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Choice with incomparable alternatives.

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Abstract

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Keywords: non-transitivity, categorization, multiple criteria decision making, incomparability.

JEL classification: D01, D11.

1 Introduction

The act of choice is based on the possibility to compare the available options. In the rational choice theory, the decision maker ranks all the alternatives and selects his most preferred one. So, he behaves as if he were capable of assessing all the items to one another. However, in reality, the comparability of the choice objects can be questioned. The adage "you cannot compare apples and oranges" advises not to draw parallel between alternatives that are popularly considered to be incomparable, even if they can be available together in a choice set. Likewise, the personal experience of a difficulty or even an impossibility to compare the options is not uncommon. In this paper, we focus on choice problems with

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incomparable alternatives: we characterize and represent the choice behavior of an agent facing this difficulty.

First, we need to introduce in concrete terms our concepts of comparability and incomparability. Three configurations with basic choice behaviors can be used:

Configuration 1:

$$\Big\{ \text{apple, yoghurt} \ \rightarrow \ \text{apple}$$ [1]

⇒ For this agent, apple and yoghurt are *comparable* and he prefers apple.

Configuration 2:

$$\begin{cases} \text{red apple, green apple} \ \rightarrow \ \text{red apple, green apple} \\ \text{red apple, yoghurt} \ \rightarrow \ \text{red apple} \\ \text{green apple, yoghurt} \ \rightarrow \ \text{green apple} \end{cases}$$

⇒ Red apple and green apple are *comparable* and they are *equivalent*: both are preferred to yoghurt.

Configuration 3:

$$\begin{cases} \text{apple, yoghurt} \ \rightarrow \ \text{apple} \\ \text{yoghurt, cheese} \ \rightarrow \ \text{yoghurt} \\ \text{apple, cheese} \ \rightarrow \ \text{apple, cheese} \end{cases}$$

⇒ Apple is comparable to yoghurt and it is preferred. Yoghurt is comparable to cheese and it is preferred. Apple and cheese are *incomparable*: none is preferred, so both are chosen.

Thus, the agent's choices reveal easily if he is facing comparable or incomparable options and these examples help identifying these notions distinctly.

The worthwhile observed choices occur when the decision maker picks several options. In our framework, this behavior can lead to opposite interpretations: both alternatives are chosen because they are *comparable-equivalent* or *incomparable*. However, depending on the relationship between these options and a third one (they dominate or are dominated by it), these situations can be separated. The alternatives are *comparable-equivalent* if they have the same relationship with the third one (cf. Configuration 2); they are *incomparable* if they have opposite relationship with the third one (cf. Configuration 3).

Moreover, this issue of incomparability of alternatives raises a phenomenon described as irrational: non-transitive preferences. Indeed, in the third configuration, apple is preferred to yoghurt which is preferred to cheese, but apple is not preferred to cheese. Although this choice behavior does not seem unreasonable, the standard theory cannot explain it. With this approach focused on incomparability, the originality of our model is to provide a credible and coherent justification for non-transitive preferences. The intuition is that

[1] It can be read as follows: when the agent faces an apple and a yoghurt, he chooses the apple.

the agent cannot compare apple and cheese because they are too different. For instance, he may consider that they belong to two categories: apple is a dessert and cheese is a dairy product. If both categories are equally important to him, then apple and cheese appear to be incomparable, so he picks both. Thus, our model can justify some violations of transitivity.

With this explanation of non-transitive phenomenons, we notice that it is quite natural to understand the decision making with incomparable alternatives by assuming that the agent uses a categorization. When he faces a choice problem, we consider that he behaves as if he partitions the set in unordered categories: he groups the comparable options together and isolates the incomparable ones. Then, he picks his most preferred alternative in each category. This decision process seems credible because cognitive science showed that using a categorization helps simplifying complex choice problem. Recently, the advantage of categorization in the decision process was also studied in the context of choice theory. Manzini and Mariotti [8] are interested in a two-stage decision process: "Categorize Then Choose". First, the agent categorizes the alternatives and eliminates the options in dominated categories. Second, he selects his preferred alternative amongst the remaining ones. The advantage of this sequential choice is to avoid pairwise comparisons which can be tedious if the size of available alternatives is large. Their model can also justify some "irrational" phenomenons: pairwise cycle of choice and menu dependence.

Our model is different from the model of Manzini and Mariotti [8]. We consider that the decision process is not sequential and there is no ranking of categories: all classes are equally important to the decision maker. Formally, this choice behavior can be summarized by a two criteria decision making[2]. The first one is a *weak order* that is a partial order on the set of all alternatives. For every pairs of alternatives, either one is preferred, or the agent has no preference between them. The second criterion is an *equivalence relation*. The set of all available alternatives is partitioned in equivalence classes which represent the unordered categories in our explanation. So, the choice resulting from the application of these two criteria corresponds to the intersection of an equivalence relation and a weak order. The properties of an equivalence relation (reflexivity, symmetry and transitivity) and of a weak order (asymmetry and negative transitivity) are simple and very common. We could expect that the properties satisfied by the result of their intersection should also be usual. Indeed, we can easily show that this binary relation is asymmetric, incomplete and transitive. However, these requirements are not sufficient for the characterization of this intersection. Therefore, we introduce a new property, called *Common Domination Implies Equivalence* (CDIE). This condition is quite intuitive: if two alternatives do not dominate each other and they have a common "dominant" or "dominated" alternative, then these two alternatives dominate the same options and are dominated by the same options. This new property helps us characterizing the natural decision process of choosing the best

[2]For the literature on multicriteria decision making, see Manzini and Mariotti [7] ; Apesteguia and Ballester [1] ; Houy and Tadenuma [5].

alternative in each category.

This choice process with a categorization can be appreciated by a multidimensional representation. Each alternative can be written as a vector of coordinates, one for each category. The alternatives are comparable if both have a positive coordinate in the same dimension. In this case, the option with the largest coordinate is chosen. Otherwise, when the alternatives do not have a positive coordinate in a common category, they are incomparable and both are chosen. We offer this representation theorem in the paper.

This approach in terms of categorization helps us providing a first solution to modelize choice behaviors with incomparable alternatives. However, despite the attractiveness of this interpretation, we show that it only corresponds to a specific case. The decision process of an agent can be reduced to a selection of the most preferred alternative in each category only if the categories are disjoint: an alternative belongs to a single category. In the general case, that is when an alternative can belong to several categories, this simplification cannot be used to characterize the agent's choices. In other words, it is not always possible to reduce the issue of incomparability of the alternatives by considering that the decision maker chooses by the application of a two criteria decison process. Therefore, we suggest a more general interpretation of this issue of incomparability, based on the gap between the alternatives. Intuitively, two options are considered incomparable if they are too different, that is if they are very far from each other.

First, we present the characterization of this choice behavior in a general framework. It is based mainly on our new property on preferences, CDIE, to which is added the most simple requirement on rationality: acyclicity. So, this characterization allows non-transitive preferences as we have shown in the Configuration 3. Here, the incomparability is based on an intuition of a "distance" between the alternatives. Therefore, we suggest an original representation theorem to give a visual conception of this issue.

This representation corresponds to a synthesis: it brings together information on comparability and superiority of the alternatives. With the function of representation, we attribute a real number to each alternative. For instance, we assume that two alternatives are available. If they are "sufficiently close" (the distance between their value is sufficiently small) then the decision maker picks the greatest one. When the values of the options are close, we consider that the alternatives are comparable. Their relative ranking by the function can be interpreted in the usual way: the alternative with the greatest real number is chosen. But if the two alternatives are "sufficiently far apart" (the distance between their value is too large) then the decision-maker picks both. When the values of the options are distant, we consider that the alternatives are incomparable for the agent.

This general representation theorem based only on our new property CDIE and acyclicity is an orginal contribution to the literature because it is a representation theorem for non-transitive strict preference relation. This possibility comes from the following fallacy: even if two alternatives are comparable to a third one, it does not imply that they should be comparable. Indeed, we can have three alternatives x, y, z such that x is sufficiently close (i.e. comparable) to y and is better, y is sufficiently close to z and is better, but x and z are too far apart: they are incomparable, both are chosen.

We also provide this representation adapted to the specific case where the categories are disjoint. We need to add a restriction on the representation function to take into consideration a transitive binary relation. The main idea is that the comparable alternatives are grouped together because they are in the same equivalence class. All comparable alternatives have sufficiently close values: thus an equivalence class is translated into a "bundle" for the function of representation. And different classes, which are incomparable, are sufficiently distant.

Finally, we wish to emphasize the originality of this interpretation: if two alternatives are too different, too far apart, the agent cannot compare them and he chooses both. Actually, there already exists an opposed interpretation with the literature of *fuzzy preferences* (Luce [6], Scott and Suppes [10] and Fishburn [4]): when two alternatives are too close, too similar, they are "equivalent" for the agent, and he picks both. This behavior is explained by limited cognitive capacities: the agent cannot discriminate between these alternatives so he takes all of them.

In the next section, we state notation and definitions. Then, we provide a general characterization of a choice behavior with incomparable options. In Section 4, we focus on the specific case of categorization. Then, we introduce our general representation theorem. Finally, we suggest some concluding remarks and we link our work with the key results of the rational choice theory. The main proofs are given in the appendix.

2 Notation and basic definitions

Let X be a finite set of alternatives. Let $\mathcal{X} = 2^X \setminus \emptyset$ be the set of all nonempty subsets of X. A binary relation P is a subset of $X \times X$ and P is:

- *reflexive* if $\forall x \in X$, $(x, x) \in P$;
- *irreflexive* if $\forall x \in X$, $(x, x) \notin P$;
- *symmetric* if $\forall x, y \in X$, $(x, y) \in P$ implies $(y, x) \in P$;
- *asymmetric* if $\forall x, y \in X$, $(x, y) \in P$ implies $(y, x) \notin P$;[3]
- *transitive* if $\forall x, y, z \in X$, $(x, y) \in P$ and $(y, z) \in P$ imply $(x, z) \in P$;
- *negatively transitive* if $\forall x, y, z \in X$, $(x, y) \notin P$ and $(y, z) \notin P$ imply $(x, z) \notin P$;[4]
- *acyclic* if $\forall n \in \mathbb{N} \setminus \{1\}$, $\forall x_1, \ldots, x_n \in X$, $[\forall i \in \{1, \ldots, n-1\}, (x_i, x_{i+1}) \in P]$ imply $x_1 \neq x_n$.[5]

[3] Note that asymmetry implies irreflexivity.

[4] Note the contrapositive is often used and is more readable: P is *negatively transitive* if $\forall x, y, z \in X$, $(x, z) \in P$ implies $(x, y) \in P$ or $(y, z) \in P$.

[5] Note that acyclicity implies asymmetry.

We also need some common compositions of these properties. A binary relation P is:

- a *weak order* if P is asymmetric and negatively transitive;
- an *equivalence relation* if P is reflexive, symmetric and transitive.

The following definitions ensure a simplification of the notation and a better comprehension of new concepts introduced below. Let P be a binary relation on X and let $x \in X$. We define ${}^{P}x$ the *set of predecessors of* x (or upper contour set), *i.e.* ${}^{P}x = \{y \in X | (y, x) \in P\}$. We define x^P the *set of successors of* x (or lower contour set), *i.e.* $x^P = \{y \in X | (x, y) \in P\}$.

Remark 1 *We recall some useful comments on the sets of predecessors and successors:*

- *if x is the predecessor of y, then y is the successor of x (and reciprocally);*
- *if P is asymmetric, then for all $x \in X$, ${}^{P}x \cap x^P = \emptyset$: an alternative cannot belong to both the set of predecessors and the set of successors of x;*
- *if P is an equivalence relation, then for all $x, y \in X$, $(x, y) \in P$ implies ${}^{P}x = {}^{P}y = x^P = y^P$: when two alternatives are in an equivalence class, they have the same set of predecessors and successors and those ones are equal.*

Given a binary relation P, we define a particular equivalence between alternatives: x and y are P*-equivalent* (denoted $\approxeq_P$) if they have the same predecessors and the same successors. That is, x and y dominate and are dominated by the same options.

Definition 1 *Let P be a binary relation on X.*
$\forall x, y \in X$, $x \approxeq_P y$ if ${}^{P}x = {}^{P}y$ and $x^P = y^P$.

Note that if P is irreflexive then 2 alternatives that are P-equivalent do not dominate each other. Indeed, in this case, for all $x, y \in X$, $x \approxeq_P y$ implies $x \notin {}^{P}y \cup y^P$: x is neither a predecessor, nor a successor of y.

3 General characterization

We characterize the choice behavior of an agent who faces a set with comparable and incomparable alternatives. First, we show the result for a general framework. In the next section, we will focus on the specific case where the decision process can be reduced to a selection by applying a categorization.

By providing a link between the choices and a preference relation, we have a first model to formalize the issue of decision making with incomparable alternatives. Here, the choice function is rationalizable by a binary relation P which is acyclic and satisfies a new property, called *Common Domination Implies Equivalence.*

3.1 *Common Domination Implies Equivalence*

Definition 2 (Common Domination Implies Equivalence, CDIE) .
P satisfies Common Domination Implies Equivalence *if*
$[\forall x, y \in X, (i)\ y \notin {}^{P}x \cup x^{P}$ *and* $(ii)\ {}^{P}x \cap {}^{P}y \neq \emptyset$ *or* $x^{P} \cap y^{P} \neq \emptyset,$ *imply* $x \approxeq_{P} y]$.

A binary relation satisfies CDIE if, when two alternatives do not dominate each other and they have a common alternative which dominates them or is dominated by both of them, then they have the same predecessors and successors.

If we refer to the examples given in the introduction, CDIE can be illustrated with the Configuration 2. Indeed, red apple and green apple are *P-equivalent* for our fictional decision maker. First, red and green apples do not dominate each other, because, facing both, the agent picks both. Second, they both dominate yoghurt. If we introduce a new alternative, let say a brownie, CDIE tells us that if the agent prefers the brownie to the red apple, he would also prefer the brownie to the green apple.

Before stating the complete characterization, we present how CDIE is related to usual properties:

Proposition 1 *Let P be an asymmetric binary relation on X.*

1. *Negative transitivity implies CDIE, but the converse is not true ;*
2. *Transitivity and CDIE are logically independent.*

Proof. Let a binary relation $P \subseteq X \times X$ be asymmetric.

1. (1.1) Let us show that: Negative transitivity $\Rightarrow$ CDIE.
   Let $X = \{x, y, z, w\}$ be the set of universal alternatives. Assume (i) $y \notin {}^{P}x \cup x^{P}$; (ii) $z \in {}^{P}x \cap {}^{P}y$ and (iii) $w \in {}^{P}x$. P negatively transitive implies $w \in {}^{P}y$ or $y \in {}^{P}x$. By assumption, $y \notin {}^{P}x$, so $w \in {}^{P}y$. Hence P sastifies CDIE.

   (1.2) Let us show that: CDIE $\not\Rightarrow$ Negative transitivity (counterexample).
   Let $X = \{w, x, y\}$ be the universal set of options. Assume $P = \{(w, x), (y, w)\}$. CDIE is vacuously satisfied by P since (i) $y \notin {}^{P}x \cup x^{P}$ and (ii) ${}^{P}x \cap {}^{P}y = \emptyset$ and $x^{P} \cap y^{P} = \emptyset$. However, P is not negatively transitive: $(w, x) \in P$ but $(x, y) \notin P$ and $(w, y) \notin P$.

2. (2.1) Let us show that: Transitivity $\not\Rightarrow$ CDIE (counterexample).
   Let $X = \{x, y, z, w\}$ be the universal set of options. Assume that we have $P = \{(w, x), (x, z), (w, z), (w, y)\}$ which implies P is transitive. We have (i) $y \notin {}^{P}x \cup x^{P}$ and (ii) ${}^{P}x \cap {}^{P}y = \{w\} \neq \emptyset$. However, $z \in x^{P}$ but $z \notin y^{P}$, that is $x^{P} \neq y^{P}$. P does not satisfy CDIE.

   (2.2) Let us show that: CDIE $\not\Rightarrow$ Transitivity.
   Since Negative Transitivity implies Transitivity, by 1.2 this proposition is true.

■

Now that we have presented CDIE, the key property satisfied by the binary relation, we introduce the axioms satisfied by the choice function.

3.2 Axiomatization

A function $C : \mathcal{X} \to \mathcal{X}$ is a *choice function* if and only if $\forall S \in \mathcal{X}$, $C(S) \subseteq S$. Note that, by definition, $\forall S \in \mathcal{X}$, $C(S) \neq \emptyset$: from any set, at least one alternative is chosen.

Let C be a choice function. Let P be an asymmetric binary relation on X. We say that P *rationalizes* C if $\forall S \in \mathcal{X}, C(S) = \{x \in S | \forall y \in S, (y,x) \notin P\}$.

The axiomatization is based on the rational choice theory. Two well-known axioms on consistency of choices are needed:

Axiom 1 (Contraction Consistency, α) .
C satisfies Contraction Consistency *if* $\forall x \in S \subseteq T \in \mathcal{X},\ x \in C(T) \Rightarrow x \in C(S)$.

This axiom, also called Chernoff axiom ([3]) or Sen's property α ([11]), imposes a condition on consistency when the feasible set is contracted. If an alternative x is chosen in a set, then x would also be chosen in a "reduction" of this set, from which some alternatives have been removed. By contrast, the following property imposes a condition on consistency when the feasible set is expanded:

Axiom 2 (Expansion Consistency, γ) .
C satisfies Expansion Consistency *if* $\forall n \in \mathbb{N}$ *and* $\forall S_1, \ldots, S_n \in \mathcal{X}$,
$x \in \bigcap_{i \in \{1,\ldots,n\}} C(S_i) \Rightarrow x \in C(\bigcup_{i \in \{1,\ldots,n\}} S_i)$.

Expansion Consistency is also known as Sen's property γ ([12]). This axiom means that if an alternative is chosen in several sets, then it would also be chosen in the union of these sets. The following axiom is the translation of CDIE into the terminology of choice function:

Axiom 3 (Revealed Equivalence, RE) .
C satisfies Revealed Equivalence *if*
$\forall x, y \in X$, *if* $\{x,y\} = C(\{x,y\})$
and there exists $z \in X$ *such that* $\{z\} = C(\{x,y,z\})$
or $[\{x\} = C(\{x,z\})$ *and* $\{y\} = C(\{y,z\})]$,
then $[\forall S \in \mathcal{X}$, *i)* $x \in C(S \cup \{x\}) \Leftrightarrow y \in C(S \cup \{y\})$
and ii) $C(S \cup \{x\}) \setminus \{x,y\} = C(S \cup \{y\}) \setminus \{x,y\}]$.

If two alternatives x and y are chosen when only both are available and if there exists a third alternative z such that z is chosen between $\{x,y,z\}$ or x and y are chosen when each is available by pair with z. Then for any set S, if x is chosen in S then y is chosen (and reciprocally). Furthermore, the chosen alternatives in S with x or S with y are the same, of course except for x and y.

3.3 Characterization

Now we have all the information needed to present the characterization of a choice behavior with incomparable alternatives:

Theorem 1 *Let C be a choice function. The following propositions are equivalent.*

1. *C satisfies Contraction Consistency, Expansion Consistency and Revealed Equivalence,*

2. *there exists an acyclic binary relation P satisfying CDIE that rationalizes C.*

4 Specific case: simplification with a categorization

In this section, we focus on a specific configuration of choice with incomparable alternatives: the choice behavior can be reduced to a selection based on a categorization of the options. Intuitively, when an agent faces a choice set, we consider that he chooses as if he partitions it into categories. With this categorization, the agent sorts out the options. If two alternatives are comparable, they belong to the same category. Otherwise, they belong to different categories. As a category is composed of comparable alternatives, the decision maker is able to rank them and he chooses his most preferred options in each class. However, he cannot rank alternatives which are not comparable: there is no cross categories comparison. At the end, he picks his most preferred alternatives in each category: for him, they are incomparable and they form his selected set of options.

Formally, this decision process corresponds to a two criteria decision making: an equivalence relation, to get the partition of the alternatives into categories, and a weak order, to rank the alternatives partially. As the categories are represented by equivalence classes, it is important to note that we focus on a special case of categorization: all categories are disjoint. Each alternative belongs to a single category. So, the characterization with an intersection of an equivalence relation and a weak order cannot be used to explain choices with alternatives which belong to several categories. We will show in section 5 how to model this general choice behavior in general.

First, we characterize the intersection of an equivalence relation and a weak order. The binary relation which results from this intersection satisfies our new property *Common Domination Implies Equivalence*. Then, we represent the decision process using a categorization with a multidimensional representation.

4.1 A specific characterization

With only disjoint categories, the choice behavior can be summarized by a two criteria decision making. Formally, it corresponds to the intersection of an equivalence relation and a weak order.

We define the intersection of two binary relations:

Definition 3 *Let R, Q, P be binary relations on X.*
$R = Q \cap P$ means that $\forall x, y \in X$, $(x, y) \in R$ if and only if $(x, y) \in Q$ and $(x, y) \in P$.

For instance, we can apply this definition to improve the understanding of the next theorem. Assume that Q is an equivalence relation and P is a weak order. Literally, x is preferred to y with respect to R if x and y are in the same equivalence class and x dominates y with respect to P. Note that we focus on the intersection of two binary relations, so both must be satisfied. That is why, R can be interpreted as a specific two criteria decision making: a partial ranking (P) and a partition in an equivalence class due to Q.

So, we get the following characterization:

Theorem 2 *Let R be a binary relation on X.*
There exists Q an equivalence relation and there exists P a weak order such that $Q \cap P = R$ if and only if R is asymmetric, transitive and satisfies CDIE.

Theorem 2 shows necessary and sufficient conditions to characterize a binary relation (denoted R) which results from the intersection of a weak order (denoted P) and an equivalence relation (denoted Q). R must satisfy three independent properties: asymmetry, transitivity and CDIE.

The intersection of an equivalence relation and a weak order corresponds to our interpretation of choice behavior of an individual who uses a categorization to simplify his decision making. With Theorem 2, we show that the characterization of the binary relation which results from this intersection is also based on our key property CDIE.

Remark 2 *Note that in Theorem 2, the weak order P does not depend on the equivalence relation Q. So, it is possible to have $x, y \in X$ such that $(x, y) \notin S$ and $(x, y) \in P$ (then $(x, y) \notin R$). We can find the same result if we define P only on the equivalence classes. In this case, the binary relation is the union of the weak order restricted on each equivalence class. With this restriction, the interpretation of this theorem as the selection of the most preferred alternative in each category is more natural. When the agent faces a set, he behaves as if the big choice problem can be divided in smaller problems. However, we decided to emphasize the most general result with Theorem 2.*

4.2 Axiomatization

We also suggest a characterization for the special case in which the preference relation can be described as the intersection of an equivalence relation and a weak order.

We need to introduce an axiom due to Plott [9] :

Axiom 4 (Path Independence) *C satisfies* Path Independence *if for all $S, T \in \mathcal{X}$, $C(C(S) \cup C(T)) = C(S \cup T)$*

This axiom requires that the final choice does not depend on any division of the set of alternatives. A choice function satifies Path Independence if, when the set of alternatives is divided, choosing separately in the subsets then in the choice set is equivalent to choose directly in the "big" set.

We propose the following characterization:

Theorem 3 *Let C be a choice function. The following propositions are equivalent.*

1. *C satisfies Path Independence, Expansion Consistency and Revealed Equivalence,*

2. *there exists an asymmetric binary relation P transitive and satisfying CDIE that rationalizes C.*

Remark 3 *Theorem 3 is a restriction of Theorem 1.*
Indeed, we can recall the following well-known results:

1. *Path Independence implies Contraction Consistency;*

2. *Asymmetry and Transitivity implies Acyclicity.*

4.3 A multidimensional representation

With the following theorem, we propose an intuitive representation of the choice behavior of an agent who applies a categorization to simplify his decision when he faces a set with incomparable alternatives.

We use a multidimensional representation: each category is considered as a dimension. We assume that an alternative can be written as a vector of coordinates. A coordinate symbolizes for the agent the "value" of the alternative in a specific dimension (i.e. category). The option belongs to a dimension if the corresponding coordinate is positive. The options are comparable in a category, so the chosen alternative has the largest coordinate in a dimension. When two alternatives have a postive coordinate in two different dimensions, they are incomparable and both are chosen.

Theorem 4 *[Conjecture] Let C be a choice function.*
The following propositions are equivalent:

1. *C satisfies Path Independence, Expansion Consistency and Revealed Equivalence;*

2. *$\exists d \in \mathbb{N}$ and $\exists \phi : X \to \mathbb{R}^d_+$ such that*

   (i) $\forall x \in X$, $\phi(x) = \big(\phi_1(x), \dots, \phi_d(x)\big)$ *with* $\exists! i \in \{1, \dots, d\}$ *such that* $\phi_i(x) > 0$ *and* $\forall j \neq i$, $\phi_j(x) = 0$*;*

   (ii) $\forall S \in \mathcal{X}$, $C(S) = \{x \in S | \exists i \in \{1, \dots, d\}$ *such that* $\forall y \in S, \phi_i(x) \geq \phi_i(y)\}$

Theorem 4 tells us that we can represent a choice function of an agent who selects by applying a categorization (i.e. an agent who has a preference relation that corresponds to the intersection of an equivalence relation and a weak order) by:

(i) all alternatives are written as a vector with d dimensions, one for each category. Since the categories are disjoint, each option has only one positive coordinate.

(ii) for all subsets of options, the chosen alternatives are those with the largest coordinate ϕ_i in a dimension i.

Remark 4 *The multidimensional representation does not work if there are alternatives that belong to several categories. In this case, the choice behavior cannot be simplified to a selection of the most preferred alternative in each category. The impossibility of such a representation is based on the satisfaction of CDIE, which is more demanding when alternatives can belong to several dimensions. Note that we could expect this impossibility since we cannot simplify the general characterization of this choice behavior with the intersection of an equivalence relation and a weak order.*

5 General representation of choices with incomparable alternatives

5.1 General case

In this section, we propose a general representation of choice with incomparable alternatives. Technically, it is a representation theorem for an acyclic binary relation which satisfies CDIE only. As a technical result, this theorem provides a representation for non-transitive preferences.

Theorem 5 *Let C be a choice function. The following propositions are equivalent:*

1. *C satisfies Contraction Consistency, Expansion Consistency and Revealed Equivalence,*

2. *there exists a function $f : X \to \mathbb{R}$ and there exists $\epsilon \in \mathbb{R}_+$ such that $\forall S \in \mathcal{X}$, $C(S) = \{x \in S | \forall y \in S, f(x) \geq f(y) \text{ or } f(x) \leq f(y) - \epsilon\}$.*

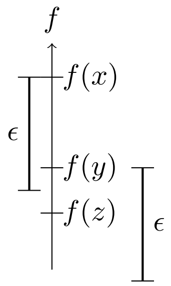
In this theorem, the couple formed by the function f and the scalar ϵ represents the choice of an agent. The choice behavior is explained by two criteria applied lexicographically: comparability and superiority. First, we focus on the comparability of two alternatives x and y. It formally depends on the difference between $f(x)$ and $f(y)$ with respect to ϵ. If $f(x)$ and $f(y)$ are "sufficiently close", that is $|f(x) - f(y)| \leq \epsilon$, then x and y are *comparable*. If $f(x)$ and $f(y)$ are "sufficiently far apart", that is $|f(x) - f(y)| > \epsilon$, then x and y are *incomparable*. Second, we focus on the superiority of an alternative. If x and

y are comparable, then the choice is as usual: the option with the greatest value by f is chosen. That is, if $f(x) > f(y)$ then x is chosen and if $f(y) > f(x)$ then y is chosen. If several alternatives have the greatest value, all are chosen. If x and y are incomparable, whatever the relation between $f(x)$ and $f(y)$ (that is $f(x) > f(y)$ or $f(x) < f(y)$), both options are chosen. The incomparability of the alternatives, that is the distance between them, prevents from an interpretation in terms of maximization. Actually, it is as if each incomparable alternative has the greatest value in each independent class of comparability. So they are separately the best, that is why all are chosen.

5.2 Remarks on this representation

Remark 1: This representation allows non-transitive preferences. Let x, y, z be three alternatives. Intuitively, it is possible that x is comparable and be preferred to y and that y is comparable and be preferred to z but x and z are too distant to be compared.

Example 1 *Let $X = \{x, y, z\}$. Let $f : X \to \mathbb{R}$ and $\epsilon \in \mathbb{R}_+$ be such that $f(x) > f(y) > f(x) - \epsilon > f(z) > f(y) - \epsilon$. The corresponding choice function is: $C(\{x, y\}) = \{x\}$, $C(\{y, z\}) = \{y\}$ and $C(\{x, z\}) = \{x, z\}$. So, $\forall P \subseteq X \times X$: $(x, y) \in P$ and $(y, z) \in P$ but $(x, z) \notin P$. Hence P is not transitive.*



Remark 2: We can focus on two extreme cases. If $\epsilon = 0$ then $\forall S \in \mathcal{X}$, $C(S) = S$ because we have : $C(S) = \{x \in S, \forall y \in S, f(x) \geq f(y) \text{ or } f(x) \leq f(y)\} = S$. Hence, in any set, the agent chooses all alternatives. Indeed, ϵ can be interpreted as the "interval of comparability". If it is null, then each alternative is incomparable with an other.

If $\epsilon \to +\infty$ then $\forall S \in \mathcal{X}$, $C(S) = \{x \in S, \forall y \in S, f(x) \geq f(y)\}$: as for the utility function, the chosen alternative has the maximal value with the function f. All alternatives are comparable because the "interval of comparability" (ϵ) is infinite so necessarily, all alternatives are "sufficiently close" to each other. Note that, in this extreme case, the binary relation is transitive.

Remark 3: Unlike the utility function, this representation is not a measure of satisfaction. Indeed, two alternatives with very different values can be chosen. The representation with f and ϵ is a composition of two criteria: comparability and superiority. The distance between alternatives indicates whether they are comparable or not. Thus, classes of comparability can be determined and there is no preference order between them. The criterion of superiority can be applied in a specific class of comparability only. Among comparable alternatives, the option with the greatest value by f is the best, so it is chosen. But with this representation, it does not make sense to compare the values by f of incomparable alternatives. Consequently, the arrangement on f's axis can be different to represent one choice behavior.

Example 2 *Let* $X = \{x, y, z\}$ *and* C *be a choice function such that :* $C(\{x, y\}) = \{x\}$, $C(\{x, z\}) = \{x, z\}$, $C(\{y, z\}) = \{y, z\}$ *and* $C(\{x, y, z\}) = \{x, z\}$.
This choice behavior can be represented in two ways:

5.3 Specific case: disjoint categories

In this section, we outline a representation theorem for a binary relation which is transitive and satisfies CDIE. In other words, this theorem represents the intersection of a weak order and an equivalence relation. This representation is based on Theorem 5 but we add conditions on the choice function and on the function of representation.

Theorem 6 *Let* C *be a choice function. The following propositions are equivalent:*

1. *C satisfies Path Independence, Expansion Consistency and Revealed Equivalence,*

2. *there exists a function* $f : X \to \mathbb{R}$ *and there exists* $\epsilon \in \mathbb{R}_+$ *such that* $\forall S \in \mathcal{X}$, $C(S) = \{x \in S | \forall y \in S, f(x) \geq f(y)$ *or* $f(x) \leq f(y) - \epsilon\}$ *and* $[\forall x, y \in S,$ *if* $f(x) - f(y) < \epsilon$ *and* $f(x) - f(z) \geq \epsilon$ *then* $f(y) - f(z) \geq \epsilon]$.

This theorem represents the binary relation which results from the intersection of a weak order and an equivalence relation. In Theorem 2, we characterize this binary relation R with three independent properties : asymmetry, transitivity and CDIE. The interpretation of the representation in terms of composition of two criteria (comparability and superiority) is similar to Theorem 5. The condition we add to get transitivity brings an additional information. The comparable alternatives form an equivalence class. When two alternatives are incomparable, they belong to two different equivalence classes. Obviously, the distance between equivalence classes is greater then ϵ.

This representation has the same limits as the theorem 5. The function of representation is not a measure of satisfaction. The ranking on f is not a representation for the weak order P. f takes the partial order and the partition of the set of alternatives into account. Consequently, the arrangement on f's axis may vary. The ranking in an equivalence class is fixed because the criterion of superiority makes sense: among comparable alternatives, the one with the greatest value by f is chosen. But there is no order for the equivalence classes thus their relative ranking on f should not be interpreted as a domination between classes.

6 Concluding remarks

In this paper, we focus on choice problem with incomparable alternatives. The origin of this interest lies in questioning the usual assumption that people are always able to assess alternatives when they must make a decision. Our main argument is that a decision maker can compare alternatives that are "sufficiently close" or that are in the same category. Then, he chooses his most preferred one in this class. On the other hand, he cannot rank two alternatives that are too "far apart", that is two alternatives which belong to two different categories: he picks both because they are incomparable for him. In order to characterize this choice behavior, we introduce a new property, *Common Domination Implies Equivalence* that is about P-equivalent alternatives. With the following diagram, we show that our characterization offers a parallel path to the usual modelization of rationality:

$$
\begin{tikzcd}
\boxed{\text{Negatively Transitive} \Leftrightarrow \text{WARP}} \arrow[dd, Rightarrow] \arrow[dr, Rightarrow] & \\
& \boxed{\text{Transitivity} + \text{CDIE} \Leftrightarrow \gamma + \text{PI} + \text{RE}} \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \\
\boxed{\text{Transitivity} \Leftrightarrow \gamma + \text{PI}} \arrow[d, Rightarrow] & \boxed{\text{CDIE} \Leftrightarrow \alpha + \gamma + \text{RE}} \arrow[dl, Rightarrow] \\
\boxed{\text{Acyclicity} \Leftrightarrow \alpha + \gamma} &
\end{tikzcd}
$$

The central part of this graph summarizes the main results of the rational choice theory:

1. "Negatively Transitive $\Leftrightarrow$ WARP[6]" corresponds to the representation by a utility function.

2. "Transitivity $\Leftrightarrow \gamma$ + PI" is due to Plott [9]

3. "Acyclicity $\Leftrightarrow \alpha + \gamma$" is due to Blair et al. [2]. This result can be interpreted as a minimal requirement for rationality.

[6]WARP: Weak Axiom of Revealed Preference

The arrows ($\Rightarrow$) represent the implications between the results.

So, our results are linked with the standard theory. However our representation is wider because it takes an information on the comparability and a kind of measure of satisfaction into account. In this framework, the common representation with a utility function corresponds to a specific case when all options are comparable.

Finally, in this paper, we adopt a nonevaluative approach of the comparability of the alternatives. Indeed, our perception is “binary”: we consider a perfect comparability in a category and an incomparability between categories. An interesting enlargement of this paper would be the introduction of a measurement of the (in)comparability. For instance, we can imagine that it could be relevant to have a “reduced” comparability cross categories.

7 Appendix

Proof of Theorem 1

Proof. 1. $\Rightarrow$ 2. Suppose that C satisfies Contraction Consistency, Expansion Consistency and Revealed Equivalence.

We define P as : $\forall x, y \in X$, $(x, y) \in P \Leftrightarrow y \notin C(\{x, y\})$[7].

1. Let us show that P is acyclic. On the contrary, assume that P is cyclic : $\exists n \in \mathbb{N} \setminus \{1\}, \exists x_1, ..., x_n \in X$, such that $[\forall i \in \{1, ..., n-1\}, (x_i, x_{i+1}) \in P$ and $x_1 = x_n]$. Then, by definition of P, $\forall i \in \{1, ..., n-1\}, x_{i+1} \notin C(\{x_i, x_{i+1}\})$ and $x_1 = x_n$. Hence by Contraction Consistency, $C(\{x_1, \ldots, x_n\}) = \emptyset$ which contradicts that C is a choice function.

2. Let us show that P satisfies CDIE. Let $x, y \in X$ be such that i) $y \notin {}^{P}x \cup x^P$ and ii) ${}^{P}x \cap {}^{P}y \neq \emptyset$ or $x^P \cap y^P \neq \emptyset$. Let us show that $[x \approxeq_P y]$.

Note that if $x = y$, then obviously ${}^{P}x = {}^{P}y$ and $x^P = y^P$, that is $[x \approxeq_P y]$. So in the following we assume $x \neq y$.

1) Assume ${}^{P}x \cap {}^{P}y \neq \emptyset$. $\exists z \in X$ such that $(z, x) \in P$ and $(z, y) \in P$. By definition of P, $\{z\} = C(\{x, z\}) = C(\{y, z\})$. By Contraction Consistency, $x \notin C(\{x, y, z\})$ and $y \notin C(\{x, y, z\})$. By definition of a choice function, $\{z\} = C(\{x, y, z\})$. Moreover, $y \notin {}^{P}x \cup x^P$ which implies $\{x, y\} = C(\{x, y\})$. Then, by Revealed Equivalence, $[\forall S \in \mathcal{X}$, $x \in C(S \cup \{x\}) \Leftrightarrow y \in C(S \cup \{y\})$ and $C(S \cup \{x\}) \setminus \{x, y\} = C(S \cup \{y\}) \setminus \{x, y\}]$. In particular, $[\forall t \in X,\ t \neq x,\ t \neq y,\ x \in C(\{x, y, t\}) \Leftrightarrow y \in C(\{x, y, t\})]$. So we can deduce that $\forall t \in X$, $C(\{x, y, t\}) \in \Big\{\{x, y\}, \{x, y, t\}, \{t\}\Big\}$.

If $C(\{x, y, t\}) = \{x, y\}$, then by Expansion Consistency $(x, t) \in P$ or $(y, t) \in P$. With no loss of generality, assume $(x, t) \in P$, that is by definition of P, $C(\{x, t\}) = \{x\}$. Applying Revealed Equivalence to $S = \{t\}$, we get $y \in C(\{y, t\})$. And since $C(\{x, t\}) \setminus \{x, y\} = C(\{y, t\}) \setminus \{x, y\}$, we know that $t \notin C(\{y, t\})$. Therefore, we necessarily have $\{y\} = C(\{y, t\})$. Thus, $\{x\} = C(\{x, t\}) \Leftrightarrow \{y\} = C(\{y, t\})$. Then by definition of P, $(x, t) \in P \Leftrightarrow (y, t) \in P$ (i.e. $x^P = y^P$). Besides, $C(\{x, t\}) = \{x\}$ and $C(\{y, t\}) = \{y\}$ imply $(t, x) \notin P$ and $(t, y) \notin P$. So $t \notin {}^{P}x \Leftrightarrow t \notin {}^{P}y$. Hence, we have $x \approxeq_P y$.

If $C(\{x, y, t\}) = \{t\}$, then $(t, x) \in P$ and $(t, y) \in P$. By the previous arguments, it can be shown that $x \approxeq_P y$.

If $C(\{x, y, t\}) = \{x, y, t\}$, by the previous arguments, it can be shown that $x \approxeq_P y$.

2) It is the same proof if we assume $x^P \cap y^P \neq \emptyset$.

3. Let us show that P rationalizes C. By definition, P rationalizes C if : $x \in C(S) \Leftrightarrow \forall y \in S$, $(y, x) \notin P$.

Let $x \in S$ be such that $\forall y \in S$, $(y, x) \notin P$. By definition of P, we have : $\forall y \in S$, $x \in C(\{x, y\})$. So, by Expansion Consistency $x \in C(S)$.

Let $x \in S$. Suppose there exists $y \in S$ such that $(y, x) \in P$. By definition of P, $x \notin C(\{x, y\})$. So by Contraction Consistency, $x \notin C(S)$. ■

[7] Notice that this definition implies P is asymmetric

Proof. 2. $\Rightarrow$ 1.
Let C be a choice function. Suppose that there exists an acyclic binary relation P satisfying CDIE that rationalizes C. So C is defined as follows: $\forall S \in \mathcal{X}$, $C(S) = \{x \in S | \forall y \in S, (y,x) \notin P\}$. Note that since P is acyclic, $\forall S \in \mathcal{X}$, $C(S) \neq \emptyset$ so C is a choice function.

1. Let us show that C satisfies Contraction Consistency.
Let $S \subseteq T \in \mathcal{X}$ be such that $\exists x \in S \subseteq T$ and $x \in C(T)$. By definition of C, $\forall y \in T$, $(y,x) \notin P$. If by contradiction, $x \notin C(S)$, it implies that $\exists t \in S$ such that $(t,x) \in P$. Since $S \subseteq T$, $t \in T$ which contradicts $x \in C(T)$. So $x \in C(T)$ implies $x \in C(S)$: C satisfies Contraction Consistency.

2. Let us show that C satisfies Expansion Consistency.
Assume $x \in \bigcap_{i \in \{1,\dots,n\}} C(S_i)$ $\forall n \in \mathbb{N}$ and $\forall S_1, \dots, S_n \in \mathcal{X}$. It implies that $x \in C(S_i)$ $\forall i \in \{1,\dots,n\}$ and by definition of C, $\forall y \in S_i$, $(y,x) \notin P$ $\forall S_i$.

By contradiction, assume that $x \notin C(\bigcup_{i \in \{1,\dots,n\}} S_i)$. So $\exists z \in \bigcup_{i \in \{1,\dots,n\}} S_i$ such that $(z,x) \in P$. Necessarily, $\exists k \in \{1,\dots,n\}$ such that $z \in S_k$. It means that $\exists S_k$ such that $(z,x) \in P$ which contradicts $x \in C(S_i)$ $\forall i \in \{1,\dots,n\}$.

3. Let us show that C satisfies Revealed Equivalence.
Let $\exists x,y,z \in X$ be such that (i) $\{x,y\} = C(\{x,y\})$ and (ii) $\{z\} = C(\{x,y,z\})$ (we only consider this case: it is the same proof if z is such that $\{x\} = C(\{x,z\})$ and $\{y\} = C(\{y,z\})$).
With (ii) $\{z\} = C(\{x,y,z\})$ we know by definition of C that $[(x,z) \notin P$ and $(y,z) \notin P]$. With (i) $\{x,y\} = C(\{x,y\})$ we know by definition of C that $[(x,y) \notin P$ and $(y,x) \notin P]$. Then, $x \notin C(\{x,y,z\})$ and $y \notin C(\{x,y,z\})$ means necessarily that $[(z,x) \in P$ and $(z,y) \in P]$ that is ${}^{P}x \cap {}^{P}y = \{z\} \neq \emptyset$. Since P satisfies CDIE, $x \approxeq_P y$ that is ${}^{P}x = {}^{P}y$ and $x^P = y^P$.

Let us show that $\forall S \in \mathcal{X}$, $x \in C(S \cup \{x\}) \Leftrightarrow y \in C(S \cup \{y\})$. $x \in C(S \cup \{x\})$ means by definition of C that $\forall t \in S$, $(t,x) \notin P$. Either $(x,t) \in P$ or $(x,t) \notin P$. In both cases, since $x \approxeq_P y$ then $(y,t) \in P$ or $(y,t) \notin P$ so, we also have $\forall t \in S$, $(t,y) \notin P$ that is $y \in C(S \cup \{y\})$.

Let us show that $\forall S \in \mathcal{X}$, $C(S \cup \{x\}) \setminus \{x,y\} = C(S \cup \{y\}) \setminus \{x,y\}$. If $\exists w \in S$ such that $w \in C(S \cup \{x\})$ but $w \notin C(S \cup \{y\})$. By definition of C, $w \in C(S \cup \{x\})$ means $\forall z \in S$, $(z,w) \notin P$ and $(x,w) \notin P$. So, $w \notin C(S \cup \{y\})$ only if $(y,w) \in P$ which is impossible because $x^P = y^P$.

■

Proof of Theorem 2

Proof. 1. $\Rightarrow$ 2. Assume that there exists S an equivalence relation and there exists P a weak order. Let us show that $R = S \cap P$ is asymmetric, transitive and satisfies CDIE.

1. Let us show that R is asymmetric. Let $x,y \in X$ be such that $(x,y) \in R$. By definition of R, $(x,y) \in S$ and $(x,y) \in P$. Since P is asymmetric, $(y,x) \notin P$, so $(y,x) \notin R$.

2. Let us show that R is transitive. Let $x, y, z \in X$ be such that $(x, y) \in R$ and $(y, z) \in R$. By definition of R, $(x, y) \in R$ implies $(x, y) \in S$ and $(x, y) \in P$. Likewise, $(y, z) \in R$ implies $(y, z) \in S$ and $(y, z) \in P$. Since S and P are transitive, we have $(x, z) \in S$ and $(x, z) \in P$ hence $(x, z) \in R$.

3. Let us show that R satisfies CDIE. Let $x, y \in X$ be such that i) $(x, y) \notin R$ and $(y, x) \notin R$ and ii) $\exists z \in X$ such that $[(z, x) \in R$ and $(z, y) \in R]$ or $[(x, z) \in R$ and $(y, z) \in R]$. Let us show that $[x \approxeq_R y]$.

We only consider the case in which $\exists z \in X$ such that $[(z, x) \in R$ and $(z, y) \in R]$: it is the same proof for $[(x, z) \in R$ and $(y, z) \in R]$. With $[(z, x) \in R$ and $(z, y) \in R]$, we know by definition of R that $[(z, x) \in S$ and $(z, y) \in S]$. And since S is symmetric and transitive, we can infer that $(x, y) \in S$ and $(y, x) \in S$. Then, the assumption i) implies $(x, y) \notin P$ and $(y, x) \notin P$.

Let $t \in X$ be such that $(t, x) \in R$ which implies $(t, x) \in P$ and $(t, x) \in S$, by definition of R. And with $(x, y) \in S$ and transitivity of S, we have $(t, y) \in S$. Furthermore, since P is negatively transitive, $(t, x) \in P$ implies $(t, y) \in P$ or $(y, x) \in P$. But by i) we know that $(y, x) \notin P$ so necessarily $(t, y) \in P$. Hence, $(t, y) \in R$.

With the same arguments, we could show that if there exists $t' \in X$ such that $(t', y) \in R$ then $(t', x) \in R$ too. Then, $\forall t \in X, (t, x) \in R \Leftrightarrow (t, y) \in R$, that is ${}^R x = {}^R y$.

It is the same proof to check that : $\forall s \in X, (x, s) \in R \Leftrightarrow (y, s) \in R$, that is $x^R = y^R$.

2. $\Rightarrow$ **1.** Assume that R is an asymmetric and transitive binary relation satisfying CDIE. Let us show that there exists S an equivalence relation and there exists P a weak order such that $R = S \cap P$.

1. Let us show the existence of an equivalence relation, denoted S, such that R is the result of the intersection of S and a weak order.
We define the dual relation of R : $R^d = \{(x, y) \in X \times X | (y, x) \in R\}$.

Definition 4 *Let $R \subseteq X \times X$ be a binary relation.*
R^t is the transitive closure *of R : $\forall x, y \in X$, $(x, y) \in R^t$ if $\exists n \in \mathbb{N} \setminus \{1\}, \exists z_1, \dots, z_n \in X$ such that $\forall i \in \{1, \dots, n-1\}, (z_i, z_{i+1}) \in R$ with $z_1 = x$ and $z_n = y$.*

We define S as $S \equiv (R \cup R^d)^t \cup \{(x, x) | x \in X\}$. Let us check that S is an equivalence relation :

- By definition, S is reflexive and transitive
- Let $x, y \in X$ be such that $(x, y) \in S$. By definitions of S and the transitive closure, it means that $\exists n \in \mathbb{N} \setminus \{1\}$, $\exists z_1, \dots z_n \in X$ such that $\forall i \in \{1, \dots, n-1\}, (z_i, z_{i+1}) \in R \cup R^d$ with $x = z_1$ and $y = z_n$. Since $R \cup R^d$ is symmetric, $(z_{i+1}, z_i) \in R \cup R^d$, $\forall i \in \{1, \dots, n-1\}$. Then we get $(y, x) \in (R \cup R^d)^t$ that is $(y, x) \in S$. Hence, S is symmetric.

So $S \equiv (R \cup R^d)^t \cup \{(x,x) | x \in X\}$ is an equivalence relation.

2. Let us show the existence of a weak order, denoted P, such that R is the result of $S \cap P$.

Let $R' = S \setminus R^d \cup \{(x,x)|x \in X\}$. Obviously, R is the asymmetric component of R'.

Let us show that R' is transitive. Let $x, y, z \in X$ be such that $(x,y) \in R'$ and $(y,z) \in R'$. If $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$ since R is transitive. Hence, $(x,z) \in R'$. If, with no loss of generality, $(x,y) \in R$ and $(y,z) \in R'$ but $(y,z) \notin R$. Then, $(z,y) \notin R$ by definition of R'. Therefore, necessarily, $(y,z) \in S$ and $\exists t \in X$ such that $[(y,t) \in R$ and $(z,t) \in R]$ or $[(t,y) \in R$ and $(t,z) \in R]$. Assume that $\exists t \in X$ such that $[(y,t) \in R$ and $(z,t) \in R]$. R satisfies CDIE and we have (i) $(y,z) \notin R$ and $(z,y) \notin R$ (i.e. $y \notin {}^{R}z \cup z^{R}$) and (ii) $y^R \cap z^R \neq \emptyset$, then ${}^{R}y = {}^{R}z$. So $(x,y) \in R$ implies $(x,z) \in R$ that is $(x,z) \in R'$. Same argument if $\exists t \in X$ such that $(t,y) \in R$ and $(t,z) \in R$. Hence, R' is transitive.

In order to use Suzumura's Theorem 3 (in [13])[8], we need to prove that R' is *consistent*.

Definition 5 *Let R be a binary relation, and let P be its asymmetric component.*
A t-tuple of alternatives $(x_1, \ldots, x_t)$ is a cycle* of order t *if $(x_1, x_2) \in P$ and $\forall i \in \{2, \ldots t-1\}, (x_i, x_{i+1}) \in R$ and $(x_t, x_1) \in R$.*
R is consistent *if there exists no cycle* of any order.*

It is straightforward to check that, since R' is transitive, there is no cycle* of any order. So R' is consistent in the sense of Suzumura. Then, there exists R^* an extended ordering (reflexive, transitive and connected[9] relation). We denote $P(R^*)$ the asymmetric part of R^*. We define P as $P \equiv P(R^*)$. So by definition P is a weak order.

3. Let us check that $R = S \cap P$. Let $x, y \in X$, $x \neq y$.
$R \subset S \cap P$ is obvious : if $(x,y) \in R$ then $(x,y) \in S$ by definition of S and $(x,y) \in P$ since $R \subseteq P$.
In order to prove $S \cap P \subset R$, assume that $(x,y) \notin R$ and $(x,y) \in S$: let us check that these assumptions imply $(x,y) \notin P$.

By definition of S, $(x,y) \in S$ imply $(x,y) \in (R \cup R^d)^t$. If $(x,y) \in R^d$ then by definition of P, $(x,y) \notin P$. So, assume that $(x,y) \notin (R \cup R^d)$. By definition of the transitive closure, $\exists n \in \mathbb{N} \setminus \{1\}$, $\exists z_1, \ldots z_n \in X$ such that $\forall i \in \{1, \ldots n-1\}$, $(z_i, z_{i+1}) \in R \cup R^d$ with $z_1 = x$ and $z_n = y$. Let $m = \min n$ such that $\forall i \in \{1, \ldots m-1\}$, $(z_i, z_{i+1}) \in R \cup R^d$ with $z_1 = x$ and $z_m = y$. If $\forall i \in \{1, \ldots m-1\}$, $(z_i, z_{i+1}) \in R$ then, since R is transitive, $(x,y) \in R$, which contradicts our hypothesis. So we deduce that $(x,y) \notin P$. If $\exists j \in \{1, \ldots m-2\}$ such that $(z_j, z_{j+1}) \in R$ and $(z_{j+1}, z_{j+2}) \in R$ then by transitivity of R, $(z_j, z_{j+2}) \in R$ which contradicts m is minimal. Then, $\forall i \in \{1, \ldots m-2\}$, $[(z_i, z_{i+1}) \in R \Rightarrow (z_{i+2}, z_{i+1}) \in R]$ or $[(z_{i+1}, z_i) \in R \Rightarrow (z_{i+1}, z_{i+2}) \in R]$. With no loss of generality, let $j \in \{1, \ldots, m-3\}$

[8] "A binary relation R has an extended ordering R^* if and only if R is consistent."
[9] P is *connected* if $\forall x, y \in X$, $x \neq y$ implies $(x,y) \in P$ or $(y,x) \in P$.

such that $(z_j, z_{j+1}) \in R, (z_{j+2}, z_{j+1}) \in R$ and $(z_{j+2}, z_{j+3}) \in R$. Thus, $z_j \notin {}^R z_{j+2} \cap z_{j+2}^R$ and $z_{j+2}^R \cap z_j^R = \{z_{j+1}\} \neq \emptyset$. Since R satisfies CDIE, $z_j^R = z_{j+2}^R$ so we must have $(z_j, z_{i+3}) \in R$ which contradicts m is minimal. So we can conclude that $(x, y) \notin P$.

■

Proof of Theorem 3

Proof. 1. ⇒ 2.

Suppose that C satisifes Path Independence, Expansion Consistency and Revealed Equivalence.

Similarly to Theorem 1, we define P as : $\forall x, y \in X$, $(x, y) \in P \Leftrightarrow y \notin C(\{x, y\})$.

Since it is well-known that Path Independence implies Contraction Consistency, we know that P is an acyclic binary relation which satisfies CDIE and rationalizes C.

Let us show that P is transitive. Let $x, y, z \in X$ be such that $(x, y) \in P$ and $(y, z) \in P$. By definition of P, $C(\{x, y\}) = \{x\}$ and $C(\{y, z\}) = \{y\}$. Since C satisfies Path Independence, $C(\{x, y, z\}) = C(C(\{x, y\}) \cup C(\{y, z\})) = C(\{x\} \cup \{y\}) = \{x\}$ (*). By Theorem 1, we know that P is acyclic hence $(z, x) \notin P$. So $x \in C(\{x, z\})$ and $C(\{x, z\}) = \Big\{\{x\}, \{x, z\}\Big\}$. Assume that $C(\{x, z\}) = \{x, z\}$. Since C satisfies Path Independence, $C(\{x, y, z\}) = C(C(\{x, y\}) \cup C(\{x, z\})) = C(\{x, z\}) = \{x, z\}$ which contradicts (*). Hence $C(\{x, z\}) = \{x\}$: $z \notin C(\{x, z\})$ and $(x, z) \in P$. P is transitive. ■

Proof. 2. ⇒ 1. Straightforward with Theorem 1 and Plott's theorem (in [9]): "*A choice function C satisfies Path Independence and Expansion Consistency if and only if there exists an asymmetric and transitive binary relation P that rationalizes C.*" ■

Proof of Theorem 4

To be completed.

Proof of Theorem 5

Proof. 1. ⇒ 2.

By theorem 1, if C satisfies Contraction Consistency, Expansion Consistency and Revealed Equivalence then there exists an acyclic binary relation P satisfying CDIE that rationalizes C.

1. We define a new binary relation $R \subseteq X \times X$ as : $\forall x, y \in X$, $(x, y) \in R$ if i) $y \notin {}^P x \cup x^P$ and ii) ${}^P x \cap {}^P y \neq \emptyset$ or $x^P \cap y^P \neq \emptyset$. Let $\overline{R}$ be the reflexive closure of R : $\overline{R} = \Big\{(x, x) | x \in X\Big\} \cup R$. Let us show that $\overline{R}$ is an equivalence relation.

1) By definition, $\overline{R}$ is reflexive.

2) Let us show that $\overline{R}$ is symmetric. Let $x, y \in X$ be such that $(x, y) \in \overline{R}$. If $x = y$ it is trivial. If $x \neq y$, we know by definition of $\overline{R}$ that i) $y \notin {}^P x \cup x^P$ which implies $x \notin {}^P y \cup y^P$. And we also know that ii) ${}^P x \cap {}^P y \neq \emptyset$ or $x^P \cap y^P \neq \emptyset$. So, by definition of $\overline{R}$, $(y, x) \in \overline{R}$.

3) Let us show that $\overline{R}$ is transitive. Let $x, y, z \in X$ be such that $(x, y) \in \overline{R}$ and $(y, z) \in \overline{R}$.

If x, y, z are not mutually different, then obviously $(x, z) \in \overline{R}$. Assume x, y, z are mutually different. By definition of $\overline{R}$, $(x, y) \in \overline{R}$ implies i) $y \notin {}^{P}x \cup x^{P}$ and ii) ${}^{P}x \cap {}^{P}y \neq \emptyset$ or $x^{P} \cap y^{P} \neq \emptyset$. P satisfies CDIE then these two requirements imply ${}^{P}x = {}^{P}y$ and $x^{P} = y^{P}$. From $(y, z) \in \overline{R}$ we know that i) $z \notin {}^{P}y \cup y^{P}$ and ii) ${}^{P}y \cap {}^{P}z \neq \emptyset$ or $y^{P} \cap z^{P} \neq \emptyset$. Since $x \approxeq_{P} y$, we can deduce i') $z \notin {}^{P}x \cup x^{P}$ and ii') ${}^{P}x \cap {}^{P}z \neq \emptyset$ or $x^{P} \cap z^{P} \neq \emptyset$. That is, by definition of $\overline{R}$, $(x, z) \in \overline{R}$.

2. $X/\overline{R}$ is the quotient set of X by $\overline{R}$: $X/\overline{R}$ forms a partition of X and an element A of $X/\overline{R}$ is an equivalence class. $\mathcal{E} = 2^{X/\overline{R}} \setminus \emptyset$ is the set of all non-empty subsets of $X/\overline{R}$.

We notice that in each equivalence class, CDIE is satisfied. Indeed, $\forall x, x' \in A \in X/\overline{R}$, with $x \neq x'$, we know that i)$x \notin {}^{P}x' \cup x'^{P}$ and ii) ${}^{P}x \cap {}^{P}x' \neq \emptyset$ or $x^{P} \cap x'^{P} \neq \emptyset$. These requirements imply $x \approxeq_{P} x'$ by CDIE. That is : ${}^{P}x = {}^{P}x'$ and $x^{P} = x'^{P}$. Therefore, every elements in a same equivalence class have the same predecessors and successors.

We define a binary relation $Q \subseteq X/\overline{R} \times X/\overline{R}$ as : $\forall A, B \in X/\overline{R}$, $(A, B) \in Q$ if $\exists x \in A$, $\exists y \in B$ such that $(x, y) \in P$.[10]

We extend the definitions of predecessor and successor. Let $T \subseteq X/\overline{R} \times X/\overline{R}$. We define ${}^{T}A$ the set of predecessors of A, *i.e.* ${}^{T}A = \{B \in X/\overline{R}, (B, A) \in T\}$. We define A^{T} the set of successors of A, *i.e.* $A^{T} = \{B \in X/\overline{R}, (A, B) \in T\}$.

Lemma 1 *$\forall A, B \in X/\overline{R}$, $(A, B) \in Q$ if and only if $\forall x \in A$ and $\forall y \in B$, $(x, y) \in P$.*

1) [$\Rightarrow$] We know that $\forall D \in X/\overline{R}$, $\forall z, z' \in D$, $z^{P} = z'^{P}$ and ${}^{P}z = {}^{P}z'$. By definition of Q, $(A, B) \in Q$ if $\exists x \in A$ and $\exists y \in B$ such that $y \in x^{P}$ then $\forall x' \in A$, $y \in x'^{P}$. And, if $(A, B) \in Q$, that is by definition, $\exists x \in A$ and $\exists y \in B$ such that $x \in {}^{P}y$ then $\forall y' \in B$, $x \in {}^{P}y'$. So, we get that if $(A, B) \in Q$ then $\forall x \in A$ and $\forall y \in B$, $(x, y) \in P$.

2) [$\Leftarrow$] Straightforward.

We extend some definitions :

Definition 6 *Q is* acyclic *if $\forall n \in \mathbb{N} \setminus \{1\}, \forall A_1, ..., A_n \in X/\overline{R}$, $[\forall i \in \{1, ..., n-1\}, (A_i, A_{i+1}) \in Q \Rightarrow A_1 \neq A_n]$.*[11]

Definition 7 (PROP1) *Q satisfies* PROP1 *if $\forall A, B \in X/\overline{R}$, $A \neq B$, such that ${}^{Q}A \cap {}^{Q}B \neq \emptyset$ or $A^{Q} \cap B^{Q} \neq \emptyset$ then $A \in {}^{Q}B \cup B^{Q}$.*

Proposition 2 *Q is acyclic and satisfies PROP1.*

1) Let us show that Q is acyclic. By contradiction, let $n \in \mathbb{N} \setminus \{1\}$ and let $A_1, \ldots, A_n \in X/\overline{R}$ be such that $\forall i \in \{1, \ldots, n-1\}$, $(A_i, A_{i+1}) \in Q$ and $A_1 = A_n$. By Lemma 1, $(A_i, A_{i+1}) \in Q$, $\forall i \in \{1, \ldots, n-1\}$ implies that $\forall x_i \in A_i$ and $\forall x_{i+1} \in A_{i+1}$, $(x_i, x_{i+1}) \in P$. So, $\forall x_1 \in A_1, \ldots, x_n \in A_n$ $\forall k \in \{1, \ldots, n-1\}$, $(x_k, x_{k-1}) \in P$. As we assume $A_1 = A_n$, $\exists x_1 \in A_1$

[10] Notice that Q depends on P, an acyclic binary relation which satisfies CDIE.

[11] With this definition, if Q is acyclic, then Q is irreflexive and asymmetric.

and $\exists x_n \in A_n$ such that $x_1 = x_n$. Then we find $x_1, \ldots, x_n \in X$, $\forall k \in \{1, \ldots, n-1\}$, $(x_k, x_{k-1}) \in P$ and $x_1 = x_n$ which contradicts P is acyclic.

2) Let us show that Q satisfies PROP1. i)By contradiction, let $A, B \in X/\overline{R}$, $A \neq B$ be such that ${}^{Q}A \cap {}^{Q}B \neq \emptyset$ and $A \notin {}^{Q}B \cup B^{Q}$. From ${}^{Q}A \cap {}^{Q}B \neq \emptyset$, we know that $\exists D \in X/\overline{R}$ such that $(D, A) \in Q$ and $(D, B) \in Q$. By Lemma 1, we obtain that $\forall x \in A$ and $\forall y \in B$, ${}^{P}x \cap {}^{P}y \neq \emptyset$. From $A \notin {}^{Q}B \cup B^{Q}$, we obtain by definition of Q : $\forall x \in A$ and $\forall y \in B$, $y \notin {}^{P}x \cup x^{P}$. But, by definition of $\overline{R}$ if $\exists x, y \in X$ such that $y \notin {}^{P}x \cup x^{P}$ and ${}^{P}x \cap {}^{P}y \neq \emptyset$ then x and y belong to the same equivalence class, that is $A \cap B \neq \emptyset$, which contradicts $A \neq B$ because A and B are equivalence classes.
ii) By the same reasoning, it is true for $A^{Q} \cap B^{Q} \neq \emptyset$.

3. We introduce new definitions :

Definition 8 *Q^t is the* transitive closure of Q *:* $\forall A, B \in X/\overline{R}$, $(A, B) \in Q^t$ *if* $\exists n \in \mathbb{N}\backslash\{1\}$, $\exists D_1, \ldots, D_n \in X/\overline{R}$ *such that* $\forall i \in \{1, \ldots, n-1\}$, $(D_i, D_{i+1}) \in Q$, $D_1 = A$ *and* $D_n = B$.

Definition 9 $\Gamma \in \mathcal{E}$ *is a* minimal component *if* $\forall A \in \Gamma$, $A \cup {}^{Q^t}A \cup A^{Q^t} \subseteq \Gamma$ *and* Γ *is minimal.*

Existence of a minimal component are straightforward.

Proposition 3 $\forall A, B \in \Gamma, \exists n \in \mathbb{N} \setminus \{1\}, \exists D_1, \ldots, D_n \in \Gamma$ *such that* $[\forall k \in \{1, \ldots, n-1\}, (D_k, D_{k+1}) \in Q$ *or* $(D_{k+1}, D_k) \in Q$ *with* $D_1 = A$ *and* $D_n = B]$.

We need to define the following sets :
$\mathcal{P}_A = \{D \in \Gamma | \exists n \in \mathbb{N}\backslash\{1\}, \exists D_1, \ldots, D_n \in \Gamma, \forall i \in \{1, \ldots, n-1\}, (D_i, D_{i+1}) \in Q$ or $(D_{i+1}, D_i) \in Q, D_1 = A$ and $D_n = D\} \cup A$.
$\mathcal{P}_B = \{E \in \Gamma | \exists m \in \mathbb{N}\backslash\{1\}, \exists E_1, \ldots, E_m \in \Gamma, \forall j \in \{1, \ldots, m-1\}, (E_j, E_{j+1}) \in Q$ or $(E_{j+1}, E_j) \in Q, E_1 = B$ and $E_n = E\} \cup B$.
If $\mathcal{P}_A \cap \mathcal{P}_B = \emptyset$ then $B \notin \mathcal{P}_A$. So $\mathcal{P}_A \subset \Gamma$ which contradicts that Γ is a minimal component. Consequently, $\mathcal{P}_A \cap \mathcal{P}_B \neq \emptyset$ which implies $B \in \mathcal{P}_A$. The proposition is true : when two equivalence classes belong to the same minimal component, there is a "path" connecting them.

Proposition 4 $\forall \Gamma \in \mathcal{E}$, *$Q^t$ is a linear order on* Γ.

1) By definition, Q^t is transitive.

2) Q^t is asymmetric, since by Proposition 2 Q is acyclic.

3) Let us show that Q^t is connected. By contradiction, let $A, B \in \Gamma$, $A \neq B$, such that $(A, B) \notin Q^t$ and $(B, A) \notin Q^t$. From Proposition 3, we know that A and B are linked : $\exists n \in \mathbb{N} \setminus \{1\}, \exists D_1, \ldots, D_n \in \Gamma$ such that $\forall k \in \{1, \ldots, n-1\}, (D_k, D_{k+1}) \in Q$ or $(D_{k+1}, D_k) \in Q$ with $D_1 = A$ and $D_n = B$. Let m be the minimal integer such that this proposition is satisfied for A and B. If $\forall i \in \{1, \ldots, m-1\}, (D_i, D_{i+1}) \in Q$ then, by

definition of Q^t, $(A, B) \in Q^t$ (and $(B, A) \in Q^t$ if $\forall i \in \{1, \ldots, m-1\}, (D_{i+1}, D_i) \in Q$). Otherwise $\exists j \in \{2, \ldots m-1\}$ such that $[(D_{j-1}, D_j) \in Q$ and $(D_{j+1}, D_j) \in Q]$ (configuration 1) or $[(D_j, D_{j-1}) \in Q$ and $(D_j, D_{j+1}) \in Q]$ (configuration 2). If $[(D_{j-1}, D_j) \in Q$ and $(D_{j+1}, D_j) \in Q]$, that is $D_{j-1}^Q \cap D_{j+1}^Q \neq \emptyset$. By PROP1 it means that $D_{j-1} \in^Q D_{j+1} \cup D_{j+1}^Q$. So $\exists m' < m, \exists D_1, \ldots D_{m'} \in \Gamma, \forall k \in \{1, \ldots, m'-1\}, (D_k, D_{k+1}) \in Q$ or $(D_{k+1}, D_k) \in Q$ with $D_1 = A$ and $D_{m'} = B$. So we obtain a contradiction : m is not the smallest. We use the same argument for $[(D_j, D_{j-1}) \in Q$ and $(D_j, D_{j+1}) \in Q]$. When we use PROP1 on every cases described by "configuration 1" and "configuration 2", we find that : $\exists n \in \mathbb{N} \setminus \{1\}, \exists D_1, \ldots D_n \in \Gamma$ with $D_1 = A$ and $D_n = B$, and either $\forall k \in \{1, \ldots n-1\}, (D_k, D_{k+1}) \in Q$, or $\forall k \in \{1, \ldots n-1\}, (D_{k+1}, D_k) \in Q$. By definition of Q^t it means that we have either $(A, B) \in Q^t$, or $(B, A) \in Q^t$, that is, Q^t is connected.

Remark 5 *If a binary relation is a linear order, its restrictions are too. Consequently, $\forall A \in X/\overline{R}$, $Q^t|_{Q^t A}$ is a linear order.*

Definition 10 *$A \in \Gamma$ is a* least element of Γ *if $A^{Q^t} = \emptyset$.*

It is straightforward to check that any linear order on a finite set has a unique least element. Therefore, $\forall \Gamma \in \mathcal{E}$, Γ has a unique least element. Likewise, $\forall A \in \Gamma$, ${}^{Q^t}A$ has a unique least element.

To number the equivalence classes, we proceed as follow. Firstly, we number the minimal components.

Let $\mathcal{M}$ be the set of all available non-empty minimal components. We define a new binary relation $\mathcal{R} \subseteq \mathcal{M} \times \mathcal{M}$. By definition of a minimal component, $\mathcal{R} = \emptyset$. So $\mathcal{R}$ is obviously a partial order (asymmetric and transitive). By Szpilarjn's theorem, there exists a linear order $\mathcal{R}^*$ which contains $\mathcal{R}$. Then, $\mathcal{M}$ is a non-empty finite linear ordered set : it has a unique least element and we number it Γ_1. Likewise, $\mathcal{M} \setminus \Gamma_1$ is a linear order and we denote Γ_2 it least element. And more generally, we denote Γ_i the least element of $\mathcal{M} \setminus \{\Gamma_j | j < i\}$.

Secondly, let Γ, $\#\Gamma = n$, be the first minimal component. We denote $A_{\Gamma,1}$ its unique least element. If $n = 1$, we go to the next minimal component. Otherwise, we number the rest of the equivalence classes of Γ as follow : $\forall i \in \{2, \ldots, n\}$, $A_{\Gamma,i}$ is the unique least element of ${}^{Q^t}A_{\Gamma,i-1}$. So $A_{\Gamma,n}$ is such that ${}^{Q^t}A_{\Gamma,n} = \emptyset$.
Let $\Delta \in \mathcal{E}$, $\#\Delta = m$, be the second minimal component, $\Delta \neq \Gamma$. We number its equivalence classes similarly : we denote $A_{\Delta,1}$ its unique least element. If $m = 1$, we go to the next minimal component. If $m > 1$, $\forall j \in \{1, \ldots, m\}$, $A_{\Delta,j}$ is the unique least element of ${}^{Q^t}A_{\Delta,j-1}$, and $A_{\Delta,m}$ is such that ${}^{Q^t}A_{\Delta,m} = \emptyset$. And we proceed like this for each equivalence classes in each minimal component…

4. Algorithm to construct the function of representation. Let $\epsilon \in \mathbb{R}_+^*$. Let Γ, $\#\Gamma = n$, be the first minimal component.

Step 1 : we allocate the real value $V_{\Gamma,1} = 0$ to the equivalence classe $A_{\Gamma,1}$.

Step i : we allocate the real value $V_{\Gamma,i}$ to the equivalence classe $A_{\Gamma,i}$, such that :

$$V_{\Gamma,i} = \frac{V_{\Gamma,k} + \epsilon + \max\left\{V_{\Gamma,j}; V_{\Gamma,l} + \epsilon\right\}}{2}$$

With :

- $$V_{\Gamma,k} = \min_{k' | A_{\Gamma,k'} \in A^Q_{\Gamma,i}} V_{\Gamma,k'}$$

- $$V_{\Gamma,j} = \max_{j' | A_{\Gamma,j'} \in A^Q_{\Gamma,i}} V_{\Gamma,j'}$$

- $$V_{\Gamma,l} = \max_{\substack{l' | A_{\Gamma,l'} \notin A^Q_{\Gamma,i} \textbf{ and} \\ \exists m < i \textbf{ s.t. } A_{\Gamma,l'} \in A^Q_{\Gamma,m}}} V_{\Gamma,l'}$$

The way to allocate real value to equivalence class in a minimal component is always the same, as we define for $V_{\Gamma,i}$. The only difference is the way to allocate the value to the first equivalence class in a minimal component. So, $V_{\Gamma,n}$ is the value allocate to the last equivalence classe of Γ. Let $\Delta \in \mathcal{E}$, be the second minimal component, $\Delta \neq \Gamma$.

Step 1 : we allocate the real value $V_{\Delta,1} = V_{\Gamma,n} + 2\epsilon$ to the equivalence classe $A_{\Delta,1}$.

Step i : we allocate the real value $V_{\Delta,i}$ to the equivalence class $A_{\Delta,i}$, as we define previously for $V_{\Gamma,i}$.

And we proceed like this for each equivalence classes in each minimal component...

There is a link between the value of an equivalence class and the value of an alternative. Indeed, let $V \in \mathbb{R}$ be the value associated to the equivalence class A. For all $x \in A$, the value is $f(x) = V$. Notice that every alternative of an equivalence class has the same value.

5. We want to check that, $\forall S \in \mathcal{X}$, $a \in C(S) \Leftrightarrow \forall b \in S, f(a) \geq f(b)$ or $f(a) \leq f(b) - \epsilon$. We know that P rationalizes C so, by definition : $\forall S \in \mathcal{X}$, $C(S) = \{a \in S, \forall b \in S, (b,a) \notin P\}$. So we need to prove : $\{a \in S, \forall b \in S, (b,a) \notin P\} = \{a \in S, \forall b \in S, f(a) \geq f(b)$ or $f(a) \leq f(b) - \epsilon\}$.

Lemma 2 $\forall a, b \in X, (b,a) \in P \Leftrightarrow f(a) + \epsilon > f(b) > f(a)$.

Let A and B be the equivalence classes respectively containing a and b. From definition of Q, $(b,a) \in P \Leftrightarrow (B,A) \in Q$. Let V_A and V_B be the values associated to A and B. Then the lemma becomes : $(B,A) \in Q \Leftrightarrow V_A + \epsilon > V_B > V_A$. From the algorithm, we know that :

$$V_B = \frac{V_k + \epsilon + \max\left\{V_j; V_l + \epsilon\right\}}{2}$$

With :

$$V_k = \min_{k'|D_{k'} \in B^Q} V_{k'}; \qquad V_j = \max_{j'|D_{j'} \in B^Q} V_{j'}; \qquad V_l = \max_{\substack{l'|D_{l'} \notin B^Q \textbf{ and} \\ \exists m<i \textbf{ s.t. } D_{l'} \in D_m^Q}} V_{l'}.$$

We know that $(B, A) \in Q$ so $V_A \in [V_k, V_j]$.
1) If $\max\{V_j; V_l + \epsilon\} = V_j$ then $V_B = \frac{V_k+\epsilon+V_j}{2}$. Since $V_k + \epsilon > V_j$, $V_k + \epsilon > V_B > V_j$. Besides, $V_A + \epsilon \geq V_k + \epsilon$ and $V_j \geq V_A$. So, $V_A + \epsilon > V_B > V_A$.
2) If $\max\{V_j; V_l + \epsilon\} = V_l + \epsilon$ then $V_B = \frac{V_k+\epsilon+V_l+\epsilon}{2}$. Since $V_k + \epsilon > V_l + \epsilon > V_j$, we have $V_k + \epsilon > V_B > V_j$. Besides, $V_A + \epsilon \geq V_k + \epsilon$ and $V_j \geq V_A$. So, $V_A + \epsilon > V_B > V_A$.
Since the values of $a \in A$ and $b \in B$ are : $f(a) = V_A$ and $f(b) = V_B$, then we find $f(a) + \epsilon > f(b) > f(a)$.

Let $a \in C(S)$. P rationalizes C so by definition, $\forall b \in S$, $(b, a) \notin P$. From Lemma 2, $(b, a) \notin P \Leftrightarrow f(a) \geq f(b)$ or $f(a) \leq f(b) + \epsilon$.
Let $a \in S$ such that $f(a) \geq f(b)$ or $f(a) \leq f(b) + \epsilon$. From Lemma 2, it means that $\forall b \in S$, $(b, a) \notin P$, that is $a \in C(S)$. ■

Proof. 2. $\Rightarrow$ **1.** Suppose that there is a function $f : X \to \mathbb{R}$ and $\epsilon \in \mathbb{R}_+$ such that $\forall S \in \mathcal{X}$, $C(S) = \{x \in S | \forall y \in S, f(x) \geq f(y) \text{ or } f(x) \leq f(y) - \epsilon\}$.

1. Let us show that C satisfies Contraction Consistency. Let $x \in S \subseteq T \in \mathcal{X}$ be such that $x \in C(T)$. By definition of C, we know that $\forall y \in T$, $f(x) \geq f(y)$ or $f(x) \leq f(y) - \epsilon$. Suppose that $x \notin C(S)$. Then $\exists z \in S$ such that $f(x) < f(z) < f(x) + \epsilon$. Since $S \subseteq T$, then $z \in T$, which contradicts $x \in C(T)$.

2. Let us show that C satisfies Expansion Consistency. Assume $x \in \bigcap_{i \in \{1,\ldots,n\}} C(S_i)$, $\forall n \in \mathbb{N}$ and $\forall S_1, \ldots, S_n \in \mathcal{X}$. That is, $\forall S_j \in \{S_i\}_{i \in \{1,\ldots,n\}}$, $x \in C(S_j)$, and by definition of C : $\forall y \in S_j$, $f(x) \geq f(y)$ or $f(x) \leq f(y) - \epsilon$. Suppose $x \notin C(\bigcup_{i \in \{1,\ldots,n\}} S_i)$. That is $\exists z \in \bigcup_{i \in \{1,\ldots,n\}} S_i$ such that $f(x) < f(z) < f(x) + \epsilon$. And in particular, $\exists S_k \in \{S_i\}_{i \in \{1,\ldots,n\}}$ such that $z \in S_k$. So, by definition, in this set, $x \notin C(S_k)$, which contradicts $x \in \bigcap_{i \in \{1,\ldots,n\}} C(S_i)$.

3. Let us show that C satisfies Revealed Equivalence.
1) Let $x, y \in X$ be such that $\{x, y\} = C(\{x, y\})$ and $\exists z \in X$ such that $\{z\} = C(\{x, y, z\})$. From $\{x, y\} = C(\{x, y\})$, we have, by definition of C : $[f(x) \geq f(y)$ or $f(x) \leq f(y) - \epsilon]$ and $[f(y) \geq f(x)$ or $f(y) \leq f(x) - \epsilon]$. So there are two possible cases : either $f(x) = f(y)$ or, with no loss of generality $f(y) \leq f(x) - \epsilon$. From $\{z\} = C(\{x, y, z\})$, we must have $f(z) > f(x) > f(z) - \epsilon$ and $f(z) > f(y) > f(z) - \epsilon$. But if $f(y) \leq f(x) - \epsilon$, then there is a contradiction because $f(z) > f(x)$ implies $f(z) - \epsilon > f(x) - \epsilon \geq f(y)$ and we need $f(y) > f(z) - \epsilon$. So necessarily $f(x) = f(y)$. And now, it is straightforward to check that $[\forall S \in \mathcal{X}, x \in C(S \cup \{x\}) \Leftrightarrow y \in C(S \cup \{y\})$ and $C(S \cup \{x\}) \setminus \{x, y\} = C(S \cup \{y\}) \setminus \{x, y\}]$.
2) Same reasonning if $x, y \in X$ be such that $\{x, y\} = C(\{x, y\})$ and $\exists z \in X$ such that $\{x\} = C(\{x, z\})$ and $\{y\} = C(\{y, z\})$. ■

Proof of Theorem 6

Proof. 1. $\Rightarrow$ **2.** To be completed.

$\blacksquare$

Proof. 2. $\Rightarrow$ **1.** Suppose that there is a function $f : X \to \mathbb{R}$ and $\epsilon \in \mathbb{R}_+$ such that $\forall S \in \mathcal{X}$, $C(S) = \{x \in S | \forall y \in S, f(x) \geq f(y) \text{ or } f(x) \leq f(y) - \epsilon\}$ and $[\forall x, y \in S,$ if $f(y) > f(x) - \epsilon$ and $f(z) \leq f(x) - \epsilon$ then $f(z) \leq f(y) - \epsilon]$.

1. By Theorem 5, we know that C satisfies Expansion Consistency and Revealed Equivalence.

2. Let us show that C satisfies Path Independence. Let $S, T \in \mathcal{X}$. C satisfies Path Independence if $C(C(S) \cup C(T)) = C(S \cup T)$.

By Theorem 5, we know that C satisfies Contraction Consistency. Since $C(S) \cup C(T) \subseteq S \cup T$ then $C(S \cup T) \subseteq C(C(S) \cup C(T))$.

Let $x \in C(C(S) \cup C(T))$. There are 3 possibilities: $x \notin T$ and $x \in C(S)$; $x \notin S$ and $x \in C(T)$; $x \in C(S) \cap C(T)$. First, with no loss of generality, assume $x \notin T$ and $x \in C(S)$. By contradiction, assume that $x \notin C(S \cup T)$. Then, $\exists t \in S \cup T$ such that $f(x) < f(t) < f(x) + \epsilon$ (*). $t \notin S$ otherwise $x \notin C(S)$ and $t \notin C(T)$ otherwise $x \notin C(C(S) \cup C(T))$. Consequently, $t \in T \setminus C(T)$. By definition of C, $C(T) \neq \emptyset$. Then $\exists z \in C(T)$ such that $f(t) < f(z) < f(t) + \epsilon$ (**). Since $x \in C(C(S) \cup C(T))$, by definition of C, $\forall y \in C(S) \cup C(T)$, $f(x) \geq f(y)$ or $f(x) \leq f(y) - \epsilon$. In particular, $f(x) \geq f(z)$ or $f(x) \leq f(z) - \epsilon$. With (*) and (**) we have $f(z) > f(x)$ so $f(x) \geq f(z)$ is impossible. Furthermore, by definition of C, if $f(x) \leq f(z) - \epsilon$ and with (**) $f(t) > f(z) - \epsilon$ then $f(x) \leq f(t) - \epsilon$ which contradicts (*). So necessarily, $x \in C(S \cup T)$.

Second, assume $x \in C(S) \cap C(T)$. By contradiction, assume that $x \notin C(S \cup T)$. Then, $\exists t \in S \cup T$ such that $f(x) < f(t) < f(x) + \epsilon$ (*'). Necessarily, $t \in S \cup T \setminus C(S) \cup C(T)$. By definition of C, $C(S) \cup C(T) \neq \emptyset$ so $\exists z \in C(S) \cup C(T)$ such that $f(t) < f(z) < f(t) + \epsilon$ (**'). Since $x \in C(C(S) \cup C(T))$, by definition of C, $\forall y \in C(S) \cup C(T)$, $f(x) \geq f(y)$ or $f(x) \leq f(y) - \epsilon$. In particular, $f(x) \geq f(z)$ or $f(x) \leq f(z) - \epsilon$. With (*') and (**') we have $f(z) > f(x)$ so $f(x) \geq f(z)$ is impossible. Furthermore, by definition of C, if $f(x) \leq f(z) - \epsilon$ and (**') $f(t) > f(z) - \epsilon$ then $f(x) \leq f(t) - \epsilon$ which contradicts (*). So necessarily, $x \in C(S \cup T)$.

Hence, $C(C(S) \cup C(T)) \subseteq C(S \cup T)$: C satisfies Path Independence. $\blacksquare$

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