_C are respectively the number of adults and children in the household and $\lambda \in [0, 1]$ is a parameter reflecting the importance of a child with respect to an adult (see Cutler and Katz (1992)). Given an

[4] This is equivalent to assume that the household maximises a symmetric and quasi-concave social welfare function of its members' utilities (see e.g. Ebert and Moyes (2009)).

arbitrary real $n \in [1, +\infty)$, it is always possible to find two non-negative integers m°, m^* and a real $\lambda \in [0, 1]$ such that $n = m^\circ + \lambda m^*$. This procedure can be generalised to an arbitrary list of individual types acknowledging for differences in age, health and the like. In what follows, we find convenient to interpret n as the household's size – measured in terms of *equivalent adults* in the household – or equivalently as the household's type.

There are two commodities: a *private good* and a *public* – to the household's members – *good*. The quantity of the private good consumed by member i in the household is denoted by x_i, while Z represents the quantity of the public good purchased by the household. The prices of the private and public goods are indicated by $p > 0$ and $q > 0$, respectively. In order to allow for the possibility of congestion we denote as

$$G = \phi(Z, \theta, n) := \frac{Z}{\psi(\theta, n)} \tag{2.1}$$

the *effective consumption of the public good* by any member of a household of type n. The parameter $\theta \in [0, 1]$ is a measure of the *degree of publicness* within the household, where $\theta = 0$ indicates the pure public good case while $\theta = 1$ corresponds to the private good case. The *congestion function* $\psi(\theta, n)$ has the following (natural) properties:

(2.2a) $\psi(\theta, n)$ is continuous in θ and n, $\forall\ \theta \in [0, 1]$, $\forall\ n \in [1, +\infty)$;

(2.2b) $\psi(\theta, n)$ is increasing in θ, $\forall\ n \in (1, +\infty)$;

(2.2c) $\psi(\theta, n)$ is increasing in n, $\forall\ \theta \in [0, 1)$;

(2.2d) $\psi(0, n) = \psi(\theta, 1) = 1$, $\forall\ \theta \in [0, 1]$, $\forall\ n \in [1, +\infty)$;

(2.2e) $\psi(1, n) = n$, $\forall\ n \in [1, +\infty)$;

(2.2f) $\psi(1, n)/n$ is strictly monotonic in n, $\forall\ n \in [1, +\infty)$.

This allows for a wide range of specifications where the congestion function can follow different patterns with respect to n (see e.g. Edwards (1990) and Reiter and Weichenrieder (1999) for details). We denote as $\boldsymbol{\Psi}$ the set of congestion functions verifying conditions (2.2a) to (2.2f). A particular instance of such congestion functions, that will be shown to be of particular interest later on, is given by $\psi(\theta, n) = n^\theta$ (see Borcherding and Deacon (1972)). The possibility of crowding is best exemplified in the case where the household consists of students sharing an apartment. While such goods as lightning and heating are clearly – at least in principle – public, other goods like access to the television, the newspaper(s), the telephone or the washing machine are partially excludable. Admittedly, congestion also happens in the family even though its extent is more limited. All individuals have the same preferences and we denote as $U(x, G)$ the utility derived by an individual consuming x units of the private good and G units of the public good, where the *individual utility function* U is assumed to be continuously differentiable, monotone and strictly concave. [5]

Different principles can be used by the household members for allocating resources among themselves and the way the household decides about this distribution has been shown to have important consequences (see e.g. Ebert and Moyes (2009)). We assume here that the individuals who constitute the household agree to allocate their resources between private and public consumption in such a way that the household's welfare is maximised. This guarantees

[5] In general, the fact that all individuals in the society have the same preferences does not imply that they have the same utility function. Since in our model individuals are alike it is quite natural to assume that they all also have the same (cardinal) utility function.

that the household members do not pursue their own self-interest but rather act in such a way as to provide all of them with the greater well-being. This cooperative behaviour may be considered a reasonable approximation of the way resources are distributed when the household consists of a single family. Formally, the household maximises the sum of the utilities derived by the equivalent adults who compose it. [6] The cooperative model and the assumption that individuals are identical may be considered reasonable approximations of the households' behaviour in modern societies when the household consists of a single family. For cultural and historical reasons, members of the same family are expected to share some common values, and cooperation among its members is the founding element of the family. One may therefore abstract from differences between the individuals who constitute the household – like adults and children – because they subscribe to the same common objectives and values. Things are quite different when the household is constituted by different families who join forces to achieve particular economic objectives. It follows from our assumptions that at the optimum all the household's members will get the same amount of the private good. The problem of a household of type n can be written as

$$\text{P(H)} \qquad \max \; n\,U(x,G) \;\; \text{s.t.} \;\; pnx + (\psi(\theta,n)\,q)\,G \leqq y,$$

where x is the consumption of the private good equal for all equivalent adults in the household, $y > 0$ is the household's total income, and $\psi(\theta,n)\,q$ can be interpreted as the price of one unit of effective consumption of the public good. Denoting as $X(p,q,y;\psi,\theta,n)$ and $G(p,q,y;\psi,\theta,n)$ the unique solution to problem P(H) and upon substitution into the utility function, we get the *representative indirect utility function*

$$V(p,q,y;\psi,\theta,n) := U(X(p,q,y;\psi,\theta,n), G(p,q,y;\psi,\theta,n)) \,. \tag{2.3}$$

Letting $u = V(p,q,y;\psi,\theta,n)$ and upon inverting, we obtain the *household expenditure function* $y = C(p,q,u;\psi,\theta,n)$, which indicates the minimum household income that guarantees that each of its members will reach the utility level u.

We follow the standard practice which involves choosing the household comprising a single individual as the reference household type ($n = 1$). Then, the *equivalent income function* $E(p,q,y;\psi,\theta,n)$ is implicitly defined by

$$V(p,q,y;\psi,\theta,n) = V(p,q,E(p,q,y;\psi,\theta,n);\psi,\theta,1), \tag{2.4}$$

which upon inverting gives

$$E(p,q,y;\psi,\theta,n) = C(p,q,V(p,q,y;\psi,\theta,n);\psi,\theta,1). \tag{2.5}$$

The *equivalent income* $E(p,q,y;\psi,\theta,n)$ represents the income that has to be given to a household of type $n = 1$ in order that its member enjoys the same utility as any member of household n with income y, given the prices p and q, the congestion function ψ, and the degree of publicness θ. It is quite standard in the literature to consider equivalence scales rather than the more general concept of the equivalent income function. The introduction of equivalence scales requires extra assumptions to be made about the relationship between the scales and the equivalent income or the household expenditure function. Starting with the household expenditure function $C(p,q,u;\psi,\theta,n)$, standard (relative) *equivalence scales* are defined by

$$M(p,q,u;\psi,\theta,n) = \frac{C(p,q,u;\psi,\theta,n)}{C(p,q,u;\psi,\theta,1)}. \tag{2.6}$$

[6] This is consistent with the maximisation of a symmetric, monotone non-decreasing and quasi-concave *social welfare function* $F(U(x_1,G),\ldots,U(x_n,G))$ when n is the number of persons in the household (see Bourguignon (1989), Blackorby and Donaldson (1993)).

The scale $M(p, q, u; \psi, \theta, n)$ measures the extra cost for a household of size n of providing each of its members with the utility level u relative to that of the single person household. Equivalence scales can also be defined by using the equivalent income function rather than the expenditure function. Upon substituting $u = V(p, q, y; \psi, \theta, n)$ into (2.6), we obtain

$$\text{(2.7)} \qquad \begin{aligned} m(p, q, y; \psi, \theta, n) :&= M(p, q, V(p, q, y; \psi, \theta, n); \psi, \theta, n) \\ &= \frac{y}{C(p, q, V(p, q, y; \psi, \theta, n); \psi, \theta, 1)} \\ &= \frac{y}{E(p, q, y; \psi, \theta, n)}, \end{aligned}$$

or equivalently

$$\text{(2.8)} \qquad E(p, q, y; \psi, \theta, n) = \frac{y}{m(p, q, y; \psi, \theta, n)}.$$

In this case the equivalent income is obtained by deflating household income by a scale factor, which may depend on household income.

Consider now an individual with utility function U who has to allocate an income y/n between the consumption of the private good and the public good whose prices are p and $\psi(\theta, n)\, q/n$, respectively. The individual's optimisation problem is given by

$$\text{P(I)} \qquad \max \; U(x, G) \quad \text{s.t.} \quad px + \frac{\psi(\theta, n)\, q}{n}\, G \leqq \frac{y}{n},$$

the solution to which is indicated by $\overline{X}(p, \psi(\theta, n)\, q/n, y/n)$ and $\overline{G}(p, \psi(\theta, n)\, q/n, y/n)$. Upon substitution and insertion into the individual utility function, we get the *individual indirect utility function*

$$\text{(2.9)} \qquad \overline{V}\left(p, \frac{\psi(\theta, n)\, q}{n}, \frac{y}{n}\right) := U\left(\overline{X}\left(p, \frac{\psi(\theta, n)\, q}{n}, \frac{y}{n}\right), \overline{G}\left(p, \frac{\psi(\theta, n)\, q}{n}, \frac{y}{n}\right)\right),$$

which upon inversion gives the usual *individual expenditure function* $\overline{C}(p, \psi(\theta, n)\, q/n, u)$. We deduce from the definitions of the household expenditure and the individual expenditure functions that

$$\text{(2.10)} \qquad y \equiv C(p, q, u; \psi, \theta, n) = n\, \overline{C}\left(p, \frac{\psi(\theta, n)\, q}{n}, u\right) \equiv n\, \frac{y}{n}.$$

Upon substituting into (2.6) and since $q \geqq (\psi(\theta, n)/n)\, q$, we finally get

$$\text{(2.11)} \qquad M(p, q, u; \psi, \theta, n) = \frac{n\, \overline{C}\left(p, \frac{\psi(\theta, n)\, q}{n}, u\right)}{\overline{C}(p, q, u)} \leqq n,$$

with a strict inequality if $\theta < 1$ and $n > 1$. It follows from Ebert and Moyes (2009) that an arbitrary small amount of publicness is a necessary and sufficient condition for $M(p, q, u; \psi, \theta, n) > n$ whenever $n > 1$ and we therefore impose $\theta \in [0, 1)$ from now on.

3. Inequality of Well-Being and Isoelastic Scales

We consider *populations* comprising H households ($H \geqq 2$), where each household is described by two attributes: its *income* and its *size*. A *heterogenous distribution* – or for short a *situation*

– is a partitioned vector $(\mathbf{y};\mathbf{n}) := (y_1,\dots,y_H;n_1,\dots,n_H)$, where $y_h > 0$ and $n_h \in [1,+\infty)$ represent respectively the income and the size – or equivalently the degree of neediness – of household h. The set of situations for a population comprising H households is indicated by $\mathscr{S}_H$. While the household population size H is fixed throughout, we insist on the fact that the distribution of households according to type $\mathbf{n} := (n_1,n_2,\dots,n_H)$ may vary. Assuming that prices p and q as well as the degree of publicness θ and the congestion function ψ are fixed, we drop these from the formulae in the most part of this section in order to simplify the notation.

For comparisons of living standards across households to be meaningful, it is necessary to correct household incomes for differences in needs and this *adjustment process* will involve two dimensions. On the one hand, the household's income is converted to an equivalent income which is the income needed by a household of type $n = 1$ (the reference household type) in order to achieve the same level of well-being as that attained by the original household. On the other hand, this equivalent income is attached a weight that is assumed to depend exclusively on household size. Formally, we associate to the situation $(\mathbf{y};\mathbf{n}) := (y_1,\dots,y_H;n_1,\dots,n_H) \in \mathscr{S}_H$ the *adjusted income distribution*

(3.1) $$(E(\mathbf{y};\mathbf{n}) \,|\, w(\mathbf{n})) := (E(y_1;n_1),\dots,E(y_H;n_H) \,|\, w(n_1),\dots,w(n_H)),$$

where $E(y_h,n_h)$ and $w(n_h) > 0$ are respectively the equivalent income and the weight assigned to household h (see e.g. Ebert and Moyes (2003)). It follows from the definition of the equivalent income and from the properties of the indirect utility function that the *equivalent income function* $E(\,\cdot\,;\,\cdot\,)$ has the following properties:

(3.2a) $E(y;n)$ is continuous in y and n, $\forall\, y > 0$, $\forall\, n \in [1,+\infty)$;

(3.2b) $E(y;n)$ is increasing in y, $\forall\, y > 0$, $\forall\, n \in [1,+\infty)$;

(3.2c) $E(y;n)$ is decreasing in n, $\forall\, y > 0$, $\forall\, n \in [1,+\infty)$; and

(3.2d) $\lim_{y\to 0} E(y;n) = 0$, $\forall\, y > 0$, $\forall\, n \in [1,+\infty)$.

The last property follows from taking the limit of (2.8) and from the fact that $m(y,n) \geqq 1$, for all $n \in [1,+\infty)$ and all $y > 0$. As far as the *weighting function* $w(\,\cdot\,)$ is concerned, we assume that

(3.3a) $w(n)$ is continuous in n, $\forall\, n \in [1,+\infty)$ and

(3.3b) $w(n)$ is non-decreasing in n, $\forall\, n \in [1,+\infty)$.

The weighting function allows for different possibilities among which is the standard one consisting in weighting the equivalent income by the number of persons in the household or the one that gives each household the same weight irrespective of its size and composition. The way in which the households' equivalent incomes are weighted is not innocuous and it has been shown to have important consequences for normative evaluation. [7] The set of (unidimensional) *income distributions* is indicated by

(3.4) $$\mathscr{D}_H := \{(\mathbf{s} \,|\, \mathbf{w}) := (s_1,\dots,s_H \,|\, w_1,\dots,w_H) \;\mid\; s_h > 0 \text{ and } w_h > 0, \forall\, h = 1,2,\dots,H\}$$

[7] This was first recognised by Glewwe (1991), who showed that a regressive transfer of income between two households might decrease the inequality of well-being when the equivalent incomes are weighted by the household sizes.

and by definition $(E(\mathbf{y};\mathbf{n})\,|\,w(\mathbf{n})) \in \mathscr{D}_H$, for all $(\mathbf{y};\mathbf{n}) \in \mathscr{S}_H$. We use $\mu(\mathbf{s}\,|\,\mathbf{w})$ to represent the (arithmetic) *mean* of the income distribution $(\mathbf{s}\,|\,\mathbf{w}) \in \mathscr{D}_H$.

We are interested in comparing situations from the point of view of inequality of well-being for populations of households whose members face different circumstances. To this end we introduce a *bidimensional inequality index* $I : \mathscr{S}_H \longrightarrow \mathbb{R}$ such that $I(\mathbf{y};\mathbf{n})$ measures the extent of inequality in situation $(\mathbf{y};\mathbf{n})$ with the property that

$$I(\mathbf{y};\mathbf{n}) = J(E(\mathbf{y};\mathbf{n})\,|\,w(\mathbf{n})), \tag{3.5}$$

where $J : \mathscr{D}_H \longrightarrow \mathbb{R}$ is a *unidimensional inequality index.* According to definition (3.5), the assessment of the inequality in the well-being of the members of households is a two-stage process, where the two dimensions of the households' heterogeneity are first aggregated into a single measure (the equivalent income) and where the distributions of these equivalent incomes – appropriately weighted – are then compared by means of a standard inequality index. This two-stage process in the measurement of income inequality for heterogenous populations is indeed rather natural when one uses equivalence scales to adjust households' incomes for differences in needs.

The preceding discussion makes clear that the extent of inequality will – in addition to the choice of the index J – depend on the equivalent income function $E(\,\cdot\,;\,\cdot\,)$ and on the weighting function $w(\,\cdot\,)$. For instance, J can be the Gini index, an Atkinson-Kolm-Sen (AKS) index or a member of the generalised entropy family. There is no need for our purpose to choose a particular unidimensional inequality index: it suffices that J verifies four natural (in the inequality literature sense) conditions. The first condition – satisfied by most unidimensional inequality indices – simply requires that the index takes the value zero only when all incomes are equal.

Normalisation (UN). For all $(\mathbf{s}\,|\,\mathbf{w}) \in \mathscr{D}_H$, we have $J(\mathbf{s}\,|\,\mathbf{w}) = 0$, if and only if $s_1 = s_2 = \cdots = s_H$.

The second condition is also standard and it states that inequality is not changed when incomes increase or decrease proportionally.

Scale Invariance (USI). For all $(\mathbf{s}\,|\,\mathbf{w}) \in \mathscr{D}_H$ and all $\zeta > 0$, we have $J(\zeta\,\mathbf{s}\,|\,\mathbf{w}) = J(\mathbf{s}\,|\,\mathbf{w})$.

The next condition is but an adaptation in our framework of the standard population invariance principle of Dalton (1920) that requires that inequality does change when the population is replicated.

Distributional Invariance (UDI). For all $(\mathbf{s}\,|\,\mathbf{w}) \in \mathscr{D}_H$ and all $\lambda > 0$, we have $J(\mathbf{s}\,|\,\lambda\,\mathbf{w}) = J(\mathbf{s}\,|\,\mathbf{w})$.

Before we turn to the fourth condition, we need to introduce a piece of additional notation. We indicate by $F^{-1}(\,\cdot\,;(\mathbf{s}\,|\,\mathbf{w}))$ the *inverse cumulative distribution function* of $(\mathbf{s}\,|\,\mathbf{w})$ obtained by letting $F^{-1}(0;(\mathbf{s}\,|\,\mathbf{w})) := \min\{s_1, s_2, \ldots, s_H\}$ and

$$F^{-1}(p;(\mathbf{s}\,|\,\mathbf{w})) := Inf\,\{z \in (-\infty,+\infty)\;|\;F(z;(\mathbf{s}\,|\,\mathbf{w})) \geq p\}\,,\ \forall\, p \in (0,1] \tag{3.6}$$

(see Gastwirth (1971)). The *Lorenz curve* of the unidimensional distribution $(\mathbf{s}\,|\,\mathbf{w}) \in \mathscr{D}_H$ – denoted as $L(p;(\mathbf{s}\,|\,\mathbf{w}))$ – is then defined by

$$L(p;(\mathbf{s}\,|\,\mathbf{w})) := \int_0^p F^{-1}(q;(\mathbf{s}\,|\,\mathbf{w}))\,dq,\ \forall\ p \in [0,1]. \tag{3.7}$$

Then, we will say that *distribution* $(\mathbf{s}^* \mid \mathbf{w}^*)$ *Lorenz dominates distribution* $(\mathbf{s}^\circ \mid \mathbf{w}^\circ)$, which we write $(\mathbf{s}^* \mid \mathbf{w}^*) \geq_L (\mathbf{s}^\circ \mid \mathbf{w}^\circ)$, if

$$(3.8) \qquad L(p; (\mathbf{s}^* \mid \mathbf{w}^*)) \geqq L(p; (\mathbf{s}^\circ \mid \mathbf{w}^\circ)), \ \forall\, p \in (0,1), \text{ and } \mu(\mathbf{s}^* \mid \mathbf{w}^*) = \mu(\mathbf{s}^\circ \mid \mathbf{w}^\circ).$$

The fourth condition captures the very idea of inequality reduction and it requires that inequality does not increase when incomes are more equally distributed in the sense that the Lorenz curve moves upwards.

Lorenz Consistency (ULC). For all $(\mathbf{s}^* \mid \mathbf{w}^*), (\mathbf{s}^\circ \mid \mathbf{w}^\circ) \in \mathscr{D}_H$, we have $J(\mathbf{s}^* \mid \mathbf{w}^*) \leqq J(\mathbf{s}^\circ \mid \mathbf{w}^\circ)$ whenever $(\mathbf{s}^* \mid \mathbf{w}^*) \geq_L (\mathbf{s}^\circ \mid \mathbf{w}^\circ)$.

Clearly, not all indices I as defined by (3.5) are suitable bidimensional inequality indices – even though the unidimensional index J possesses the four properties above – and we require I to satisfy two conditions. The first condition is merely a restatement of the standard *scale invariance* property in the unidimensional setting and it requires that proportional changes in the households' incomes have no impact on the inequality of well-being in the population.

Income Scale Invariance (BISI). For all $(\mathbf{y}; \mathbf{n}) \in \mathscr{S}_H$ and all $\zeta > 0$, we have $I(\zeta\mathbf{y}; \mathbf{n}) = I(\mathbf{y}; \mathbf{n})$.

Similarly, our second condition requires that demographic changes that result in proportional shifts in the distribution of sizes across households have no impact in the inequality of well-being.

Neediness Scale Invariance (BNSI). For all $(\mathbf{y}; \mathbf{n}) \in \mathscr{S}_H$ and all $\lambda > 0$, we have $I(\mathbf{y}; \lambda\mathbf{n}) = I(\mathbf{y}; \mathbf{n})$.

The two conditions above are concerned with the way the marginal distributions of household income and household size change and with their impact on the inequality of well-being. Formally, they impose that changes that do not modify the (relative) inequality of the marginal distributions of household income and household size have no impact on the inequality of well-being. Combining these two conditions, we obtain the kind of invariance property considered in the standard multidimensional literature, where different scalings are used for different attributes (see e.g. Tsui (1995)). The difference is the particular nature of our second variable that refers to demographic changes in the households' composition. However, these two conditions are silent about the incidence on the inequality of well-being of modifications of the joint distributions of income and neediness that leave unchanged their marginal distributions.

At this stage, and in the absence of further restrictions placed on the adjustment process – namely the equivalent income function and the weighting function – there is no guarantee that conditions BISI and BNSI be satisfied. Nor it is clear what the interest is of requiring that the unidimensional inequality index J is Lorenz consistent. Indeed, the latter property has no particular meaning in the present context, unless one is able to relate the *shifts of the Lorenz curves* of the adjusted income distributions to *particular modifications* of the joint distributions of household income and neediness that result in an uncontroversial reduction of bidimensional inequality. The next transformation (see e.g. Ebert (2000), Gravel and Moyes (2008)), which fully exploits the bidimensionality of a situation, constitutes in our model a natural generalisation of the notion of a (unidimensional) progressive transfer. Given two situations $(\mathbf{y}^*; \mathbf{n}^*), (\mathbf{y}^\circ; \mathbf{n}^\circ) \in \mathscr{S}_H$, we will say that $(\mathbf{y}^*; \mathbf{n}^*)$ is obtained from $(\mathbf{y}^\circ; \mathbf{n}^\circ)$ by means of a *between-type progressive transfer*, if there exists $\Delta > 0$ and two households h, k such that

(3.9a) $$y_h^\circ < y_h^* \leqq y_k^* < y_k^\circ;\; n_k^\circ = n_k^* < n_h^* = n_h^\circ;$$

(3.9b) $$y_h^* - y_h^\circ = y_k^\circ - y_k^* = \Delta;\; \text{and}$$

(3.9c) $$(y_g^*, n_g^*) = (y_g^\circ, n_g^\circ),\; \forall\; g \neq h, k.$$

A between-type progressive transfer fully recognises the multidimensional nature of inequality – household h is more deprived than household k in both income and need – but only the first attribute is used for reducing inequality. To this extent, a between-type progressive transfer is a particular case of the more general transformation introduced by Kolm (1977) who requires that transfers take place in all attributes. [8] Given the nature of the need variable, it does not make sense to redistribute needs from needy to less needy households and we therefore prevent ourselves from so doing. Hence, by definition a between-type progressive transfer does not modify the distribution of household types in the population: if $(\mathbf{y}^\circ; \mathbf{n}^\circ)$ is converted into $(\mathbf{y}^*; \mathbf{n}^*)$ by means of a between-type progressive transfer, then $\mathbf{n}^* = \mathbf{n}^\circ$. Consider now the following condition:

Weak Equity (WE). For all $(\mathbf{y}^*; \mathbf{n}^*), (\mathbf{y}^\circ; \mathbf{n}^\circ) \in \mathscr{S}_H$, if $(\mathbf{y}^*; \mathbf{n}^*)$ is obtained from $(\mathbf{y}^\circ; \mathbf{n}^\circ)$ by means of a between-type progressive transfer, then $(E(\mathbf{y}^*; \mathbf{n}^*) \,|\, w(\mathbf{n}^*)) \geq_L (E(\mathbf{y}^\circ; \mathbf{n}^\circ) \,|\, w(\mathbf{n}^\circ))$.

This condition – combined with the fact that the equivalent income function is increasing with income and decreasing with neediness – implies that $\min\{E(y_g^*; n_g^*)\} \geqq \min\{E(y_g^\circ; n_g^\circ)\}$ and

(3.10) $$\mu(E(\mathbf{y}^*; \mathbf{n}^*) \,|\, w(\mathbf{n}^*)) = \mu(E(\mathbf{y}^\circ; \mathbf{n}^\circ) \,|\, w(\mathbf{n}^\circ)),$$

where $\mathbf{n}^* = \mathbf{n}^\circ$. Although WE appears at first sight to be a mild condition, it has important consequences for the structure of the adjustment process as the next result demonstrates.

Proposition 3.1. *The adjustment process* $(E(\,\cdot\,;\,\cdot\,); w(\,\cdot\,))$ *verifies condition* WE *if and only if*

(3.11a) $$E(y; n) = \frac{y}{K(n)},\; \forall\; y > 0,\; \forall\; n \in [1, +\infty),\; \textit{and}$$

(3.11b) $$w(n) = \eta K(n),\; \forall\; n \in [1, +\infty)\; \textit{and for some}\; \eta > 0.$$

According to Proposition 3.1, the imposition of WE restricts the way the adjusted income distributions are derived in two respects. Firstly, the equivalence scale must be independent of household income: in other terms, the equivalence scale verifies the *relative equivalence scale exactness* condition of Blackorby and Donaldson (1993), or equivalently the *independence of base level* condition due to Lewbel (1989) (see also Blundell and Lewbel (1991)). Secondly, the weights associated to the households' equivalent incomes must be proportional to the equivalence scale. It follows that the equivalent income function and the weighting function cannot be chosen independently from each other if we want the adjustment process to verify WE. [9]

[8] It must also be stressed that Kolm (1977) imposes no particular restrictions concerning the respective positions with respect to the different attributes of the households involved in this generalised transfer. In particular, it is not necessary that one household is richer than another in all attributes for the transfer to make sense.

[9] This result is reminiscent of Ebert and Moyes (2003) who obtained similar restrictions on the adjustment process but using a slightly different approach.

On the other hand, Proposition 3.1 does not give any indication about the form of the size dependency of the equivalence scale and there are a number possibilities consistent with (3.11). Upon substituting (3.11) into (3.5), we obtain

$$I(\mathbf{y};\mathbf{n}) = J\left(\frac{y_1}{K(n_1)},\dots,\frac{y_H}{K(n_H)}\,\middle|\,\eta K(n_1),\dots,\eta K(n_H)\right) =: J\left(\frac{\mathbf{y}}{K(\mathbf{n})}\,\middle|\,\eta K(\mathbf{n})\right). \tag{3.12}$$

Clearly, since by definition the unidimensional inequality index J is *scale invariant*, all bidimensional inequality indices I as defined by (3.12) verify BISI. But *scale invariance* (USI) – even combined with *distributional invariance* (UDI) – does not guarantee that a bidimensional inequality indices I of the form (3.12) will obey BNSI. However, WE in conjunction with the four standard properties UN, USI, UDI and ULC ensures that the bidimensional inequality index I verifies BNSI as the next result formally demonstrates.

Proposition 3.2. *Consider a bidimensional inequality index I as defined by* (3.5)*, where the unidimensional inequality index J verifies* UN, USI, UDI *and* ULC, *and assume in addition that the adjustment process $(E(\,\cdot\,;\,\cdot\,);w(\,\cdot\,))$ verifies* WE. *Then, the condition that*

$$K(n) = \xi\, n^{\epsilon}, \text{ where } \xi,\epsilon > 0, \tag{3.13}$$

is necessary and sufficient for BNSI *to be fulfilled.*

While the imposition of BNSI in a heterogenous setting is admittedly open to debate, we insist on the fact that it is this condition which – combined with standard properties of unidimensional inequality indices and a natural restriction imposed on the adjustment process – precipitates isoelastic equivalence scales.

Upon substituting $\xi =: \tilde{g}(p,q,y;\psi,\theta,1)$ and $\epsilon =: \tilde{f}(p,q,y;\psi,\theta,1)$ into (3.13), we obtain

$$m(p,q,y;\psi,\theta,n) = \tilde{g}(p,q,y;\psi,\theta,1)\, n^{\tilde{f}(p,q,y;\psi,\theta,1)}, \tag{3.14}$$

which holds for all $\psi \in \mathbf{\Psi}$, $n \geqq 1$, $\theta \in [0,1)$, $y > 0$ and all $(p,q) \gg (0,0)$. Finally, appealing to (2.7), we get a similar condition when the equivalence scale refers to the household representative member's utility. More precisely:

$$M(p,q,u;\psi,\theta,n) = g(p,q,u;\psi,\theta,1)\, n^{f(p,q,u;\psi,\theta,1)}, \tag{3.15}$$

for all $\psi \in \mathbf{\Psi}$, $n \geqq 1$, $\theta \in [0,1)$, $u \in \mathbb{R}$ and all $(p,q) \gg (0,0)$.

4. Recovering the Household's Members' Preferences

By definition the equivalence scale $M(p,q,u;\psi,\theta,n)$ is isoelastic if (3.13) holds, for all $\psi \in \mathbf{\Psi}$, all $n \in [1,+\infty)$, all $\theta \in [0,1)$, and all $u \in \mathbb{R}$ and all $(p,q) \gg (0,0)$. In this case, the equivalent income is proportional to household income, where the factor of proportionality is a non-increasing function of household size. This is equivalent to requiring that the relative change in the equivalence scale resulting from a proportional increase of household size is independent of household size. Formally

$$\frac{M(p,q,u;\psi,\theta,\lambda n^{\circ})}{M(p,q,u;\psi,\theta,n^{\circ})} = \frac{M(p,q,u;\psi,\theta,\lambda n^{*})}{M(p,q,u;\psi,\theta,n^{*})},\ \forall\, n^{\circ},n^{*} \geqq 1,\ \forall\,\lambda > 1,\ \forall\,(p,q,u;\psi,\theta). \tag{4.1}$$

It can be shown (see Aczel (1966, Chapter 3)) that the solution of (4.1) is precisely (3.15). Now consider an individual who has Cobb-Douglas preferences represented by the utility function

$U(x, G) = x^{\vartheta} G^{1-\vartheta}$, where $0 < \vartheta < 1$. Letting $n = 1$, the solutions to problem P(I) are $x = \overline{X}(p, q, y) = \vartheta y/p$ and $G = \overline{G}(p, q, y) = (1 - \vartheta)y/q$. Upon substitution into $U(x, G)$, we obtain the individual indirect utility function

$$\overline{V}(p, q, y) = \left(\frac{\vartheta}{p}\right)^{\vartheta} \left(\frac{1-\vartheta}{q}\right)^{1-\vartheta} y, \tag{4.2}$$

which upon inverting gives the individual expenditure function

$$\overline{C}(p, q, u) = \left(\frac{p}{\vartheta}\right)^{\vartheta} \left(\frac{q}{1-\vartheta}\right)^{1-\vartheta} u. \tag{4.3}$$

Substituting $\psi(\theta, n)q/n$ for q into (4.3) and using (2.10), we get the household expenditure function

$$C\,(p, q, u; \psi, \theta, n) = n\,\overline{C}\left(p, \frac{\psi(\theta, n)\,q}{n}, u\right) = n\,\left(\frac{p}{\vartheta}\right)^{\vartheta} \left(\frac{\psi(\theta, n)\,q}{(1-\vartheta)\,n}\right)^{1-\vartheta} u, \tag{4.4}$$

which can be rewritten as

$$C\,(p, q, u; \psi, \theta, n) = n\,\left(\frac{\psi(\theta, n)}{n}\right)^{1-\vartheta} \left(\frac{p}{\vartheta}\right)^{\vartheta} \left(\frac{q}{1-\vartheta}\right)^{1-\vartheta} u. \tag{4.5}$$

We derive the household (relative) equivalence scale

$$M\,(p, q, u; \psi, \theta, n) = n\,\left(\frac{\psi(\theta, n)}{n}\right)^{1-\vartheta}, \tag{4.6}$$

which is independent of the household representative utility. If $\psi(\theta, n) = n^{\rho(\theta)}$, then we get $M\,(p, q, u; \psi, \theta, n) = n^{\vartheta + (1-\vartheta)\,\rho(\theta)}$: the equivalence scale is isoelastic with respect to household size. We conclude that, (i) if preferences are Cobb-Douglas, and (ii) if the congestion function is isoelastic, then the (relative) equivalence scale is isoelastic. Actually, these two conditions are also necessary as indicated in the following result.

Proposition 4.1. *Assume that $\psi \in \mathbf{\Psi}$ and $\theta \in [0, 1)$. Then, $M\,(p, q, u; \psi, \theta, n)$ verifies condition* (4.1) *if and only if:*

(a) *There exists $\rho(\theta) \in [0, 1)$ such that $\psi(\theta, n) = n^{\rho(\theta)}$, and*

(b) *$U(x, G) = x^{\vartheta} G^{1-\vartheta}$, for some $\vartheta \in (0, 1)$.*

5. Concluding Remarks

We have argued in the paper that, under mild requirements concerning the inequality index, isoelastic equivalence scales have the property that inequality of well-being is not affected when both households' incomes and household's needs vary proportionally. Within a simple model where all individuals have identical preferences and households maximise a symmetric, monotone and quasi-concave social welfare function, we have also shown that Cobb-Douglas preferences are necessary and sufficient for such equivalence scales to arise. While it is generally claimed that isoelastic scales provide reasonable approximations of the scales currently used in practice, our second result uncovers the restrictions one has to place implicitly on the household's members preferences as well as on the way the intra-allocation of resources is determined by the household in order to generate such scales. It is interesting to note that isoelastic scales can also be recovered from personal judgements within the subjective approach model of Kapteyn and van Praag (1976).

According to definition (2.5), the equivalence scale depends in theory on the household welfare measured by the utility of a representative member and on the prices of the private and public goods. There is experimental evidence that suggests that equivalence scales are not independent of household income – or equivalently of household welfare – as shown for instance by Koulovatianos *et al.* (2005a,b). This is supported by econometric studies that indicate that the scales values vary significantly with the income of the household (see Donaldson and Pendakur (2004)). Equation (4.6) makes clear that, if the household maximises the sum of utilities of its members and if the latter have the same Cobb-Douglas preferences, then the resulting equivalence scales are independent of the utility achieved by a typical household member or equivalently of household income. Isoelastic equivalence scales verify the condition of *relative equivalence scale exactness* (Blackorby and Donaldson (1993)) or *independence of base level* (Lewbel (1989), Blundell and Lewbel (1991)). Proposition 4.1 indicates in addition that, if households behave cooperatively and individuals have the same Cobb-Douglas preferences, then the equivalence scales are also independent of prices.

Crucial for our result are the assumptions that the household members behave in a cooperative way and that they are identical in all respects. Suppose the household members do not cooperate – every household member decides in isolation the amount she is willing to contribute to the household public good – and that preferences are identical and Cobb-Douglas. Then, the (relative) equivalence scale is

$$M\left(p,q,u;\psi,\theta,n\right)=\left(1+\vartheta(n-1)\right)\psi(\theta,n)^{1-\vartheta}, \tag{5.1}$$

which reduces to

$$M\left(p,q,u;\psi,\theta,n\right)=(1-\vartheta)n^{(1-\vartheta)\theta}+\vartheta n^{(1-\vartheta)\theta+1}, \tag{5.2}$$

if the congestion function is isoelastic. Thus, as far as our first assumption is concerned, we note that in the non-cooperative case Cobb-Douglas preferences no longer lead to isoelastic scales. However, equivalence scales are isoelastic for sufficiently small values of ϑ. But, the main limitation of our approach is certainly the strong assumption that individuals are perfectly identical. This is at variance with the real world where households typically consist of individuals of different types, for instance adults and children. Allowing for such a heterogeneity into the model and investigating its implications for the structure of the equivalence scales is certainly the next step to be taken.

6. Proofs

Before we proceed to the proof of Proposition 3.1, we find convenient to introduce two technical results, the first one being a generalisation of the Cauchy equation.

Lemma 6.1. *Let* $F:[1,+\infty)\longrightarrow\mathbb{R}$ *and* $H:\mathbb{R}_{++}\times[1,+\infty)\longrightarrow\mathbb{R}$. *Then:*

$$F(m)\,H(u+\Delta;m)+F(q)\,H(v-\Delta;q)=F(m)\,H(u;m)+F(q)\,H(v;q), \tag{6.1}$$

for all $\Delta>0$, *all* $u,v>0$ *such that* $u+\Delta\leqq v-\Delta$ *and all* $m,q\in[1,+\infty)$, *if and only if, there exist* $\chi:\mathbb{R}_{++}\times[1,+\infty)\longrightarrow\mathbb{R}$ *and* $\eta>0$ *such that*

$$F(m)\,H(y;m)=\chi(m)+\eta y,\ \forall\ y>0,\ \forall\ m\in[1,+\infty). \tag{6.2}$$

Proof. Since it is obvious that (6.2) implies (6.1), we only prove the converse implication. Let us rewrite (6.1) as follows:

$$F(m)\,H(u+\Delta;m) - F(q)\,H(v;q) = F(m)\,H(u;m) - F(q)\,H(v-\Delta;q). \tag{6.3}$$

Fix $v = \tilde{v} > 0$ and consider the following functions:

$$f(\tilde{v},t;m,q) \equiv \tilde{f}(t) := F(m)\,H(t;m) - F(q)H(\tilde{v};q), \tag{6.4a}$$

$$g(t;m) \equiv \tilde{g}(t) := F(m)\,H(t;m), \text{ and} \tag{6.4b}$$

$$\phi(\tilde{v},t;q) \equiv \tilde{\phi}(t) := -F(q)\,H(\tilde{v}-t;q). \tag{6.4c}$$

Then, condition (6.3) implies that

$$\tilde{f}(u+\Delta) = \tilde{g}(u) + \tilde{\phi}(\Delta),\ \forall\ u,\Delta > 0 \text{ such that } u+\Delta \leqq v-\Delta. \tag{6.5}$$

This is a Pexider equation the solution of which (Aczel (1966, Theorem 1, p. 142)) is

$$\tilde{f}(t) = \tilde{\alpha} + \tilde{\gamma} + \tilde{\beta}t \equiv f(\tilde{v},t;m,q), \tag{6.6a}$$

$$\tilde{g}(t) = \tilde{\alpha} + \tilde{\beta}t \equiv g(\tilde{v},t;m) \text{ and} \tag{6.6b}$$

$$\tilde{\phi}(t) = \tilde{\gamma} + \tilde{\beta}t \equiv \phi(\tilde{v},t;q), \tag{6.6c}$$

from which we deduce that $\tilde{\alpha} = \alpha(\tilde{v};m)$, $\tilde{\gamma} = \gamma(\tilde{v};q)$ and $\tilde{\beta} = \beta(\tilde{v})$. Substituting into (6.4a), (6.4b) and (6.4c), we get

$$\tilde{f}(t) := F(m)\,H(t;m) - F(q)\,H(\tilde{v};q) = \tilde{\alpha} + \tilde{\gamma} + \tilde{\beta}t = \alpha(\tilde{v};m) + \gamma(\tilde{v};q) + \beta(\tilde{v})t, \tag{6.7a}$$

$$\tilde{g}(t) := F(m)\,H(t;m) = \tilde{\alpha} + \tilde{\beta}t = \alpha(\tilde{v};m) + \beta(\tilde{v})t, \tag{6.7b}$$

$$\tilde{\phi}(t) := -\,F(q)\,H(\tilde{v}-t;q) = \tilde{\gamma} + \tilde{\beta}t = \gamma(\tilde{v};q) + \beta(\tilde{v})t. \tag{6.7c}$$

Subtracting (6.7b) from (6.7a), we obtain

$$\tilde{\gamma} \equiv \gamma(\tilde{v};q) = -F(q)\,H(\tilde{v};q). \tag{6.8}$$

Fixing now $v = \hat{v} \neq \tilde{v}$ and using a similar reasoning as above, we get

$$\hat{f}(t) := F(m)\,H(t;m) - F(q)H(\hat{v};q) = \hat{\alpha} + \hat{\gamma} + \hat{\beta}t = \alpha(\hat{v};m) + \gamma(\hat{v};q) + \beta(\hat{v})t, \tag{6.9a}$$

$$\hat{g}(t) := F(m)\,H(t;m) = \hat{\alpha} + \hat{\beta}t = \alpha(\hat{v};m) + \beta(\hat{v})t, \tag{6.9b}$$

$$\hat{\phi}(t) := F(q)\,H(\hat{v}-t;q) = \hat{\gamma} + \hat{\beta}t = \gamma(\hat{v};q) + \beta(\hat{v})t, \tag{6.9c}$$

where $\hat{\alpha} = \alpha(\hat{v};m)$, $\hat{\gamma} = \gamma(\hat{v};q)$, $\hat{\beta} = \beta(\hat{v})$, and we deduce that

$$\hat{\gamma} \equiv \gamma(\hat{v};q) = -F(q)\,H(\hat{v};q). \tag{6.10}$$

Subtracting (6.9b) from (6.7b), we get

$$[\tilde{\alpha} - \hat{\alpha}] + \left[\tilde{\beta} - \hat{\beta}\right] t = 0,\ \forall\ t > 0, \tag{6.11}$$

which implies that $\hat{\alpha} \equiv \alpha(\hat{v},m) = \alpha(\tilde{v},m) \equiv \tilde{\alpha} = \alpha$ and $\hat{\beta} \equiv \beta(\hat{v}) = \beta(\tilde{v}) = \tilde{\beta} = \beta$. Substituting the value of $\tilde{\gamma}$ given by (6.8) into (6.7c), we obtain

$$F(q)\,H(\tilde{v}-t;q) = F(q)\,H(\tilde{v};q) - \beta t. \tag{6.12}$$

Letting $s = \tilde{v} - t$, this can be rewritten as

(6.13) $$F(q)\,H(s+t;q) = F(q)\,H(s;q) + \beta t.$$

Defining

(6.14a) $$f^*(z) := F(q)\,H(z;q),$$

(6.14b) $$g^*(z) := F(q)\,H(z;q), \text{ and}$$

(6.14c) $$\phi^*(z) = \beta z,$$

condition (6.13) can be rewritten as

(6.15) $$f^*(s+t) = g^*(s) + \phi^*(t), \ \forall\ s,t > 0.$$

Appealing to Aczel (1966, Theorem 1, p. 142) again, we obtain

(6.16a) $$f^*(z) := F(q)\,H(z;q) = a + c + bz,$$

(6.16b) $$g^*(z) := F(q)\,H(z;q) = a + bz,$$

(6.16c) $$\phi^*(z) := \beta z = c + bz,$$

where $a = \chi(q)$ while b and c are independent of q. Comparing (6.16a) and (6.16b), we conclude that $c = 0$, which upon substituting into (6.14c) and letting $b = \eta$ implies that

(6.17) $$f^*(z) := F(q)\,H(z;q) = \chi(q) + \eta z, \ \forall\ z > 0,$$

and the proof is complete. □

Our second technical result exploits the separability of the Lorenz quasi-ordering and it shows that the unconcerned individuals play no role in the ranking of the distributions under comparison.

Lemma 6.2. *Let* $(\mathbf{s}^* \,|\, \mathbf{w}^*), (\mathbf{s}^\circ \,|\, \mathbf{w}^\circ) \in \mathscr{D}_m$ $(m \geqq 2)$ *such that* $\mathbf{w}^* = \mathbf{w}^\circ$ *and let* $(\tilde{\mathbf{s}} \,|\, \tilde{\mathbf{w}}) \in \mathscr{D}_q$ $(q \geqq 1)$. *Then:*

(6.18) $$((\mathbf{s}^* \,|\, \mathbf{w}^*); (\tilde{\mathbf{s}} \,|\, \tilde{\mathbf{w}})) \geq_L ((\mathbf{s}^\circ \,|\, \mathbf{w}^\circ); (\tilde{\mathbf{s}} \,|\, \tilde{\mathbf{w}}))$$

if and only if

(6.19) $$(\mathbf{s}^* \,|\, \mathbf{w}^*) \geq_L (\mathbf{s}^\circ \,|\, \mathbf{w}^*).$$

Proof. Given two distributions $(\hat{\mathbf{s}}^* \,|\, \hat{\mathbf{w}}), (\hat{\mathbf{s}}^\circ \,|\, \hat{\mathbf{w}}) \in \mathscr{D}_r$ $(r \geqq 2)$, we know that $(\hat{\mathbf{s}}^* \,|\, \hat{\mathbf{w}}) \geq_L (\hat{\mathbf{s}}^\circ \,|\, \hat{\mathbf{w}})$ if and only if

(6.20) $$\sum_{h=1}^{r} \hat{w}_h\, \phi\left(\hat{s}^*_h\right) \geqq \sum_{h=1}^{r} \hat{w}_h\, \phi\left(\hat{s}^\circ_h\right), \ \forall\ \phi \text{ concave}$$

(see e.g. Ebert and Moyes (2002, Prop. 3.1), Marshall and Olkin (1979, Chap 4)). The result follows from noticing that

(6.21) $$\sum_{h=1}^{m} w^*_h\, \phi\left(s^*_h\right) + \sum_{h=1}^{q} \tilde{w}_h\, \phi\left(\tilde{s}_h\right) \geqq \sum_{h=1}^{m} w^\circ_h\, \phi\left(s^\circ_h\right) + \sum_{h=1}^{q} \tilde{w}_h\, \phi\left(\tilde{s}_h\right), \ \forall\ \phi,$$

is equivalent to

$$\sum_{h=1}^{m} w_h^* \,\phi\,(s_h^*) \geqq \sum_{h=1}^{m} w_h^\circ \,\phi\,(s_h^\circ)\,,\ \forall\ \phi, \tag{6.22}$$

where $w_h^* = w_h^\circ$, for all $h = 1, 2, \ldots, m$. □

Proof of Proposition 3.1.

Sufficiency. We show that, if the equivalent income function and the weighting functions verify conditions (3.11a) and (3.11b), respectively, then the adjustment process satisfies WE. Consider two situations $(\mathbf{y}^*; \mathbf{n}^*), (\mathbf{y}^\circ; \mathbf{n}^\circ) \in \mathscr{S}_H$ such that $(\mathbf{y}^*; \mathbf{n}^*)$ is obtained from $(\mathbf{y}^\circ; \mathbf{n}^\circ)$ by means of a between-type progressive transfer, hence there exists $\Delta > 0$ and two households h, k such that conditions (3.9a), (3.9b) and (3.9c) hold. To simplify notation, let us indicate by

$$(\mathbf{e}^* \,|\, \mathbf{w}^*) = (e_1^*, \ldots, e_H^* \,|\, w_1^*, \ldots, w_H^*) := (E(\mathbf{y}^*; \mathbf{n}^*) \,|\, w(\mathbf{n}^*)) \text{ and} \tag{6.23a}$$

$$(\mathbf{e}^\circ \,|\, \mathbf{w}^\circ) = (e_1^\circ, \ldots, e_H^\circ \,|\, w_1^\circ, \ldots, w_H^\circ) := (E(\mathbf{y}^\circ; \mathbf{n}^\circ) \,|\, w(\mathbf{n}^\circ)) \tag{6.23b}$$

the corresponding adjusted income distributions. By definition of a between-type progressive transfer, we have $\mathbf{n}^* = \mathbf{n}^\circ = \mathbf{n}$, hence $\mathbf{w}^* = \mathbf{w}^\circ = \mathbf{w}$. Furthermore, since $E(\,\cdot\,;\,\cdot\,)$ is increasing in household income and decreasing in household size (conditions (3.2a) and (3.2b), respectively), we have

$$E(y_h^\circ; n_h) < E(y_h^*; n_h) < E(y_k^*; n_h) < E(y_k^*; n_k) < E(y_k^\circ; n_h), \tag{6.24}$$

hence $e_h^\circ < e_h^* < e_k^* < e_k^\circ$. Using (3.11a) and (3.11b), we verify that

$$\frac{w_h^*}{w_h^* + w_h^*}\, e_h^* = \frac{y_h^\circ + \Delta}{K(n_h^*) + K(n_k^*)} > \frac{y_h^\circ}{K(n_h^\circ) + K(n_k^\circ)} = \frac{w_h^\circ}{w_h^\circ + w_h^\circ}\, e_h^\circ \ \text{ and} \tag{6.25a}$$

$$\mu\,(\mathbf{e}^* \,|\, \mathbf{w}^*) = \frac{(y_h^\circ + \Delta) + (y_k^\circ - \Delta)}{K(n_h^*) + K(n_k^*)} = \frac{y_h^\circ + y_k^\circ}{K(n_h^*) + K(n_k^*)} = \mu\,(\mathbf{e}^\circ \,|\, \mathbf{w}^\circ)\,, \tag{6.25b}$$

hence $(e_h^*, e_k^* \,|\, w_h^*, w_k^*) \geq_L (e_h^\circ, e_k^\circ \,|\, w_h^\circ, w_k^\circ)$. Invoking Lemma 6.2, we conclude that

$$(E(\mathbf{y}^*; \mathbf{n}^*) \,|\, w(\mathbf{n}^*)) = (\mathbf{e}^* \,|\, \mathbf{w}^*) \geq_L (\mathbf{e}^\circ \,|\, \mathbf{w}^\circ) = (E(\mathbf{y}^\circ; \mathbf{n}^\circ) \,|\, w(\mathbf{n}^\circ)), \tag{6.26}$$

and condition WE is verified.

Necessity. Consider now the situations

$$(\mathbf{y}^*; \mathbf{n}^*) := (u + \Delta, v - \Delta, v, \ldots, v; n, m, m, \ldots, m) \text{ and} \tag{6.27a}$$

$$(\mathbf{y}^\circ; \mathbf{n}^\circ) := (u, v, v, \ldots, v; n, m, m, \ldots, m), \tag{6.27b}$$

where $u < u + \Delta \leqq v - \Delta < v$ and $1 \leqq m < n$. Clearly, $(\mathbf{y}^*; \mathbf{n}^*)$ is obtained from $(\mathbf{y}^\circ; \mathbf{n}^\circ)$ by means of a between-type progressive transfer involving households $h = 1$ and $k = 2$. Assuming that condition WE holds, we must have

$$(E(\mathbf{y}^*; \mathbf{n}^*) \,|\, w(\mathbf{n}^*)) \geq_L (E(\mathbf{y}^\circ; \mathbf{n}^\circ) \,|\, w(\mathbf{n}^\circ)). \tag{6.28}$$

This implies that

$$w(n)\, E(u + \Delta; n) + w(m)\, E(v - \Delta; m) = w(n)\, E(u; n) + w(m)\, E(v; m), \tag{6.29}$$

which holds for all $u < u + \Delta \leqq v - \Delta < v$ and $1 \leqq m < n$. Application of Lemma 6.1 gives

$$w(n)\, E(y;n) = \chi(n) + \eta\, y,\ \forall\, y > 0,\ \forall\, n \in [1,+\infty). \tag{6.30}$$

Letting $L(n) := \chi(n)/w(n)$ and $H(n) := \eta/w(n)$, (6.30) can be equivalently rewritten as

$$E(y;n) = \frac{\chi(n)}{w(n)} + \frac{\eta}{w(n)}\, y = L(n) + H(n)\, y,\ \forall\, y > 0,\ \forall\, n \in [1,+\infty). \tag{6.31}$$

Now we have to show that $L(n) := \chi(n)/w(n) = 0$, for all $n \in [1,+\infty)$. Clearly, we cannot have $L(n) < 0$, for some $n \in [1,+\infty)$. Indeed, if it were the case, then for sufficiently small values of $y > 0$ we would get $E(y;n) < 0$, which is excluded since, by definition of the equivalent income function, $E(y;n) > 0$, for all $y > 0$ and all $n \in [1,+\infty)$. Suppose next that $L(n) = \xi > 0$, for some $n \in [1,+\infty)$. Then, $E(y;n) \longrightarrow \xi \neq 0$ whenever $y \longrightarrow 0$, which contradicts condition (3.2d). Thus, we have $L(n) := \chi(n)/w(n) = 0$, for all $n \in [1,+\infty)$. Letting $K(n) = 1/H(n)$, we conclude that $E(y;n) = y/K(n)$ and $w(n) = \eta\, K(n)$, for all $y > 0$ and all $n \in [1,+\infty)$, which makes the proof complete. □

Proof of Proposition 3.2.

Necessity. Assume that the unidimensional inequality index J verifies UN, USI, UDI and ULC, and that the adjustment process $(E(\,\cdot\,;\,\cdot\,); w(\,\cdot\,))$ verifies WE. We have to show that, if the bidimensional inequality index J fulfills BNSI, then the equivalence scale is isoelastic. Choosing $(\mathbf{y}^\circ;\mathbf{n}^\circ) = (K(n_1),\ldots,K(n_H);n_1,\ldots,n_H)$, we get

$$\begin{aligned} I(\mathbf{y}^\circ;\mathbf{n}^\circ) &= J\left(E(y_1^\circ;n_1^\circ),\ldots,E(y_H^\circ;n_H^\circ) \mid w(n_1^\circ),\ldots,w(n_H^\circ)\right) && \text{(by (3.3))} \\ &= J\left(\frac{K(n_1)}{K(n_1)},\ldots,\frac{K(n_H)}{K(n_H)} \,\middle|\, \eta\, K(n_1),\ldots,\eta\, K(n_H)\right) && \text{(by Prop. 3.1)} \\ &= J\left(1,\ldots,1 \mid \eta\, K(n_1),\ldots,\eta\, K(n_H)\right) \\ &= J\left(1,\ldots,1 \mid K(n_1),\ldots,K(n_H)\right) && \text{(by UDI)} \\ &= 0. && \text{(by UN)} \end{aligned} \tag{6.32}$$

Choosing next $(\mathbf{y}^*;\mathbf{n}^*) = (K(n_1),\ldots,K(n_H);\lambda\, n_1,\ldots,\lambda\, n_H)$ with $\lambda > 0$, we obtain by a similar reasoning

$$\begin{aligned} I(\mathbf{y}^*;\mathbf{n}^*) &= J\left(E(y_1^*;n_1^*),\ldots,E(y_H^*;n_H^*) \mid w(n_1^*),\ldots,w(n_H^*)\right) && \text{(by (3.3))} \\ &= J\left(\frac{K(n_1)}{K(\lambda\, n_1)},\ldots,\frac{K(n_H)}{K(\lambda\, n_H)} \,\middle|\, \eta\, K(\lambda\, n_1),\ldots,\eta\, K(\lambda\, n_H)\right) && \text{(by Prop. 3.1)} \\ &= J\left(\frac{K(n_1)}{K(\lambda\, n_1)},\ldots,\frac{K(n_H)}{K(\lambda\, n_H)} \,\middle|\, K(\lambda\, n_1),\ldots,K(\lambda\, n_H)\right). && \text{(by UDI)} \end{aligned} \tag{6.33}$$

Invoking BNSI, we have $I(\mathbf{y}^*;\mathbf{n}^*) = I(\mathbf{y}^\circ;\mathbf{n}^\circ)$, which, upon using (6.32) and (6.33), implies that

$$J\left(\frac{K(n_1)}{K(\lambda\, n_1)},\ldots,\frac{K(n_H)}{K(\lambda\, n_H)} \,\middle|\, K(\lambda\, n_1),\ldots,K(\lambda\, n_H)\right) = 0, \tag{6.34}$$

which holds true whatever the equivalence scale function $K(\cdot)$, the distribution of size $\mathbf{n} := (n_1, \ldots, n_H)$ and $\lambda > 0$. This implies that

$$\frac{K(\lambda\, n_1)}{K(n_1)} = \frac{K(\lambda\, n_2)}{K(n_2)} = \cdots = \frac{K(\lambda\, n_H)}{K(n_H)}. \tag{6.35}$$

Because (6.35) holds true for all $\mathbf{n} = (n_1, \ldots, n_H) \in [1, +\infty)^H$ and all $\lambda > 0$, we deduce that

$$K(\lambda\, n) = \phi(\lambda)\, K(n),\ \forall\ \lambda > 0,\ \forall\ n \in [1, +\infty), \tag{6.36}$$

a functional equation the solution of which (see Aczel (1966, Chapter 2)) is precisely (3.11).

Sufficiency. Assume that condition (3.13) holds and consider an arbitrary situation $(\mathbf{y}; \mathbf{n}) := (y_1, \ldots, y_H; n_1, \ldots, n_H)$. Choosing any $\lambda > 0$, we have

$$\begin{aligned}
I(\mathbf{y}; \lambda\,\mathbf{n}) &= J\left(E(y_1; \lambda\, n_1), \ldots, E(y_H; \lambda\, n_H) \mid w(\lambda\, n_1), \ldots, w(\lambda\, n_H)\right) && \text{(by (3.3))}\\
&= J\left(\frac{y_1}{K(\lambda\, n_1)}, \ldots, \frac{y_H}{K(\lambda\, n_H)} \;\middle|\; \eta\, K(\lambda\, n_1), \ldots, \eta\, K(\lambda\, n_H)\right) && \text{(by Prop. 3.1)}\\
&= J\left(\frac{y_1}{\xi(\lambda\, n_1)^\epsilon}, \ldots, \frac{y_H}{\xi(\lambda\, n_1)^\epsilon} \;\middle|\; \eta\, \xi(\lambda\, n_1)^\epsilon, \ldots, \eta\, \xi(\lambda\, n_H)^\epsilon\right) && \text{(by assumption)}\\
&= J\left(\frac{1}{\lambda^\epsilon}\,\frac{y_1}{\xi n_1^\epsilon}, \ldots, \frac{1}{\lambda^\epsilon}\,\frac{y_H}{\xi n_H^\epsilon} \;\middle|\; \eta\, \lambda^\epsilon (\xi n_1^\epsilon), \ldots, \eta\, \lambda^\epsilon (\xi n_H^\epsilon)\right)\\
&= J\left(\frac{1}{\lambda^\epsilon}\,\frac{y_1}{\xi n_1^\epsilon}, \ldots, \frac{1}{\lambda^\epsilon}\,\frac{y_H}{\xi n_H^\epsilon} \;\middle|\; \eta\, \xi n_1^\epsilon, \ldots, \eta\, \xi n_H^\epsilon\right) && \text{(by UDI)}\\
&= J\left(\frac{y_1}{\xi n_1^\epsilon}, \ldots, \frac{y_H}{\xi n_H^\epsilon} \;\middle|\; \eta\, \xi n_1^\epsilon, \ldots, \eta\, \xi n_H^\epsilon\right) && \text{(by USI)}\\
&= J\left(\frac{y_1}{K(n_1)}, \ldots, \frac{y_H}{K(n_H)} \;\middle|\; \eta\, K(n_1), \ldots, \eta\, K(n_H)\right) && \text{(by assumption)}\\
&= J\left(E(y_1; n_1), \ldots, E(y_H; n_H) \mid w(n_1), \ldots, w(n_H)\right) && \text{(by Prop. 3.1)}\\
&= I(\mathbf{y}; \mathbf{n}), && \text{(by (3.3))}
\end{aligned} \tag{6.37}$$

hence condition BNSI is verified. □

Before we proceed to the proof of Proposition 4.1, we find it convenient to introduce the following intermediate result that will be used repeatedly.

Lemma 6.3. *Let $J := [1, c)$ whenever $c > 1$ and $J := (d, 1]$ whenever $0 \leqq d < 1$. Then, there exist a non-constant function h and a function k such that*

$$h(ab) = h(a) + k(a)\, h(b),\ \forall\ a, b, ab \in J, \tag{6.38}$$

if and only if

$$\text{either } h(a) = \alpha \ln a \text{ and } k(a) = 1,\ \forall\ a \in J, \tag{6.39a}$$

$$\text{or } h(a) = \alpha\,[a^\eta - 1] \text{ and } k(a) = a^\eta\ (\lambda \neq 0),\ \forall\ a \in J. \tag{6.39b}$$

Furthermore, the functions h and k defined in (6.39) satisfy (6.38) for all $a, b, ab > 0$.

Proof of Lemma 6.3. The proof is analogous to that of Aczel (1984, Lemma 3) to which we refer the interested reader and it is therefore omitted. □

Proof of Proposition 4.1. We have already established that conditions (a) and (b) are sufficient for the (relative) equivalence scale to be isoelastic and we therefore confine ourselves to showing that they are also necessary. While the function $\psi \in \mathbf{\Psi}$ and the parameter $\theta \in [0,1)$ are kept fixed throughout the proof, we maintain them for clarity even though that makes notation more complicated. Suppose that $M(p,q,u;\psi,\theta,n)$ is isoelastic with respect to household size n, in which case there exist continuous functions $g(p,q,u;\psi,\theta,1)$ and $f(p,q,u;\psi,\theta,1)$ such that

$$M(p,q,u;\psi,\theta,n) = g(p,q,u;\psi,\theta,1)\, n^{f(p,q,u;\psi,\theta,1)}, \tag{6.40}$$

for all $n \geqq 1$ and all (p,q,u). The proof proceeds in four successive steps. In Step 1 we derive the implications of our assumptions for the expenditure function $\overline{C}(p,\psi(\theta,n)\,q/n,u)$. We obtain a complicated functional equation which is simplified in Step 2, and its solutions are derived in Step 3. Finally, the consequences for the expenditure function are examined in Step 4 and it turns out that preferences must be Cobb-Douglas and the congestion function isoelastic.

Step 1. Making use of (2.10) and (6.40), we obtain the functional equation

$$C(p,q,u;\psi,\theta,n) = C(p,q,u;\psi,\theta,1)\, g(p,q,u;\psi,\theta,1)\, n^{f(p,q,u;\psi,\theta,1)},\ \forall\, n \geqq 1,\ \forall\,(p,q,u). \tag{6.41}$$

Setting $n=1$ in (6.41) implies that

$$g(p,q,u;\psi,\theta,1) = 1,\ \forall\,(p,q,u), \tag{6.42}$$

which upon substituting into (6.41) and making use of (2.10) and (2.2d) gives

$$n\,\overline{C}\left(p, \frac{\psi(\theta,n)\,q}{n}, u\right) = \overline{C}(p,q,u)\, n^{f(p,q,u;\psi,\theta,1)},\ \forall\, n \geqq 1,\ \forall\,(p,q,u). \tag{6.43}$$

Substituting $(1,q/p)$ for (p,q) in (6.43) and dividing both sides by n, we obtain

$$\overline{C}\left(1, \frac{\psi(\theta,n)}{n}\frac{q}{p}, u\right) = \overline{C}\left(1, \frac{q}{p}, u\right) n^{f\left(1,\frac{q}{p},u;\psi,\theta,1\right)-1},\ \forall\, n \geqq 1,\ \forall\,(p,q,u). \tag{6.44}$$

Letting $\xi(\theta,n) := n/\psi(\theta,n)$, equation (6.44) rewrites

$$\overline{C}\left(1, \frac{1}{\xi(\theta,n)}\frac{q}{p}, u\right) = \overline{C}\left(1, \frac{q}{p}, u\right) n^{f\left(1,\frac{q}{p},u;\psi,\theta,1\right)-1},\ \forall\, n \geqq 1,\ \forall\,(p,q,u). \tag{6.45}$$

Since $\xi(\theta,n)$ is strictly monotone in n, it has an inverse $\varphi(\theta,r)$ defined by

$$\xi(\theta,\varphi(\theta,r)) = r,\ \forall\, r \in I(\theta) := \{s \mid \exists\, n \in [1,+\infty) : \xi(\theta,n) = s\}, \tag{6.46}$$

where

$$I(\theta) \subseteq (0,1] \text{ if } \xi(\theta,n) \text{ is decreasing in } n, \text{ and} \tag{6.47a}$$

$$I(\theta) \subseteq [1,+\infty) \text{ if } \xi(\theta,n) \text{ is increasing in } n. \tag{6.47b}$$

For later use, we find convenient to introduce the set

$$J(\theta) := \{s \mid \exists\, r \in I(\theta) : s = 1/r\}, \tag{6.48}$$

where

(6.49a) $$J(\theta) \subseteq (0,1] \text{ if } \xi(\theta,n) \text{ is decreasing in } n, \text{ and}$$

(6.49b) $$J(\theta) \subseteq [1,+\infty) \text{ if } \xi(\theta,n) \text{ is increasing in } n.$$

Replacing n by $\varphi(\theta,r)$ in (6.45), we obtain

(6.50) $$\overline{C}\left(1, \frac{q}{p}\frac{1}{r}, u\right) = \overline{C}\left(1, \frac{q}{p}, u\right) \varphi(\theta,r)^{f\left(1,\frac{q}{p},u;\psi,\theta,1\right)-1}, \ \forall\, r \in I(\theta), \ \forall\, (p,q,u).$$

Step 2. Now define

(6.51a) $$h(a,u) := \ln \overline{C}(1,a,u), \qquad (a = q/p)$$

(6.51b) $$k(\theta;a,u) := f(1,a,u;\psi,\theta,1) - 1,$$

(6.51c) $$\ell(\theta;b) := \ln\, \varphi(\theta,1/b). \qquad (b = 1/r)$$

For later use, we note that the function $h(a,u)$ inherits the properties of the (individual) expenditure function: in particular, it is increasing in a. Similarly, $\ell(\theta;b)$ is strictly monotonic in b since the inverse $\varphi(\theta,r)$ of $\xi(\theta,n)$ is strictly monotonic in r. Then (6.50) can be rewritten as

(6.52) $$h\left(\frac{q}{p}\frac{1}{r}, u\right) = h\left(\frac{q}{p}, u\right) + k\left(\theta; \frac{q}{p}, u\right) \ell\left(\theta; \frac{1}{r}\right), \ \forall\, r \in I(\theta), \ \forall\, (p,q,u),$$

which is equivalent to the following functional equation

(6.53) $$h(ab,u) = h(a,u) + k(\theta;a,u)\, \ell(\theta;b), \ \forall\, b \in J(\theta), \ \forall\, a > 0, \ \forall\, u.$$

Setting $a = 1$ in the preceding equation, we get

(6.54) $$h(b,u) = h(1,u) + k(\theta;1,u)\, \ell(\theta;b), \ \forall\, b \in J(\theta), \ \forall\, u.$$

Because $h(\,\cdot\,,u)$ is increasing, we must have $k(\theta;1,u) \neq 0$, and we deduce from (6.54) that

(6.55) $$\ell(\theta;b) = \frac{h(b,u) - h(1,u)}{k(\theta;1,u)}, \ \forall\, b \in J(\theta), \ \forall\, u.$$

Substituting into (6.53) and subtracting $h(1,u)$ from both sides, we get

(6.56) $$h(ab,u) - h(1,u) = h(a,u) - h(1,u) + \frac{k(\theta;a,u)}{k(\theta;1,u)}\, [h(b,u) - h(1,u)],$$

for all $b \in J(\theta)$, all $a > 0$, and all u. Now define

(6.57a) $$\hat{h}(a,u) := h(a,u) - h(1,u),$$

(6.57b) $$\hat{k}(\theta;a,u) := k(\theta;a,u)/k(\theta;1,u),$$

and substitute into (6.56) to get

(6.58) $$\hat{h}(ab,u) = \hat{h}(a,u) + \hat{k}(\theta;a,u)\, \hat{h}(b,u), \ \forall\, a > 0, \forall\, b \in J(\theta), \ \forall\, u,$$

which implies that $\hat{k}(\theta;a,u)$ is independent of θ, hence $\hat{k}(\theta;a,u) = \tilde{k}(a,u)$.

Step 3. Given $\theta \in [0,1)$ and $u \in \mathbb{R}$, we want to solve the functional equation

$$\hat{h}(ab,u) = \hat{h}(a,u) + \tilde{k}(a,u)\,\hat{h}(b,u), \text{ where } a,b,ab \in J(\theta). \tag{6.59}$$

Invoking Lemma 6.3 and letting

$$\beta(u) := h(1,u) \quad \text{and} \quad \gamma(\theta;u) := k(\theta;1,u), \tag{6.60}$$

there are two cases to be considered.

Case 1: $\hat{h}(a,u) = \alpha(u)\ln a$ and $\tilde{k}(a,u) = 1$, for all $a \in \mathbb{R}_{++}$.
Making use of (6.57a), (6.57b), (6.60), and upon substituting into (6.51a), (6.51b) and (6.51c), we obtain

$$h(a,u) = \hat{h}(a,u) + h(1,u) = \alpha(u)\,\ln a + \beta(u),\ \forall\, a \in \mathbb{R}_{++}; \tag{6.61a}$$

$$k(\theta;a,u) = \tilde{k}(a,u)\,k(\theta;1,u) = \gamma(\theta;u),\ \forall\, a \in \mathbb{R}_{++}; \tag{6.61b}$$

$$\ell(\theta;a) = \frac{h(a,u)-h(1,u)}{k(\theta;1,u)} = \frac{\alpha(u)\,\ln a}{\gamma(\theta;u)},\ \forall\, a \in \mathbb{R}_{++}. \tag{6.61c}$$

Case 2: $\hat{h}(a,u) = \alpha[a^\eta - 1]$ and $\tilde{k}(a,u) = a^\eta$ ($\eta \neq 0$), for all $a \in \mathbb{R}_{++}$. Making use again of (6.57a), (6.57b), (6.60), and upon substituting into (6.51a), (6.51b) and (6.51c), we obtain

$$h(a,u) = \hat{h}(a,u) + h(1,u) = \alpha(u)\,[\,a^{\eta(u)} - 1\,] + \beta(u),\ \forall\, a \in \mathbb{R}_{++}; \tag{6.62a}$$

$$k(\theta;a,u) = \tilde{k}(a,u)\,k(\theta;1,u) = \gamma(\theta;u),\ \forall\, a \in \mathbb{R}_{++}; \tag{6.62b}$$

$$\ell(\theta;a) = \frac{h(a,u)-h(1,u)}{k(\theta;1,u)} = \alpha(u)\,[\,a^{\eta(u)} - 1\,],\ \forall\, a \in \mathbb{R}_{++}. \tag{6.62c}$$

Step 4. Now we examine the implications for the expenditure function and the congestion function of the two solutions we have obtained above.

Case 1. $h(t,u) = \alpha(u)\,\ln t + \beta(u)$. Then we have

$$e^{h(t,u)} = e^{\alpha(u)\,\ln t + \beta(u)} = e^{\beta(u)}\,e^{\ln t^{\alpha(u)}} = \delta(u)\,t^{\alpha(u)}, \tag{6.63}$$

where $\delta(u) := e^{\beta(u)}$, for all $t > 0$ and all u. Substituting into (6.51a) and acknowledging the linear homogeneity in prices of the expenditure function, we get

$$\overline{C}(p,q,u) = p\,\overline{C}\left(1,\frac{q}{p},u\right) = p\,e^{h\left(\frac{q}{p},u\right)} = p\,\delta(u)\left(\frac{q}{p}\right)^{\alpha(u)} = \delta(u)\,p^{1-\alpha(u)}\,q^{\alpha(u)}, \tag{6.64}$$

for all $(p,q) \gg (0,0)$ and all u. The monotonicity of the expenditure function in prices implies that $1-\alpha(u) > 0$ and $\alpha(u) > 0$, hence $0 < \alpha(u) < 1$, for all u. By definition, the individual expenditure function is strictly increasing in u, which implies that

$$\begin{aligned}\frac{\partial\,\overline{C}(p,q,u)}{\partial\,u} &= \delta'(u)\,p^{1-\alpha(u)}\,q^{\alpha(u)} + \delta(u)\,p^{1-\alpha(u)}\,q^{\alpha(u)}\,[\,\alpha'(u)\ln q - \alpha'(u)\ln p\,] \\ &= p^{1-\alpha(u)}\,q^{\alpha(u)}\left[\delta'(u) + \delta(u)\,\alpha'(u)\,\ln\left(\frac{q}{p}\right)\right] > 0,\end{aligned} \tag{6.65}$$

where $\alpha'(u)$ is the derivative of $\alpha(u)$ with respect to u. Since $\ln(q/p) \in (-\infty, +\infty)$, it is necessary for (6.65) to hold that $\alpha'(u) = 0$, hence $\alpha(u)$ is independent of u and $\alpha(u) = \epsilon \in (0,1)$, for all u. Furthermore, we have

$$\ell(\theta; s) = \ln s^{\alpha(u)/\gamma(\theta;u)} = \ln s^{\epsilon/\gamma(\theta;u)}, \ \forall\, s > 0, \ \forall\, u, \tag{6.66}$$

and we conclude that $\gamma(\theta; u)$ is independent of u, hence $\gamma(\theta; u) = \zeta(\theta)$. By definition

$$\ell(\theta; s) := \ln\, \varphi(\theta, 1/s) = \ln s^{\epsilon/\zeta(\theta)}, \ \forall\, s > 0, \tag{6.67}$$

which implies that

$$\varphi(\theta, 1/s) = s^{\epsilon/\zeta(\theta)}, \ \forall\, s > 0, \tag{6.68}$$

or equivalently

$$\varphi(\theta, r) = (1/r)^{\epsilon/\zeta(\theta)} = r^{-\epsilon/\zeta(\theta)}, \ \forall\, r > 0. \tag{6.69}$$

Using the fact that by definition $\varphi(\theta, r) = n$, we obtain

$$\xi(\theta, n) = \xi(\theta, \varphi(\theta, r)) = r = n^{-\zeta(\theta)/\epsilon}, \tag{6.70}$$

and finally

$$\psi(\theta, n) = \frac{n}{\xi(\theta, n)} = \frac{n}{n^{-\zeta(\theta)/\epsilon}} = n^{\rho(\theta)}, \tag{6.71}$$

where $\rho(\theta) := 1 + \zeta(\theta)/\epsilon$, and it is an admissible congestion function.

Case 2. $h(t,u) = \alpha(u)\,[\,t^{\eta(u)} - 1\,] + \beta(u)$. Then we have

$$e^{h(t,u)} = e^{\alpha(u)\,[\,t^{\eta(u)}-1]+\beta(u)} = e^{\beta(u)}\, e^{\alpha(u)\,[\,t^{\eta(u)}-1]} =: \delta(u)\, e^{\alpha(u)\,[\,t^{\eta(u)}-1]}, \tag{6.72}$$

for all $t > 0$ and all u. Substituting into (6.51a) and using the linear homogeneity in prices of the expenditure function, we obtain

$$\overline{C}(p,q,u) = p\,\overline{C}\left(1, \frac{q}{p}, u\right) = p\, e^{h\left(\frac{q}{p}, u\right)} = \delta(u)\, p\, e^{\alpha(u)\,\left[\left(\frac{q}{p}\right)^{\eta(u)} - 1\right]}, \tag{6.73}$$

for all $(p,q) \gg 0$ and all u. By definition, the expenditure function $\overline{C}(p,q,u)$ must be increasing in prices. On the one hand, we must have

$$\frac{\partial\, \overline{C}(p,q,u)}{\partial\, q} = \delta(u)\, p\, e^{\alpha(u)\,\left[\left(\frac{q}{p}\right)^{\eta(u)} - 1\right]}\, \alpha(u)\, \frac{\eta(u)}{p} \left(\frac{q}{p}\right)^{\eta(u)-1} > 0, \tag{6.74}$$

which simplifies to

$$\frac{\partial\, \overline{C}(p,q,u)}{\partial\, q} = \delta(u)\, e^{\alpha(u)\,\left[\left(\frac{q}{p}\right)^{\eta(u)} - 1\right]}\, \alpha(u)\, \eta(u) \left(\frac{q}{p}\right)^{\eta(u)-1} > 0. \tag{6.75}$$

Since $\delta(u) > 0$, it is necessary for (6.75) to hold that $\alpha(u)\,\eta(u) > 0$. On the other hand, it must be the case that

$$\frac{\partial\, \overline{C}(p,q,u)}{\partial\, p} = \delta(u)\, e^{\alpha(u)\,\left[\left(\frac{q}{p}\right)^{\eta(u)} - 1\right]} - \delta(u)\, e^{\alpha(u)\,\left[\left(\frac{q}{p}\right)^{\eta(u)} - 1\right]}\, \alpha(u)\, \eta(u) \left(\frac{q}{p}\right)^{\eta(u)} > 0, \tag{6.76}$$

which simplifies to

$$\frac{\partial\,\overline{C}(p,q,u)}{\partial\,p} = \delta(u)\,e^{\alpha(u)\left[\left(\frac{q}{p}\right)^{\eta(u)}-1\right]}\left[1-\alpha(u)\,\eta(u)\,\left(\frac{q}{p}\right)^{\eta(u)}\right] > 0. \tag{6.77}$$

Since $\delta(u) > 0$, it is necessary for (6.77) to hold that the term within squared brackets is positive. However, depending on the values of q/p and $\eta(u) - 1$, this term can be positive, negative, or zero, and we therefore conclude that Case 2 is impossible.

To sum up, we have shown that

$$\overline{C}(p,q,u) = \delta(u)\,p^{1-\epsilon}\,q^{\epsilon}\ (0 < \epsilon < 1),\ \forall\ (p,q) \gg (0,0),\ \forall\ u,\ \text{and} \tag{6.78}$$

$$\psi(\theta,n) = n^{\rho(\theta)}\ \text{ where } \theta \in [0,1),\ \forall\ n \in [1,+\infty), \tag{6.79}$$

and the proof is complete. □

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