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Probabilistic Assignment of Objects: Characterizing the Serial Rule

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Abstract

We study the problem of assigning a set of objects to a set of agents when each agent is supposed to receive only one object and has strict preferences over the objects. We focus on the probabilistic methods. We present a characterization of the serial rule proposed by Bogomolnaia and Moulin (2001): the serial rule is the only rule satisfying sd-efficiency, sd no-envy, and bounded invariance. A special representation of feasible assignments that we develop, as preference-decreasing consumption schedules is the key to the simple proof of our main result.

JEL classification: C70, D61, D63.

Keywords: probabilistic assignment; serial rule; sd-efficiency; sd no-envy; bounded invariance; consumption schedule

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1. Introduction

We study the problem of assigning a set of "objects" to a set of agents, when each agent is supposed to receive only one object and has strict preferences over the objects. An example is the problem of assigning apartments to students. There are two types of methods to do this. The traditional method is to assign each object to an agent. However, the indivisibility of objects prevents such methods from achieving any notion of fairness. Suppose, for instance, that two objects are to be assigned to two agents who happen to have the same preference: one agent has to receive his most preferred object, and the other his least preferred object. To restore some form of fairness, probabilistic methods have been proposed and they are commonly used in practice. We focus on such methods.

In the first paper on probabilistic assignments, each agent is assumed to have cardinal preferences, namely, *von Neumann-Morgenstern* preferences over lotteries (Hylland and Zeckhauser, 1979) and assignments are evaluated "ex ante" before the lottery is drawn. A rule defined by a market-clearing price algorithm achieves efficiency and "no-envy," the requirement that each agent should find his assignment at least as desirable as that of each other agent. However, this rule does not always give agents the incentive to report their true preferences. In fact, it turns out that the incentive requirement is not satisfied by any efficient and envy-free rule (Zhou, 1990).[1]

Alternatively, we turn to the way that lotteries are assigned on the basis of the ordinal preferences over the objects. This can be justified by the limited knowledge of agents evaluating objects: an agent could rank objects linearly but might not have an access to the objects so as to evaluate them exactly. It is also difficult to elicit the exact *von Neumann-Morgenstern* preferences from agents (Che and Kojima, 2010). Thus, we rest on the ordinal information of preferences. Then the issue arises of how to compare *lotteries* given the ordinal preferences over the *objects*. We proceed as in Bogomolnaia and Moulin (2001) and assume that each agent compares lotteries by means of the first-order stochastic dominance relation associated with his strict ordinal preferences over objects. Given two distinct

[1] If we weaken no-envy to "equal treatment of equals," this impossibility result still holds.

lotteries, he first compares the probabilities of his receiving his most preferred object in the two lotteries. Next, he compares the sums of the probabilities of his receiving his two most preferred objects. Then, he compares the sums of the probabilities of his receiving his three most preferred objects, and so on. If at each step, the sum of the probabilities, say at the first lottery, is at least as large as that at the second lottery, we say that the first lottery first-order stochastically dominates the other at his preference.[2]

Several axioms have been formulated in terms of this stochastic dominance relation (Bogomolnaia and Moulin, 2001). The first is "sd-efficiency,"[3] the requirement that an assignment should not be first-order stochastically dominated for *all* agents by each other assignment. Next is a fairness requirement, "sd no-envy," that each agent's assignment should stochastically dominate each other agent's assignment at his preference. The last two are strategic requirements. First is "sd-strategy-proofness": when an agent reports his true preference, his assignment should weakly stochastically dominate[4] his assignment when he misrepresents his preferences, no matter what this misrepresentation is.[5] Second is "weak sd-strategy-proofness": when an agent reports his true preference, his assignment should not be stochastically dominated by his assignment when he misrepresents his preferences, no matter what this misrepresentation is.

The "serial rule," introduced by Bogomolnaia and Moulin (2001) satisfies sd-efficiency and sd no-envy. The rule is described by means of an algorithm along which an agent consumes probabilities at an equal rate at each moment. The probability supply of each object is one. Each agent starts with his most preferred object. When the supply of the object that he is consuming is exhausted, he switches to his next most preferred object among the available objects. His

[2]When a lottery stochastically dominates the other at an ordinal preference, the expected utility of the first is larger than that of the other, no matter what *von Neumann-Morgenstern* preference is (as long as it is consistent with the ordinal preference).

[3]For short, we use the prefix "sd" for stochastic dominance in other expressions below. In Bogomolnaia and Moulin (2001), this requirement is referred to as *ordinal efficiency.* In this paper, we adopt the terminology and the notation of Thomson (2010).

[4]An assignment weakly stochastically dominates another if the former stochastically dominates the other or they are the same.

[5]Sd-strategy-proofness is referred to as "straightforwardness" in Gibbard (1978).

assignment is defined as the list of probabilities that he has consumed. The rule can also be defined by means of an exchange algorithm when the endowment of each agent consists of an equal share of all the objects (Kesten, 2009).

Although extensively studied,[6] it is only recently that attempts have been made to characterize the serial rule.[7] In our model, we have two different strategic requirements: sd-strategy-proofness and weak sd-strategy-proofness. It turns out that any sd-efficient and sd envy-free rule violates sd-strategy-proofness (Bogomolnaia and Moulin, 2001). The serial rule satisfies sd-efficiency, sd no-envy, and weak sd-strategy-proofness, but it is not the only one (Kesten et al., 2010).

Several characterizations of the serial rule have been obtained instead from "invariance" requirements imposed together with efficiency and fairness (Hashimoto and Hirata (2010), Heo (2010), and Kesten et al. (2010)). Other invariance axioms have also been formulated for implementation in the same model (Heo and Manjunath, 2010).

In this paper, we propose a new invariance axiom of a rule, which we call "bounded invariance". It says the following. Let k be an integer no greater than the number of objects. Fix the preferences of all agents but one, say agent i. Consider two preference relations for agent i that coincide from his most preferred object down to his k-th most preferred object. Then, apply the rule to the two resulting profiles, thereby obtaining two assignment matrices. Consider agent i's k most preferred objects in the two preferences (they are the same). Then, the probabilities assigned to each agent receiving each of these objects should coincide in the two matrices. This requirement is not very demanding: it is satisfied by all of the rules that have been discussed in the literature (Section 2.3). Moreover, it pertains to a simpler and narrower class of preference changes than "upper

[6]For the strategic aspects of the serial rule, see Che and Kojima (2010), Ekici and Kesten (2010), Heo and Manjunath (2010), and Kesten (2009). For a generalization of this rule to the universal preference domain, namely, the preference domain allowing indifferences, see Katta and Sethuraman (2006). For a generalization of this rule when each agent receives more than one object, see Kojima (2009), Kasajima (2009), and Heo (2010). When each agent has a private endowment, the rule can be adapted so as to meet "endowment lower bound" (Yilmaz (2010) and Athanassoglou and Sethuraman (2010).

[7]The serial rule is characterized by sd-efficiency and sd no-envy only in the special case where each agent has the same preference over all the objects except for the "null object," that is, "receiving nothing." (Bogomolnaia and Moulin, 2002)

invariance" of Kesten et al. (2010) (Section 2.1). Indeed, we show that bounded invariance is weaker than upper invariance.

Our main result is a characterization of the serial rule by means of sd-efficiency, sd no-envy, and bounded invariance. The key to our result is a special representation of feasible assignments that we develop, as "preference-decreasing consumption schedules": given a preference, we represent a lottery as a sequence of time intervals such that an agent with that preference consumes his most preferred object in the first interval, his second most preferred object in the second interval, and so on.

This paper is organized as follows. We describe the formal model in Section 2. We define the axioms in Section 3. Our main result is in Section 4. We conclude with a discussion of two possible generalizations of the model.

2. Model

Let $A \equiv \{o_1, \cdots, o_{|A|}\}$ be a set of objects and $N \equiv \{1, 2, \cdots, n\}$ a set of agents. We assume that $|A| = |N|$. Each agent $i \in N$ has a strict preference R_i over A. Let $\mathcal{R}$ be the set of all such preferences. Let $R \equiv (R_i)_{i \in N} \in \mathcal{R}^N$ be a preference profile. As we keep A and N fixed, we define an **economy** as a preference profile R. For each $S \subseteq A$, let $R_i|_S$ be the preference R_i restricted to S. For each $R_0 \in \mathcal{R}$ and each $a \in A$, denote by $U^0(R_0, a) \equiv \{o \in A : o\ R_0\ a\}$ the strict upper contour set of R_0 at a, and by $U(R_0, a) \equiv \{o \in A : o\ R_0\ a\} \cup \{a\}$ the weak upper contour set of R_0 at a. Similarly, denote by $L^0(R_0, a) \equiv \{o \in A : a\ R_0\ o\}$ the strict lower contour set of R_0 at a, and by $L(R_0, a) \equiv \{o \in A : a\ R_0\ o\} \cup \{a\}$ the weak lower contour set of R_0 at a.

A **probabilistic assignment matrix** is a $|N| \times |A|$ matrix $P \equiv (p_{ia})_{i \in N, a \in A}$ where p_{ia} is the probability of receiving object a by agent i. Given $R \in \mathcal{R}^N$, a probabilistic assignment matrix P is **feasible**, if (i) for each $i \in N$ and each $a \in A$, $p_{ia} \in [0, 1]$, (ii) $\sum_{i \in N} p_{ia} = 1$, and (iii) $\sum_{a \in A} p_{ia} = 1$. By the Birkhoff-von Neumann theorem (Birkhoff (1946), von Neumann (1953)), each feasible assignment matrix can be represented as a convex combination of degenerate probabilistic assignment matrices, i.e., deterministic assignments.[8] Let $\mathcal{P}$ be the set of all feasible assignment matrices. A **rule** is a mapping from $\mathcal{R}^N$ to $\bigcup_{R\in\mathcal{R}^N}\mathcal{P}$. Denote a generic rule by φ.

2.1. Axioms

We assume that each agent compares lotteries by means of the first-order stochastic dominance relation associated with his strict preferences over objects. Suppose that he is given two lotteries. The agent first compares the probabilities of his receiving his most preferred object in the two lotteries. Next, he compares the sums of the probabilities of his receiving his two most preferred objects. Then, he compares the sums of the probabilities of his receiving his three most preferred objects, and so on. If at each step, the sum of the probabilities, say at the first lottery, is at least as large as that at the second lottery and two lotteries are different, we say that the first lottery dominates the other in the first-order stochastic dominance sense, given his preference over objects.

The formal definition is as follows. For each $i \in N$, let $R_i \in \mathcal{R}$, $p_i \equiv (p_{ia})_{a\in A}$, and $p'_i \equiv (p'_{ia})_{a\in A}$. We say that $\boldsymbol{p_i}$ **weakly stochastically dominates** $\boldsymbol{p'_i}$ **at** $\boldsymbol{R_i}$, if for each $a \in A$, $\sum_{b:bR_ia} p_{ib} \geq \sum_{b:bR_ia} p'_{ib}$. We write this as $\boldsymbol{p_i}\ \boldsymbol{R_i^{sd}}\ \boldsymbol{p'_i}$. Let $R \in \mathcal{R}^N$ and $P, P' \in \mathcal{P}$. We say that $\boldsymbol{P}$ **stochastically dominates** $\boldsymbol{P'}$ **at** $\boldsymbol{R}$, if for each $i \in N$, $p_i\ R_i^{sd}\ p'_i$ and $P \neq P'$. We write this as $\boldsymbol{P}\ \boldsymbol{R^{sd}}\ \boldsymbol{P'}$. For short, we use the prefix "sd" for stochastic dominance in other expressions below.

Let φ be a rule and $R \in \mathcal{R}^N$. An assignment matrix $P \in \mathcal{P}$ is "stochastic dominance efficient at R," or simply, **sd-efficient at** $\boldsymbol{R}$, if it is not stochastically dominated by each other $P' \in \mathcal{P}$.[9] The corresponding property of a rule is "stochastic dominance efficiency," or simply,

Sd-efficiency: For each $R \in \mathcal{R}^N$, $\varphi(R)$ is sd-efficient at R.

The following is a fairness axiom. Each agent should find his assignment

[8]Degenerate probabilistic assignment matrix $P \in \mathcal{P}$ is such that for each $i \in N$ and each $a \in A$, $p_{ia} \in \{0, 1\}$.

[9]As mentioned earlier, this requirement is referred to as *ordinal efficiency* in Bogomolnaia and Moulin (2001). However, we adopt the terminology and the notation of Thomson (2010).

at least as desirable as that of each other agent. That is, $P \in \mathcal{P}$ is "stochastic dominance envy-free at R," or simply, **sd envy-free at $\boldsymbol{R}$**, if for each pair $i, j \in N$, $p_i \ R_i^{sd} \ p_j$. The corresponding property of a rule is "stochastic dominance no-envy," or simply,

Sd no-envy: For each $R \in \mathcal{R}^N$, $\varphi(R)$ is sd envy-free at R.

Next are three invariance axioms. The first invariance axiom is proposed by Heo (2010). Let $k \in \{1, 2, \cdots, |A|\}$. Fix the preferences of all agents but one, say agent i. Consider two preference relations for agent i that coincide from his most preferred object down to his k-th most preferred object. Then, apply the rule to the two resulting profiles, thereby obtaining two matrices. Consider agent i's k most preferred objects in the two preferences (they are the same). Then, the probabilities assigned to agent i receiving each of these objects should coincide in the two matrices. Although this requirement does not have a direct strategic implication, it is implied by an important strategic requirement, which we call "sd-strategy-proofness."[10] The formal definition is as follows. For each $o \in A$, let $R_i(o)$ be the preference R_i truncated at o: for each $a, b \in U(R_i, o)$, $a \ R_i \ b$ if and only if $a \ R_i(o) \ b$.

Limited invariance: Let $R \in \mathcal{R}^N$, $i \in N$, $a \in A$, and $R'_i \in \mathcal{R}$ be such that $R_i(a) = R'_i(a)$. Then, for each $o \in U(R_i, a)$, $\varphi_{io}(R) = \varphi_{io}(R'_i, R_{-i})$.

Example 1. Illustration of limited invariance. Let $N = \{1, 2, 3\}$, $A = \{a, b, c, d, e\}$, and φ be a rule satisfying *limited invariance*. Consider the following preference profiles:

[10] When an agent reports his true preference, his assignment should weakly stochastically dominates his assignment when he misrepresents his preferences, no matter what this misrepresentation is.

$\boldsymbol{R_1}$	R_2	R_3		$\boldsymbol{R'_1}$	R_2	R_3
$\boldsymbol{a}$	a	b		$\boldsymbol{a}$	a	b
$\boldsymbol{c}$	c	a	and	$\boldsymbol{c}$	c	a
d	d	d		b	d	d
b	b	e		e	b	e
e	e	c		d	e	c

Let $P \equiv \varphi(R_1, R_2, R_3)$ and $P' \equiv \varphi(R'_1, R_2, R_3)$. Since

$R_1(c)$	$=$	$R'_1(c)$
a		a
c		c

limited invariance requires that $p_{1a} = p'_{1a}$ and $p_{1c} = p'_{1c}$. □

The second invariance axiom adds a flavor of "non-bossiness" (Satterthwaite and Sonnenschein, 1981)[11] to *limited invariance*. Suppose that the hypothesis of *limited invariance* holds. Consider again agent i's k most preferred objects in the two preferences. *Limited invariance* requires that the probabilities assigned to *agent* i receiving each of these objects should coincide in the two matrices. The next requirement additionally requires that the probabilities assigned to *each other agent* receiving each of these objects should coincide in the two matrices. This requirement is not very demanding: it is satisfied by all of the rules that have been discussed in the literature.[12]

Bounded invariance: Let $R \in \mathcal{R}^N$, $i \in N$, $a \in A$, and $R'_i \in \mathcal{R}$ be such that $R_i(a) = R'_i(a)$. Then, for each $o \in U(R_i, a)$ and each $j \in N$, $\varphi_{jo}(R) = \varphi_{jo}(R'_i, R_{-i})$.

[11]If an agent receives the same assignment by misreporting his preferences, then the other agent should also receive the same assignment.

[12]The rules include the class of sequential dictatorial rules, the random priority rule, and the serial rule. We discuss these rules in Section 2.3.

A requirement in the spirit of *bounded invariance* has also been formulated in the probabilistic voting model (Gibbard, 1977). Agents have strict preferences over "public" alternatives and an outcome is a lottery over alternatives. Fix the preferences of all agents but one, say agent i. Consider an alternative, say a, and two preference relations for agent i that have the same upper contour sets at a. Then, apply the rule to the two resulting profiles, thereby obtaining two lotteries. Then, the requirement is that the total probability placed on this common upper contour set should remain the same. Note that, in this model, a lottery is a collective decision: it applies in common to *all* agents. In our model, this is not the case: assignments differ from agent to agent. However, each assignment matrix, i.e., the collection of agents' assignments, can still be viewed as a collective decision. Then, Gibbard (1977)'s requirement can be rephrased in our model as follows. Fix the preferences of all agents but one, say agent i. Consider an object, say a, and two preference relations for agent i that have the same upper contour sets at a. Then, apply the rule to the two resulting profiles, thereby obtaining two matrices. Consider agent i's upper contour sets at a in the two preferences (they are the same). Then, the total probability placed on *each agent* (agent i as well as each other agent) receiving the objects in agent i's upper contour set at a should remain the same in the two matrices.

Localization: Let $R \in \mathcal{R}^N$, $i \in N$, $a \in A$, and $R'_i \in \mathcal{R}$ be such that $U(R_i, a) = U(R'_i, a)$. Then, for each $j \in N$, $\sum_{o \in U(R_i,a)} \varphi_{jo}(R) = \sum_{o \in U(R_i,a)} \varphi_{jo}(R'_i, R_{-i})$.

There is a clear logical relation between this requirement and our invariance requirement.

Proposition 1. *Localization* implies *bounded invariance.*

Proof. Let φ be a *localized* rule. Let $R \in \mathcal{R}^N$, $i \in N$, $a \in A$, and $R'_i \in \mathcal{R}$ be such that $R_i(a) = R'_i(a)$. For each $o \in U(R_i, a)$, we have $U(R_i, o) = U(R'_i, o)$. By *localization*, for each $o \in U(R_i, a)$ and each $j \in N$, $\sum_{o' \in U(R_i,o)} \varphi_{jo'}(R) = \sum_{o' \in U(R_i,o)} \varphi_{jo'}(R'_i, R_{-i})$. This implies that for each $o \in U(R_i, a)$ and each $j \in N$, $\varphi_{jo}(R) = \varphi_{jo}(R'_i, R_{-i})$. $\square$

The last invariance axiom is independently proposed by Kesten et al. (2010).

Let $R \in \mathcal{R}^N$, $P \in \mathcal{P}$, $i \in N$, and $a \in A$. We say that $\boldsymbol{R_i'} \in \boldsymbol{\mathcal{R}}$ **is an upper invariant transformation of** $\boldsymbol{R_i}$ **at** $\boldsymbol{(a, P)}$ if for some $S \subseteq \{c \in A : p_{ic} = 0\}$, $U^0(R_i', a) = U^0(R_i, a) \setminus S$ and for each pair $b, c \in U^0(R_i', a)$, $b\ R_i\ c$ if and only if $b\ R_i'\ c$.

Upper invariance: Let $R \in \mathcal{R}^N$, $i \in N$, $a \in A$, and $R_i' \in \mathcal{R}$ be an upper invariant transformation of R_i at $(a, \varphi(R))$. Then, for each $j \in N$, $\varphi_{ja}(R) = \varphi_{ja}(R_i', R_{-i})$.

The logical relations among three invariance axioms are as follows:[13]

Proposition 2. *Upper invariance* implies *bounded invariance*, which in turn implies *limited invariance*.

Proof. Let φ be a rule satisfying *upper invariance*. Let $R \in \mathcal{R}^N$, $P \in \mathcal{P}$, $i \in N$, $a \in A$, and $R_i' \in \mathcal{R}$ be an upper invariant transformation of R_i at (a, P) such that $U^0(R_i', a) = U^0(R_i, a)$ and $R_i(a) = R_i'(a)$. Then, for each $j \in N$, we obtain that $\varphi_{ja}(R) = \varphi_{ja}(R_i', R_{-i})$. Similarly, for each $b \in U^0(R_i, a)$, P_i' is an upper invariant transformation of R_i at (b, P), and thus for each $j \in N$, we obtain that $\varphi_{jb}(R) = \varphi_{ja}(R_i', R_{-i})$. Thus, φ satisfies *bounded invariance*. The relation between *bounded invariance* and *limited invariance* follows directly from the definitions. □

Remark: The serial rule is also characterized by *sd-efficiency*, *sd no-envy*, *limited invariance*, and an additional axiom, "consistency." (Heo, 2010)

2.2. Consumption Schedule

In this section, we introduce an alternative representation of feasible assignment matrices, which is the key to the proof of our main result. Suppose that a preference profile and a feasible assignment matrix are given. Recall that each row

[13]Note that there is no logical relation between *upper invariance* and *localization*, although both imply *bounded invariance*. The serial rule satisfies *upper invariance* but not *localization*. Next, consider the following rule for an economy with $N = \{1, 2\}$ and $A = \{a, b\}$. If $R_1 = R_2$, then the rule coincides with the serial dictatorial rule associated with $1 \succ 2$ (Section 2.3). Otherwise, the rule assigns to each agent his least preferred object. This rule satisfies *localization* but not *upper invariance*.

of the matrix is the probabilities of the corresponding agent receiving each of the objects. In each row, these probabilities are listed for a fixed order of the objects. The order is independent of the preference profile. In what follows, however, we reorder the entries of each row according to the preference of the agent indexed in that row, in the decreasing order of his preferences.

Example 2. An assignment matrix in decreasing order of preferences Let $A \equiv \{a, b, c\}, N \equiv \{1, 2, 3\}, R \in \mathcal{R}^N$, and $P \in \mathcal{P}$ be such that

$$\begin{array}{ccc} R_1 & R_2 & R_3 \\ \hline a & a & b \\ b & c & a \\ c & b & c \end{array} \quad \text{and} \quad P = \begin{pmatrix} \frac{3}{4}(a) & 0(b) & \frac{1}{4}(c) \\ \frac{1}{4}(a) & 0(b) & \frac{3}{4}(c) \\ 0(a) & 1(b) & 0(c) \end{pmatrix}$$

In P, the objects are listed in the order $a - b - c$. If we reorder the entries of each row according to the preference of the agent indexed in that row, in the decreasing order of his preferences, then we obtain the following matrix:

$$\begin{pmatrix} \frac{3}{4}(a) & 0(b) & \frac{1}{4}(c) \\ \frac{1}{4}(a) & \frac{3}{4}(c) & 0(b) \\ 1(b) & 0(a) & 0(c) \end{pmatrix}$$

Note that the second and the third rows of P have changed. This new matrix is such that the first column consists of the probabilities assigned to the most preferred objects of the agents, the second column consists of the probabilities assigned to their second most preferred objects, and so on.

We proceed one step further. We introduce a consumption process over time. Imagine that each agent consumes probability shares of objects in decreasing order of his preference at a speed of 1. The (time) interval during which he consumes an object is set equal to the probability assigned to his receiving the object. In example 2, agent 2 first consumes his most preferred object a during an interval of length $\frac{1}{4}$, his second most preferred object c during an interval of length $\frac{3}{4}$, and his least preferred object b during an interval of length 0. This is equivalent to saying that agent 2 consumes a during $[0, \frac{1}{4}]$, c during $[\frac{1}{4}, 1]$, and b during $[1, 1]$. Then, we obtain a list of times at which agent 2 switches from one object to another, $(\frac{1}{4}, 1, 1)$. Doing this for each agent results in the following matrix:

$$\begin{pmatrix} \frac{3}{4}(a) & \frac{3}{4}(b) & 1(c) \\ \frac{1}{4}(a) & 1(c) & 1(b) \\ 1(b) & 1(a) & 1(c) \end{pmatrix}$$

To illustrate, agent 1 consumes a during $[0, \frac{3}{4}]$, b during $[\frac{3}{4}, \frac{3}{4}]$, and c during $[\frac{3}{4}, 1]$. The formal definition is as follows. Consider a list of $|A|$ numbers, $t = (t^1, t^2, \cdots, t^{|A|})$ such that

(i) $t^{|A|} = 1$, and

(ii) for each $k \in \{1, \cdots, |A| - 1\}$, $0 \leq t^k \leq t^{k+1} \leq 1$.

We call t a **consumption schedule** if it satisfies (i) and (ii). Let $\mathcal{T}$ be the set of all consumption schedules. Without loss of generality, let $t^0 = 0$. Let $i \in N$ and $R_i \in \mathcal{R}$. For each $k \in \{1, \cdots, |A|\}$, denote by o_i^k the k-th most preferred object of agent i with preference R_i.[14]

Let $i \in N$, $t \in \mathcal{T}$ and $R_i \in \mathcal{R}$. We call (t, R_i) the **preference-decreasing consumption schedule $\boldsymbol{t}$ at $\boldsymbol{R_i}$** if agent i consumes probabilities according to t at a speed of 1 in the decreasing order of R_i. That is, the agent starts with his most preferred object at time 0 and consumes it during $[0, t^1]$. Then, he switches to his second most preferred object and consumes it during $[t^1, t^2]$. $\cdots$ In general, for each $k \in \{1, \cdots, |A|\}$, the probability that he consumes his k-th most preferred object is $t^k - t^{k-1}$. For simplicity, we shorten the expression, "preference-decreasing consumption schedule" to "consumption schedule." Let f be a mapping that associates each consumption schedule at each preference to an assignment:

$$\text{for each } k \in \{1, \cdots, |A|\},\ f_{o_i^k}(t, R_i) \equiv t^k - t^{k-1}.$$

[14]Formally, it should be $o^k(R_i)$, but for simplicity, we omit R_i and use o_i^k instead, otherwise specified.

For each $i \in N$, each $R_i \in \mathcal{R}$, and each assignment p_i, we call $t(p_i, R_i) \in \mathcal{T}$ the **preference-decreasing consumption schedule at R_i representing p_i** if

$$t(p_i, R_i) = (p_{io_i^1}, \sum_{t=1}^{2} p_{io_i^t}, \sum_{t=1}^{3} p_{io_i^t}, \cdots, \sum_{t=1}^{|A|-1} p_{io_i^t}, \sum_{t=1}^{|A|} p_{io_i^t} = 1).$$

Note that $f(t(p_i, R_i), R_i) = p_i$: for each $k \in \{1, \cdots, |A|\}$,

$$f_{o_i^k}(t(p_i, R_i), R_i) = \sum_{t=1}^{k} p_{io_i^t} - \sum_{t=1}^{k-1} p_{io_i^t} = p_{io_i^k}.$$

With a slight abuse of notation, we denote by $t(p_i, R_i)$ the consumption schedule at R_i representing p_i.[15] (That is, we let $(t(p_i, R_i), R_i) \equiv t(p_i, R_i)$.)

Let $a \in A$. We define $\boldsymbol{s(t(p_i, R_i), a)}$ as the time at which agent i starts consuming a under $t(p_i, R_i)$ and $\boldsymbol{e(t(p_i, R_i), a)}$ the time at which agent i ends consuming a under $t(p_i, R_i)$. Obviously, $s(t(p_i, R_i), a) \leq e(t(p_i, R_i), a)$.

For each $R \in \mathcal{R}^N$ and each $P \in \mathcal{P}$, let $\boldsymbol{t(P, R) \equiv (t(p_i, R_i))_{i \in N}}$ be the **profile of consumption schedules at R representing P**: for each $P \in \mathcal{P}$ and $R \in \mathcal{R}$, $t(P, R)$ is uniquely defined. Let $t_i(P, R) \equiv t(p_i, R_i)$.

Let $\tau \in\ ]0, 1]$. Now, we define a consumption schedule during a certain time interval, $[0, \tau]$. Let $R \in \mathcal{R}^N$, $P \in \mathcal{P}$, and $i \in N$. Let $A|_\tau^{t_i(P,R)} \equiv \{a \in A : s(t_i(P, R), a) < \tau\}$ be the set of objects that agent i consumes during $[0, \tau]$ under $t_i(P, R)$.[16] Let $A|_\tau^{t(P,R)} \equiv \bigcup_{i \in N} A|_\tau^{t_i(P,R)}$ be the set of objects at least one agent consumes during $[0, \tau]$ under $t(P, R)$. Let $R_i|_\tau^{t(P,R)} \equiv R_i|_{A|_\tau^{t_i(P,R)}}$ be the preference of R_i restricted to $A|_\tau^{t_i(P,R)}$. Let $R|_\tau^{t(P,R)} \equiv (R_i|_\tau^{t(P,R)})_{i \in N}$. We call $t_i(P, R)|_\tau$ the **consumption schedule $t_i(P, R)$ truncated at τ** if

[15] That is, we omit the second element R_i from $(t(p_i, R_i), R_i)$, since R_i is already embedded in $t(p_i, R_i)$.

[16] Agent i starts consuming each of these objects before τ, but might end consuming it after τ.

(i) the associated preference is $R_i|_{\tau}^{t(P,R)}$,

(ii) for each $i \in N$ and each $a \in A$ with $e(t_i(P,R),a) < \tau$,

$$s(t_i(P,R),a)|_{\tau} = s(t_i(P,R),a), \text{ and}$$
$$e(t_i(P,R),a)|_{\tau} = e(t_i(P,R),a)$$

(iii) for each $i \in N$ and each $a \in A$ with $s(t_i(P,R),a) < \tau \leq e(t_i(P,R),a)$,

$$s(t_i(P,R),a)|_{\tau} = s(t_i(P,R),a), \text{ and}$$
$$e(t_i(P,R),a)|_{\tau} = \tau.$$

Let $t(P,R)|_{\tau} \equiv (t_i(P,R)|_{\tau})_{i \in N}$ be the **profile of consumption schedules $\boldsymbol{t(P,R)}$ truncated at $\boldsymbol{\tau}$**.

Example 2. (continued) Illustrating a consumption schedule

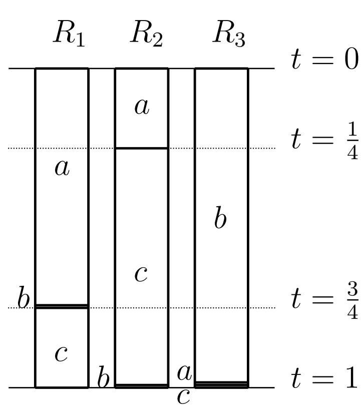
We have $t_1(P,R) = (\frac{3}{4}, \frac{3}{4}, 1)$: agent 1 consumes

a during $[0, \frac{3}{4}]$ (equivalently, $s(t_1(P,R),a) = 0$ and $e(t_1(P,R),a) = \frac{3}{4}$),

b during $[\frac{3}{4}, \frac{3}{4}]$ (equivalently, $s(t_1(P,R),b) = e(t_1(P,R),b) = \frac{3}{4}$), and

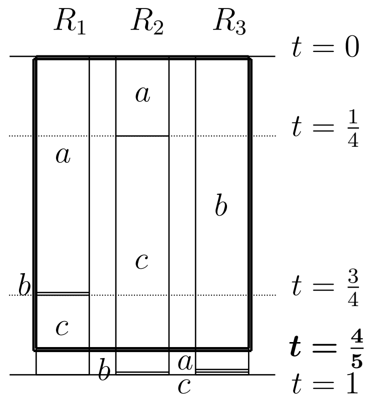
c during $[\frac{3}{4}, 1]$ (equivalently, $s(t_1(P,R),c) = \frac{3}{4}$ and $e(t_1(P,R),c) = 1$).

Similarly, we obtain $t_2(P,R) = (\frac{1}{4}, 1, 1)$ and $t_3(P,R) = (1,1,1)$. The profile of consumption schedules $t(P,R)$ is illustrated in the following figure: a thick horizontal line denoted by an object means that the consumption of that object is 0.



Note that at $t = \frac{1}{4}$, agent 2 ends consuming a, and starts consuming c although the supply of a is not yet exhausted then.

Lastly, consider the profile of consumption schedules truncated at $\frac{4}{5}$. It corresponds to the profile of consumption consumption in the box:



Then, $t(P,R)|_{\frac{4}{5}}$ is

$$\begin{pmatrix} t_1(P,R)|_{\frac{4}{5}} & = & (\frac{3}{4},\frac{3}{4},\frac{4}{5}) \\ t_2(P,R)|_{\frac{4}{5}} & = & (\frac{1}{4},\frac{4}{5}) \\ t_3(P,R)|_{\frac{4}{5}} & = & (\frac{4}{5}) \end{pmatrix} \quad \text{and} \quad \begin{array}{ccc} R_1|_{\frac{4}{5}}^{t(P,R)} & R_2|_{\frac{4}{5}}^{t(P,R)} & R_3|_{\frac{4}{5}}^{t(P,R)} \\ \hline a & a & b \\ b & c & \\ c & & \end{array}$$

□

2.3. The Serial Rule

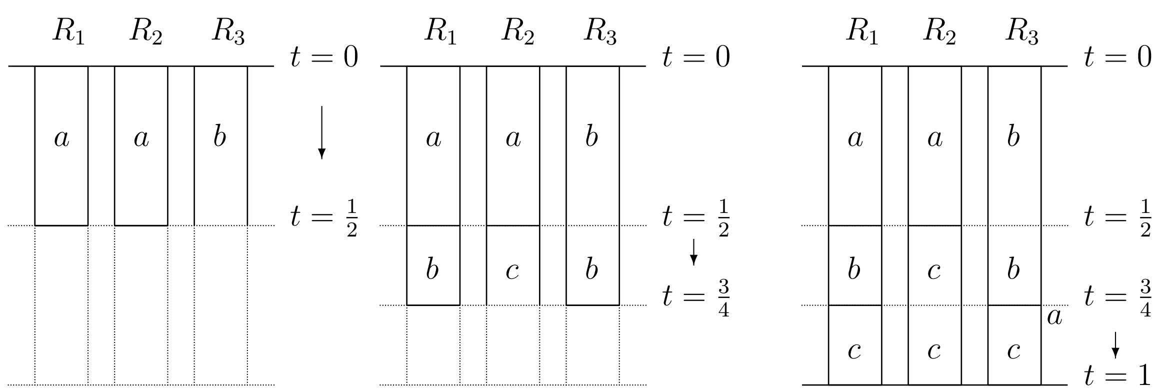
We now define the rule introduced by Bogomolnaia and Moulin (2001). It is defined by the following algorithm.

The serial rule, $\boldsymbol{S}$: the probability supply of each object is one. Given a preference profile, at time $t=0$, each agent starts with his most preferred object in A. Consumption rates are equal across agents (we normalize this common rate to be 1). When the supply of his most preferred object is exhausted, he switches to his second most preferred object among the available objects and starts consuming it. When the supply of his second most preferred object is exhausted, he switches to his third most preferred object and starts consuming it, and so on. His assignment is defined as the list of probabilities that he has consumed.[17]

Note that an agent switches from one object to another *only when* the supply of the object he is currently consuming is exhausted.

Example 3. Illustrating the serial rule.
Let $A \equiv \{a, b, c\}$, $N \equiv \{1, 2, 3\}$, $R \in \mathcal{R}^N$ be such that

R_1	R_2	R_3
a	a	b
b	c	a
c	b	c



Step 1 **Step 2** **Step 3**

Step 1: at $t = 0$, agents 1 and 2 start consuming a, and agent 3 starts consuming b. At $t = \frac{1}{2}$, the supply of a is exhausted.

Step 2: at $t = \frac{1}{2}$, agent 1 switches to his second most preferred object, b and agent 2 switches to c. Agent 3 continues to consume b. At $t = \frac{3}{4}$, the supply of b is exhausted.

[17] There are several rules other than the serial rule in this model. We introduce two of them. Given an ordering $\succ$ over N, the **sequential dictatorial rule associated with** $\succ$, $\boldsymbol{SD^{\succ}}$ is defined as follows: if $\succ = (1 \succ 2 \succ \cdots \succ (n-1) \succ n)$, then agent 1 first receives his most preferred object with probability 1. Then, agent 2 chooses his most preferred lottery among the remaining objects with probability one. The assignment is determined by repeating this process up to agent n. The **random priority rule**, $\boldsymbol{RP}$, is defined by averaging the sequential dictatorial rules over all possible $\succ$'s.

Step 3: at $t = \frac{3}{4}$, agent 1 switches to his least preferred object, c and agent 2 continues to consume c. Agent 3 switches to his second most preferred object, a, but since the supply of a was exhausted in Step 1, he switches to c. In the end, we obtain the following assignment matrix:

$$S(e) = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

$\square$

As in Section 2.2, we represent each rule by means of profiles of consumption schedules, as follows:

$\boldsymbol{\varphi}$-profile of consumption schedules, $\boldsymbol{t\varphi}$: for each $R \in \mathcal{R}^N$, $t\varphi(R) \equiv t(\varphi(R), R)$.

For each $R \in \mathcal{R}^N$, $t\varphi(R)$ is uniquely determined, and thus $t\varphi$ is uniquely defined. Let $t\varphi_i(R) \equiv t_i(\varphi(R), R)$.

In particular, the S-profile of consumption schedules, denoted by $\boldsymbol{tS}$, is for each $R \in \mathcal{R}^N$, $tS(R) = t(S(R), R)$. By definition of the serial rule, for each $R \in \mathcal{R}^N$, $tS(R)$ is such that for each $k \in \{1, \cdots, |A| - 1\}$, each $i \in N$ turns from o_i^k to o_i^{k+1} when the supply of o_i^k is exhausted. Formally, $\varphi = S$ if and only if for each $R \in \mathcal{R}^N$, $tS(R) \equiv T$ is such that there are no $i, j \in N$ and $a \in A$ such that $e(T_i, a) < e(T_j, a)$ and $S_{ja}(R) > 0$. That is, it should not be the case that agent j consumes a positive amount of object a after agent i ends consuming a.

Remark: Kesten et al. (2010) offer another characterization of the serial rule by means of the following axiom.

> **Ordinal fairness**: For each $R \in \mathcal{R}^N$, each pair $i, j \in N$, and each $a \in A$ with $\varphi_{ia}(R) > 0$, $\sum_{b \in U(R_i, a)} \varphi_{ib}(R) \leq \sum_{b \in U(R_j, a)} \varphi_{jb}(R)$.

By using the profile of consumption schedules, we are able to prove it directly.

Theorem 1. (**Kesten et al. (2010)**) The serial rule is the only rule satisfying *ordinal fairness.*

Proof. Let φ be a rule satisfying this axiom. Suppose, by contradiction, that $\varphi \neq S$. Then, there is $R \in \mathcal{R}^N$ such that $\varphi(R) \neq S(R)$. There are a pair $i, j \in N$ and $a \in A$ such that $e(t\varphi_i(R), a) < e(t\varphi_j(R), a)$ and $\varphi_{ja}(R) > 0$. Thus,

$$e(t\varphi_j(R), a) = \sum_{b \in U(R_j, a)} \varphi_{jb}(R) > \sum_{b \in U(R_i, a)} \varphi_{ia}(R) = e(t\varphi_i(R), a),$$

in violation of *ordinal fairness*.[18] $\square$

3. Results

Our main result is a characterization of the serial rule.

Theorem 2. The serial rule is the only rule satisfying *sd-efficiency, sd no-envy*, and *bounded invariance.*

Proof. We start by two properties of the serial rule. Let $P \in \mathcal{P}$, $R \in \mathcal{R}^N$, and $i \in N$. We say that **agent $\boldsymbol{i}$ does not do his best on $\boldsymbol{a}$ at $\boldsymbol{\tau}$ under $\boldsymbol{t(P, R)}$** if

$$\left(\begin{array}{c} \text{for some } \tau \in [0,1[\text{ and some } a \in A, \text{ agent } i \text{ ends consuming } a \text{ at } \tau \\ \text{even though the supply of } a \text{ is not exhausted at } \tau \text{ under } t(P,R). \end{array} \right)$$

We say that **agent $\boldsymbol{i}$ does his best during $\boldsymbol{[0, \tau]}$ under $\boldsymbol{t(P, R)}$** if, under $t(P, R)$, for each $\tau' \leq \tau$, there is no object $b \in A$ such that agent i does not do his best on b at τ'. Note that for each $R \in \mathcal{R}^N$, under $tS(R)$, each agent does his best during $[0, 1]$.

Let $\tau \in [0, 1]$, $P \in \mathcal{P}$, and $R \in \mathcal{R}^N$. The first lemma says that if, under $t(P, R)$, each agent does his best during $[0, \tau]$, then $t(P, R)$ truncated at τ is the same as for the serial rule.

[18] Since we assume that $|A| = |N|$, *ordinal fairness* itself characterizes the serial rule. Suppose that for each $a \in A$, there are q_a copies of a and $\sum_{a \in A} q_a > |N|$ as in Kesten et al. (2010). Then, *ordinal fairness*, together with the following axiom, characterizes the same rule.

Non-wastefulness: For each $R \in \mathcal{R}^N$, each $i \in N$, and each $a \in A$ with $\varphi_{ia}(R) > 0$, if $b \; R_i \; a$, then $\sum_{j \in N} \varphi_{jb}(R) = 1$.

The proof is very similar to the one stated above. If for some $R \in \mathcal{R}^N$, $\varphi(R) \neq S(R)$, then either (1) there are a pair $i, j \in N$ and $a \in A$ such that $e(t\varphi_i(R), a) < e(t\varphi_j(R), a)$ and $\varphi_{ja}(R) > 0$, or (2) there are $i \in N$ and $a \in A$ such that $e(t\varphi_i(R), a) < 1$ and $\sum_{j \in N} \varphi_{ja}(R) \neq q_a$. However, (1) contradicts *ordinal fairness*, and (2) contradicts *non-wastefulness*.

Lemma 1. Let $\tau \in]0,1]$, $P \in \mathcal{P}$, and $R \in \mathcal{R}^N$. Then, $t(P,R)$ is such that each agent does his best during $[0,\tau]$ under $t(P,R)$ if and only if $t(P,R)|_\tau = tS(R)|_\tau$.

Proof. This comes from the definition of the serial rule. □

Let $\tau \in]0,1]$, $P \in \mathcal{P}$, $R \in \mathcal{R}^N$, and $R'_i \in \mathcal{R}$. Suppose that, under $t(P,R)$ and $t(P,(R'_i,R_{-i}))$, each agent does his best during $[0,\tau]$. Suppose that, under $t(P,R)$ and $t(P,(R'_i,R_{-i}))$, the rankings of the objects that agent i consumes during $[0,\tau]$ are the same. Then, the profiles of consumption schedules, $t(P,R)$ and $t(P,(R'_i,R_{-i}))$ truncated at τ are the same.

Lemma 2. Let $\tau \in]0,1]$, $P \in \mathcal{P}$, and $R \in \mathcal{R}^N$, $i \in N$, and $R'_i \in \mathcal{R}$ be such that $t(P,R)|_\tau = tS(R)|_\tau$ and $t(P,(R'_i,R_{-i}))|_\tau = tS(R'_i,R_{-i})|_\tau$.

$$\text{If } R'_i|_\tau^{t(P,(R'_i,R_{-i}))} = R_i|_\tau^{t(P,R)}, \text{ then } t(P,(R'_i,R_{-i}))|_\tau = t(P,R)_\tau.$$

Proof. This comes from the definition of the serial rule: for each $\tau \in [0,1[$, the consumption schedule during $[0,\tau]$ is independent of the consumption schedule after τ. □

We show that the serial rule satisfies three axioms. Bogomolnaia and Moulin (2001) show that the serial rule satisfies *sd-efficiency* and *sd no-envy*. Let $R \in \mathcal{R}^N$, $i \in N$, $a \in A$, and $R'_i \in \mathcal{R}$ be such that $R_i(a) = R'_i(a)$. Let $T \equiv tS(R)$ and $T' \equiv tS(R'_i,R_{-i})$. Let $\tau_a \equiv e(T_i,a)$. By Lemma 2, $T|_{\tau_a} = T'|_{\tau_a}$. Then, for each $o \in U(R_i,a)$ and each $j \in N$ such that $S_{jo}(R) > 0$,

$$\begin{aligned} e(T_j,o) = e(T'_j,o) &\leq e(T_j,a) = e(T'_j,a) \text{ and} \\ s(T_j,o) = s(T'_j,o) &\leq s(T_j,a) = s(T'_j,a). \end{aligned}$$

Thus, for each $o \in U(R_i,a)$ and each $j \in N$, $S_{jo}(R) = S_{jo}(R'_i,R_{-i})$.

Conversely, we show that if φ satisfies the three axioms, then $\varphi = S$. We start from the following lemma. Let $\tau \in]0,1]$, $P,P' \in \mathcal{P}$, and $R \in \mathcal{R}^N$. Suppose that, under $t(P,R)|_\tau$, each agent does his best during $[0,\tau]$. Let $a \in A$ be an object that at least one agent starts consuming before τ. Let $i \in N$ be an agent who consumes a positive amount of a under $t(P,R)|_\tau$. Let τ_a be the time at which this agent ends consuming a under $t(P,R)|_\tau$ (Note that $\tau_a \le \tau$). Now, consider another profile of consumption schedules, $t(P',R)|_\tau$. (Note that the preference profile R does not change.) Suppose that, under $t(P',R)|_\tau$, agent i ends consuming a earlier than he does under $t(P,R)|_\tau$. Then, under $t(P',R)|_\tau$, there is an agent (not necessarily agent i) who does not do his best on some object (not necessarily object a) even earlier than τ_a.

Lemma 3. Let $\tau \in]0,1]$, $P \in \mathcal{P}$, and $R \in \mathcal{R}^N$ be $t(P,R)|_\tau = tS(R)|_\tau$. Let $i \in N$ and $a \in A$ be such that $s(t_i(P,R),a) < \tau$ and $p_{ia} > 0$. If

$$e(t_i(P',R)|_\tau, a) < e(t_i(P,R)|_\tau, a),$$

then there are $j \in N$, $b \in A|_\tau^{t_j(P,R)}$, and $\tau' < e(t_i(P,R)|_\tau, a)$ such that agent j does not do his best on b at τ' under $t(P',R)|_\tau$.

Proof. Let $\tau_a \equiv e(t_i(P,R)|_\tau, a)$ and $\tau'_a \equiv e(t_i(P',R)|_\tau, a)$. If $\tau'_a < \tau_a$, then there are two cases.

Case 1. $t(P,R)|_{\tau'_a} = t(P',R)|_{\tau'_a}$. Since, under $t(P,R)$, each agent does his best during $[0,\tau_a]$, he does so $[0,\tau'_a]$ under $t(P,R)$. By Lemma 1, $t(P,R)|_{\tau'_a} = tS(R)|_{\tau'_a}$, and thus $t(P',R)|_{\tau'_a} = tS(R)|_{\tau'_a}$. Since $\tau'_a < \tau_a$, the supply of object a is not exhausted at τ'_a under $tS(R)|_\tau$. Thus, by feasibility, there is $j \in N$ such that $p'_{ja} > 0$ and $e(t_j(P',R)|_\tau, a) > \tau'_a$. Equivalently, under $t(P',R)|_\tau$, agent i does not do his best on a at τ'_a.

Case 2. $t(P,R)|_{\tau'_a} \neq t(P',R)|_{\tau'_a}$. By Lemma 1, $t(P',R)|_{\tau'_a} \neq tS(R)|_{\tau'_a}$. Thus, there are $j \in N$, $b \in A|_{\tau'_a}^{t_j(P',R)}$, and $\bar{\tau} < \tau'_a$ such that agent j does not do his best on b at $\bar{\tau}$. □

Let $k,l \in N$, $a \in A$, and $R_k, R_l \in \mathcal{R}$. We say that $\hat{R}_k \in \mathcal{R}$ is an **upward transformation of $\boldsymbol{R_k}$ at $\boldsymbol{a}$ with respect to $\boldsymbol{R_l}$** if (i) $R_k(a) = \hat{R}_k(a)$ and (ii) for each $b \in L^0(R_k,a) \cap U^0(R_l,a)$ and each $c \in L^0(R_k,a) \setminus U^0(R_l,a)$, $a \ \hat{R}_k \ b \ \hat{R}_k \ c$. That is, (i) the rankings from agent k's most preferred object down to a are the same at R_k and $\hat{R}_k$, but (ii) all the objects that are preferred to a at R_l but not at R_k, move upward to be placed just below a in agent k's ranking. Note that if $\hat{R}_k$ is an upward transformation of R_k at a with respect to R_l, then $U^0(\hat{R}_k,a) \cup [L^0(R_k,a) \cap U^0(R_l,a)] \supseteq U^0(R_l,a)$.

Example 4. Illustration of an upward transformation.
Consider the following preferences, R_k and R_l.

R_k	R_l		$\hat{R}_k$	R_l		$\hat{R}'_k$	R_l
a	f		a	$\boldsymbol{f}$		a	$\boldsymbol{f}$
b	a		b	a		b	a
$\boxed{c}$	d		$\boxed{c}$	$\boldsymbol{d}$		$\boxed{c}$	$\boldsymbol{d}$
d	$\boxed{c}$		$\boldsymbol{d}$	$\boxed{c}$		$\boldsymbol{f}$	c
e	e		$\boldsymbol{f}$	e		$\boldsymbol{d}$	e
f	b		e	b		e	b

The preferences $\hat{R}_k$ and $\hat{R}'_k$ are the upward transformations of R_k at c with respect to R_l (note that there are only two upward transformations in this case). The objects in $L^0(R_k, c) \setminus U^0(R_l, c)$ (that is, d and f) move upward in agent k's preference to be placed just below c. □

Let $R \in \mathcal{R}^N$, $P \equiv \varphi(R)$, and $T \equiv t\varphi(R)$. Suppose that $T \neq tS(R)$. Then, under T, there is an agent who does not do his best. Choose an agent i who does not do his best at the earliest time under T. Then, there are $j \in N$ and $a \in A$ such that $e(T_i, a) < e(T_j, a)$, $p_{ja} > 0$, and $T|_{e(T_i,a)} = tS(R)|_{e(T_i,a)}$.[19] Let $\hat{R}_i \in R$ be an upward transformation of R_i at a with respect to R_j. Let $P' \equiv \varphi(\hat{R}_i, R_{-i})$ and $T' \equiv t\varphi(\hat{R}_i, R_{-i})$.

Claim 1. For each $o \in L^0(R_i, a) \cap U^0(R_j, a)$, $p_{io} = p'_{io} = 0$.

Proof. Since for each $o \in L^0(R_i, a) \cap U^0(R_j, a)$,

$$a \ R_i \ o \ R_j \ a, \text{ and } p_{ja} > 0,$$

we obtain, by *sd-efficiency*, that $p_{io} = 0$. By *bounded invariance*, for each $o \in L(R_i, a)$ and each $k \in N$, $p_{ko} = p'_{ko}$. Thus, $p'_{ja} > 0$. Since for each $o \in L^0(\hat{R}_i, a) \cap U^0(R_j, a)$, $a \ \hat{R}_i \ o \ R_j \ a$, and $p'_{ja} > 0$, we obtain, by *sd-efficiency*, that $p'_{io} = 0$. ■

[19]Agent i is chosen so that each other agent does his best during $[0, e(T_i, a)]$ under T.

Let $\tau \equiv e(T_i, a)$ be the earliest time at which an agent does not do his best under T. Then, $T|_\tau = tS(R)|_\tau$. Our next claim is that, if the preference of agent i changes from R_i to $\hat{R}_i$, then, under $T'|_\tau$, there is an agent who does not do his best earlier than τ.

Claim 2. Under $T'|_\tau$, there are $\tau' < \tau$, $k \in N$, and $b \in A$ such that agent k does not do his best on b at τ'.

The statement in Claim 2 is equivalent to the following: there are $k, l \in N$ and $b \in A$ such that $e(T'_k|_\tau, b)(= \tau') < e(T'_l|_\tau, b)$ and $p'_{lb} > 0$.

Proof. By Claim 1 and *bounded invariance*,

$$\begin{aligned} \textstyle\sum_{o \in U(\hat{R}_i, a)} p'_{io} + \sum_{o \in L^0(R_i, a) \cap U^0(R_j, a)} p'_{io} &= \textstyle\sum_{o \in U(\hat{R}_i, a)} p'_{io} \\ &= \textstyle\sum_{o \in U(R_i, a)} p_{io} = e(T_i, a). \end{aligned}$$

Since $U(R_j, a) \subseteq U(\hat{R}_i, a) \cup [L^0(R_i, a) \cap U^0(R_j, a)]$, by *sd no-envy*,

$$\textstyle\sum_{o \in U(R_j, a)} p'_{jo} \leq \sum_{o \in U(\hat{R}_i, a) \cup [L^0(R_i, a) \cap U^0(R_j, a)]} p'_{io} = e(T_i, a) = \tau,$$

and, by *bounded invariance*, $p'_{ja} = p_{ja} > 0$. We show that there is $o \in A|_\tau^{T_j} \setminus \{a\}$ such that $p_{jo} > 0$ and $e(T'_j|_\tau, o) < e(T_j|_\tau, o)$.[20] Then, by applying Lemma 3, we complete the proof of Claim 2. Suppose, by contradiction, that for each $o \in A|_\tau^{T_j} \setminus \{a\}$, either $p_{jo} = 0$ or $e(T_j|_\tau, o) \leq e(T'_j|_\tau, o)$. There are two cases:

Case 1. $a \notin A|_\tau^{T_j}$. Then,

$$\begin{aligned} \max_{o \in A|_\tau^{T_j} \setminus \{a\}} e(T'_j|_\tau, o) &= \max_{o \in A|_\tau^{T_j}} e(T'_j|_\tau, o) \\ &\geq \max_{o \in A|_\tau^{T_j}} e(T_j|_\tau, o) = \tau, \qquad \cdots\cdots\cdots\cdots\cdots\cdots (*) \end{aligned}$$

where the inequality comes from the fact that for each $o \in A|_\tau^{T_j} \setminus \{a\}$, either $p_{jo} = 0$ or $e(T'_j|_\tau, o) \geq e(T_j|_\tau, o)$. The last equality comes from the definitions of $A|_\tau^{T_j}$ and $T_j|_\tau$. Thus, we obtain that

$$\begin{aligned} \textstyle\sum_{o \in U(R_j, a)} p'_{jo} &\geq \textstyle\sum_{o \in A|_\tau^{T_j} \cup \{a\}} p'_{jo} = \sum_{o \in A|_\tau^{T_j}} p'_{jo} + p'_{ja} \\ &\geq \max_{o \in A|_\tau^{T_j}} e(T'_j|_\tau, o) + p'_{ja} \geq \tau + p'_{ja} > \tau, \end{aligned}$$

[20]Note that since $o \in A|_\tau^{T_j} \setminus \{a\}$, $e(T_j|_\tau, o) \leq \tau$.

where the first inequality comes from the fact that $U(R_j, a) \supseteq A|_\tau^{T_j} \cup \{a\}$ and the second inequality comes from $\sum_{o \in A|_\tau^{T_j}} p'_{jo} = \max_{o \in A|_\tau^{T_j}} e(T'_j, o) \geq \max_{o \in A|_\tau^{T_j}} e(T'_j|_\tau, o)$. The third inequality comes from $(*)$ and the last comes from $p_{ja} = p'_{ja} > 0$. However, it contradicts $\tau \geq \sum_{o \in U(R_j,a)} p'_{jo}$.

Case 2. $a \in A|_\tau^{T_j}$. Then, $U(R_j, a) = A|_\tau^{T_j}$ and we obtain that

$$\begin{aligned}\sum_{o \in U(R_j,a)} p'_{jo} = \sum_{o \in A|_\tau^{T_j}} p'_{jo} &= \sum_{o \in A|_\tau^{T_j} \setminus \{a\}} p'_{jo} + p'_{ja} \\ &\geq \max_{o \in A|_\tau^{T_j} \setminus \{a\}} e(T'_j|_\tau, o) + p'_{ja} \\ &\geq \max_{o \in A|_\tau^{T_j} \setminus \{a\}} e(T_j|_\tau, o) + p_{ja} > \tau,\end{aligned}$$

where the first inequality comes from

$$\sum_{o \in A|_\tau^{T_j} \setminus \{a\}} p'_{jo} = \max_{o \in A|_\tau^{T_j} \setminus \{a\}} e(T'_j, o) \geq \max_{o \in A|_\tau^{T_j} \setminus \{a\}} e(T'_j|_\tau, o)$$

and the second inequality comes from $p_{ja} = p'_{ja} > 0$ and that for each $o \in A|_\tau^{T_j} \setminus \{a\}$, either $p_{jo} = 0$ or $e(T_j|_\tau, o) \leq e(T'_j|_\tau, o)$. The last inequality comes from

$$\max_{o \in A|_\tau^{T_j} \setminus \{a\}} e(T_j|_\tau, o) = \max_{o \in A|_\tau^{T_j} \setminus \{a\}} e(T_j, o) = \sum_{o \in A|_\tau^{T_j} \setminus \{a\}} p_{ja}$$

and $\sum_{o \in A|_\tau^{T_j}} p_{jo} = \sum_{o \in U(R_j,a)} p_{jo} > \tau$. However, it contradicts $\tau \geq \sum_{o \in U(R_j,a)} p'_{jo}$. ■

By Claim 2, under $T'|_\tau$, there is an agent who does not do his best before τ. Choose an agent k^* who does not do his best at the earliest time under $T'|_\tau$. Let $\tau^* \equiv e(T'_{k^*}|_\tau, b^*)$ (Note that $\tau^* < \tau$). Then, there are $l^* \in N$ and $b^* \in A$ such that $\tau^* < e(T'_{l^*}|_\tau, b^*)$, $p'_{l^* b^*} > 0$, and $T'|_{\tau^*} = tS(\hat{R}_i, R_{-i})|_{\tau^*}$.

Claim 3. $T|_{\tau^*} = T'|_{\tau^*}$.

Proof. We set τ^* so that $T'|_{\tau^*} = tS(\hat{R}_i, R_{-i})|_{\tau^*}$. From $\tau^* < \tau$ and $T|_\tau = tS(R)|_\tau$, it follows that $T|_{\tau^*} = tS(R)|_{\tau^*}$. By Lemma 2, $tS(R)|_{\tau^*} = tS(\hat{R}_i, R_{-i})|_{\tau^*}$. ■

Now, we prove Theorem 1. Suppose, by contradiction, that $\varphi \neq S$. Then, for some $R^0 \in \mathcal{R}^N$, $\varphi(R^0) \neq S(R^0)$. Under $t\varphi(R^0)(\equiv T^0)$, there is an agent who does not do his best. Choose an agent i^0 who does not do his best at the earliest time under T^0. Then, there are $j^0 \in N$ and $a^0 \in A$ such that

$$e(T^0_{i^0}, a^0) < e({T^0}_{j^0}, a^0) \text{ and } \varphi_{j^0 a^0}(R^0) > 0.$$

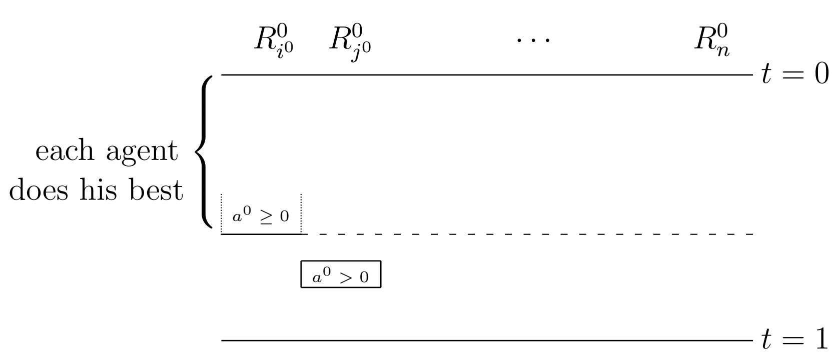
We refer to this as Step 0.

Step 0: $R^0 \in \mathcal{R}^N$, $i^0, j^0 \in N$, $a^0 \in A$, and $T^0 \equiv t(\varphi(R^0), R^0)$:

$$\left\{ \begin{array}{l} \text{(i) } e(T^0_{i^0}, a^0) < e(T^0_{j^0}, a^0), \\ \text{(ii) } \varphi_{j^0 a^0}(R^0) > 0, \\ \text{(iii)} T^0|_{e(T^0_{i^0}, a^0)} = tS(R^0)|_{e(T^0_{i^0}, a^0)}, \end{array} \right\}$$

and let $\hat{R}^0_{i^0} \in \mathcal{R}$ be an upward transformation of $R^0_{i^0}$ at a^0 with respect to $R^0_{j^0}$.

Statements (i) and (ii) together say that agent j^0 consumes a positive fraction of a^0 after agent i^0 ends consuming it, thus, under T^0, agent i^0 does not do his best on a^0 at $e(T^0_{i^0}, a^0)$. Statement (iii) says that, under T^0, i^0 does not do his best at the earliest time.



Step 0

In the next step, we replace $R^0_{i^0}$ by $\hat{R}^0_{i^0}$. Let $R^1 \equiv (\hat{R}^0_{i^0}, R^0_{-i^0})$. By Claim 3, under $t\varphi(R^1)(\equiv T^1)$, there is an agent who does not do his best earlier than agent i^0 does under T^0. Choose an agent i^1 who does not do his best at the earliest time under T^1. Then, there are $j^1 \in N$ and $a^1 \in A$ such that

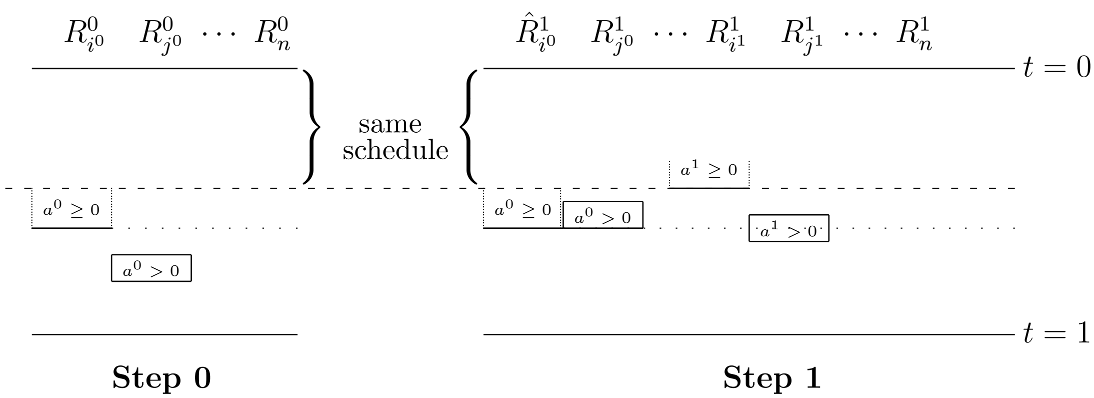
$$e(T^1_{i^1}, a^1) < e({T^1}_{j^1}, a^1) \text{ and } \varphi_{j^1 a^1}(R^1) > 0.$$

Step 1: Let $R^1 \equiv (\hat{R}^0_{i^0}, R^0_{-i^0}) \in \mathcal{R}^N$. There are $i^1, j^1 \in N$, $a^1 \in A$, and $T^1 \equiv t(\varphi(R^1), R^1)$ such that

$$\left\{\begin{array}{l} \text{(i) } e(T^1_{i^1}, a^1) < e(T^1_{j^1}, a^1), \\ \text{(ii) } \varphi_{j^1 a^1}(R^1) > 0, \\ \text{(iii) } T^1|_{e(T^1_{i^1}, a^1)} = tS(R^1)|_{e(T^1_{i^1}, a^1)}, \\ \text{(iv) } e(T^1_{i^1}, a^1) < e(T^0_{i^0}, a^0), \\ \text{(v) } T^1|_{e(T^1_{i^1}, a^1)} = T^0|_{e(T^1_{i^1}, a^1)}, \end{array}\right\}$$

and let $\hat{R}^1_{i^1} \in \mathcal{R}$ be an upward transformation of $R^1_{i^1}$ at a^1 with respect to $R^1_{j^1}$.

Statements (i) and (ii) together say that agent j^1 consumes a positive fraction of a^1 after agent i^1 ends consuming it, thus, under T^1, agent i^1 does not do his best on a^1 at $e(T^1_{i^1}, a^1)$. Statement (iii) says that, under T^1, agent i^1 does not do his best at the earliest time. Statement (iv) comes from Claim 2: the time at which some agent does not do his best decreases. Statement (v) comes from Claim 3: since i^1 is an agent who does not do his best at the earliest time under T^0, the profile of consumption schedules truncated at $e(T^1_{i^1}, a^1)$ are the same for both T^0 and T^1.



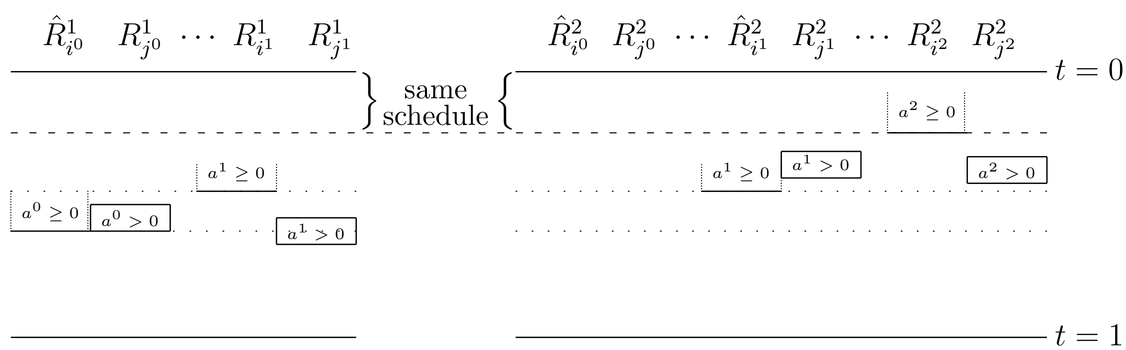
In the next step, we replace $R^1_{i^1}$ by $\hat{R}^1_{i^1}$. Let $R^2 \equiv (\hat{R}^1_{i^1}, R^1_{-i^1})$. By Claim 3, under $t\varphi(R^2)(\equiv T^2)$, there is an agent who does not do his best earlier than agent i^1 does under T^1. Choose an agent i^2 who does not do his best at the earliest time under T^2. Then, there are $j^2 \in N$ and $a^2 \in A$ such that

$$e(T^2_{i^2}, a^2) < e({T^2}_{j^2}, a^2) \text{ and } \varphi_{j^2 a^2}(R^2) > 0.$$

Step 2: Let $R^2 \equiv (\hat{R}^1_{i^1}, R^1_{-i^1}) \in \mathcal{R}^N$. There are $i^2, j^2 \in N$, $a^2 \in A$, and $T^2 \equiv t(\varphi(R^2), R^2)$ such that

$$\left\{ \begin{array}{l} \text{(i) } e(T^2_{i^2}, a^2) < e(T^2_{j^2}, a^2), \\ \text{(ii) } \varphi_{j^2 a^2}(R^2) > 0, \\ \text{(iii) } T^2|_{e(T^2_{i^2}, a^2)} = tS(R^2)|_{e(T^2_{i^2}, a^2)}, \\ \text{(iv) } e(T^2_{i^2}, a^2) < e(T^1_{i^1}, a^1), \\ \text{(v) } T^2|_{e(T^2_{i^2}, a^2)} = T^1|_{e(T^2_{i^2}, a^2)}, \end{array} \right\}$$

and let $\hat{R}^2_{i^2} \in \mathcal{R}$ be an upward transformation of $R^2_{i^2}$ at a^2 with respect to $R^2_{j^2}$.



Step 1 **Step 2**

$\vdots$

In general, we have

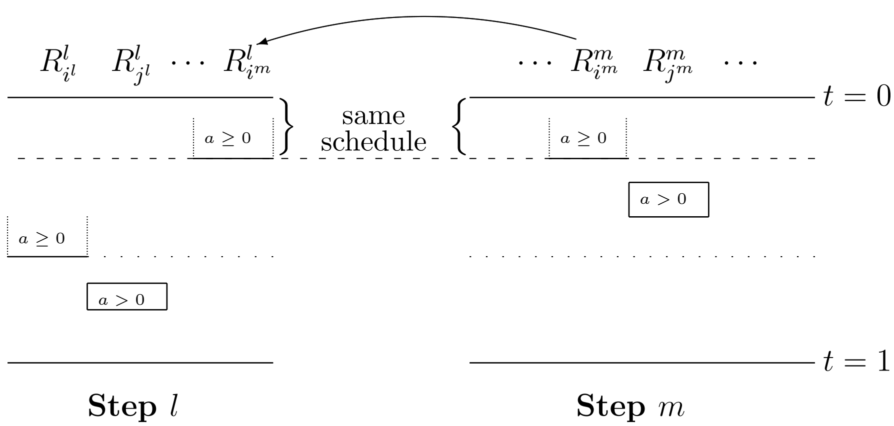
Step k: Let $R^k \equiv (\hat{R}^{k-1}_{i^{k-1}}, R^{k-1}_{-(i^{k-1})}) \in \mathcal{R}^N$. There are $i^k, j^k \in N$, $a^k \in A$, and $T^k \equiv t(\varphi(R^k), R^k)$ such that

$$\left\{\begin{array}{l}\text{(i) } e(T^k_{i^k}, a^k) < e(T^k_{j^k}, a^k),\\ \text{(ii) } \varphi_{j^k a^k}(R^k) > 0,\\ \text{(iii) } T^k|_{e(T^k_{i^k}, a^k)} = tS(R^k)|_{e(T^k_{i^k}, a^k)},\\ \text{(iv) } e(T^k_{i^k}, a^k) < e(T^{k-1}_{i^{k-1}}, a^{k-1}),\\ \text{(v) } T^k|_{e(T^k_{i^k}, a^k)} = T^{k-1}|_{e(T^k_{i^k}, a^k)},\end{array}\right\}$$

and let $\hat{R}^k_{i^k} \in \mathcal{R}$ be an upward transformation of $R^k_{i^k}$ at a^k with respect to $R^k_{j^k}$.

$$\vdots$$

Since this process runs infinitely and $|A| < \infty$, there are two distinct steps $l, m \in \mathbb{N}$ (without loss of generality, let $l < m$) such that $a^l = a^m (\equiv a)$. Note that $e(T^m_{i^m}, a) < e(T^l_{i^l}, a)$ and $T^l|_{e(T^m_{i^m}, a)} = T^m|_{e(T^m_{i^m}, a)}$. Note that, under T^l, agent l is an agent who does not do his best at the earliest time (specifically, at $e(T^l_{i^l}, a)$). Thus, at Step l, agent i^m does not do his best at $e(T^m_{i^m})$ earlier than agent i^l does. That is, i^l is not an agent who does not do his best at the earliest time at step l, a contradiction.



$\square$

Remark: The independence of the axioms listed in Theorem 2 can be established as follows: the equal division rule satisfies all three except for *sd-efficiency*. Each sequential dictatorial rule satisfies all three axioms except for *sd no-envy*. The following rule, ψ, satisfies all three axioms except for *bounded invariance*.

Example 5. A rule satisfying all but bounded invariance.

Let $A \equiv \{a, b, c, d\}$, $N \equiv \{1, 2, 3, 4\}$, $\bar{R} \in \mathcal{R}^N$ and $\bar{P} \in \mathcal{P}$ be such that

$\bar{R}_1$	$\bar{R}_2$	$\bar{R}_3$	$\bar{R}_4$
a	a	a	c
b	c	c	b
c	d	d	a
d	b	b	d

$$\bar{P} = \left(\begin{array}{l|cccc} & a & b & c & d \\ \hline \text{agent 1:} & \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{6} \\ \text{agent 2:} & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \text{agent 3:} & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \text{agent 4:} & 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \end{array} \right)$$

Let ψ be defined as $\psi(R) = \bar{P}$ if $R = \bar{R}$, and $\psi(R) = S(R)$, otherwise. If agent 4 changes his preference $\bar{R}_4$ to $\bar{R}'_4$ such that $c\ \bar{R}'_4\ a\ \bar{R}'_4\ b\ \bar{R}'_4\ d$, then $\psi_{4c}(\bar{R}'_4, \bar{R}_{-4}) \neq \psi_{4c}(\bar{R})$, a violation of *bounded invariance*. $\square$

4. Concluding Remarks

We conclude by discussing two possible generalizations of the model. The first is to introduce (finitely) multiple copies of each object. By adapting the feasibility condition of assignment matrices in a way that the sum of the probabilities assigned for each object is the number of its copies, our main result still holds. The second is to accommodate the possibility that each agent receives more than one object. If agents receive the same number of objects, then all of the axioms listed in Theorem 1 are still meaningful. Otherwise, it becomes difficult to compare agents' assignments directly, since the sum of probabilities assigned to each agent can differ from that of each other agent. Thus, *sd no-envy* has to be adapted. Heo (2010) introduces the notion of *normalized sd no-envy*: each agent first normalizes his assignment by the number of objects that he receives, and then compares his normalized assignment to that of each other agent. The serial rule can also be adapted to handle the generalization: the "generalized serial rule" is defined by the algorithm along which each agent consumes probabilities at a rate proportional to the number of objects that he receives. We are able to adapt Theorem 1, thereby obtaining a characterization of this rule by means of *sd-efficiency*, *normalized sd no-envy*, and *bounded invariance.*

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