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On Sectoral Supply Functions, Leisure & Uniqueness in Life-Cycle Productive Economies

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This contribution introduces a benchmark general approach of equilibrium dynamics in the context of a simple model of overlapping generations with heterogeneous goods that is based upon the properties of a pair of sectoral supply functions. The class of preferences that is considered hinges upon consumption and endogenous leisure motives and encapsulates earlier characterisations as special cases. The existence and stability properties of wealth-capital equilibria and monetary equilibria are successively considered, the elasticities of substitution between the two inputs being emphasised to play a key-role for that purpose together with the curvatures of the utilities of the three arguments of the utility functions. This provides a canonical understanding of a benchmark environments, the current uniqueness and determinacy results being potentially of great use for a wide area of future applications.

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I – Introduction

This contribution introduces an approach of equilibrium dynamics in the context of the model of overlapping generations based upon the properties of a pair of sectoral supply functions. The class of preferences that is considered hinges upon an endogenous labour supply and an elementary savings behaviour that comes as an alternative to the Diamond tradition retained by Galor in his benchmark contribution. The stability properties of wealth-capital equilibria and golden rule equilibria are successively considered, the elasticities of substitution between the two inputs being emphasised to play a key-role for that purpose. A generic appraisal is followed by geometric appraisals for two and three-dimmensional dynamical systems. An eventual attention is devoted to the intertemporal efficiency of the equilibria under consideration.
The main ambition of the current contribution is to progress towards an understanding of the essence of phenomena which may result in endogenous fluctuations within competitive economies. It is more precisely interested in the articulation between the elasticity properties of preferences as described by the offer curve and the ones of a basic technological set as described by a production possibility frontier with two inputs and two outputs. It is also intended to precisely characterise the nature of earlier phenomena in terms of elasticities of substitution.
Dating back from Grandmont [9], numerous contributions have been intended at circumscribing the role of an explicitly formulated technological facet in the possible occurrence of these phenomena. Reichlin [12, 13] emphasised the role of weak arbitrages in the production set when the latter builds from one or two-sector environments with fixed coefficients technologies. These conclusions have been generalised to positive elasticities of substitution between the productive factors for barter and monetary equilibria through the contribution of Cazzavillan & Pintus [5] and Benhabib & Laroque [1]. Parallely and relying on a somewhat distinct perspective, Galor [8] and Drugeon [7] have been aimed at a generalised perspective with two goods and positive elasticities of substitution between the productive factors. More specifically, the latter contributions introduce an articulation between this literature and the multisector insights of Benhabib & Nishimura [4, 5] in the optimal growth literature. It further provides a detailed decentralised account of the restrictions on sectoral elasticities of substitution which underlie the scope for endogenous fluctuations.
Section II introduces the basic framework. Section III characterises wealth-capital equilibria. Section IV analyses golden rule equilibria.

II – The Model

II.1 – The Technology

Time is discrete. There are two sectors $j = 0, 1$ in the economy. The first produces a pure consumption good in amount Y^0 whereas the second produces a pure capital good in amount Y^1. Any of the sectors uses labour and capital as inputs and the outputs of the consumption and investment good sectors respectively satisfy :

$$(1) \quad Y_t^j \leq F^j(X_{0j,t}, X_{1j,t}),$$

where $X_{ij,t}$ denotes the amount of input $i, i = 0, 1$, employed in sector $j, j = 0, 1$ at date $t = 0, 1, \ldots$. Letting $X_{0,t}$ and $X_{1,t}$ respectively denote the amount of available labour units and the aggregate capital stock at date $t = 0, 1, \ldots$, both inputs are freely shiftable at any date between the two sectors :

$$(2) \quad \sum_{j=0}^{1} X_{ij,t} \leq X_{i,t}, \quad i = 0, 1,$$

whereas the value of next period capital stock is subject to :

$$(3) \quad X_{1,t+1} \leq Y_t^1 + (1-\eta) X_{1,t},$$

for $\eta \in]0, 1]$ the depreciation rate of the capital stock.
The properties of the production technologies are restricted to the following list of assumptions:

Assumption T.1 : $F^j(\cdot, \cdot)$, $j = 0, 1$, is homogeneous of degree one, concave and continuous over $\mathbb{R}_+ \times \mathbb{R}_+$.

Assumption T.2 : $\forall j \in \{0, 1\}$, $\forall X_{1j} \in \mathbb{R}_+$, $F^j(0, X_{1j}) = 0$.

Assumption T.3 : There exists a level $\bar{X}_1 \in \mathbb{R}_+^*$ such that, for any $X_0 \in \mathbb{R}_+^*$, $X_1 \in \mathbb{R}_+^*$, $F^1(X_0, X_1) > \eta X_1$ for any $X_1 < \bar{X}_1$ and $F^1(X_0, X_1) < \eta X_1$ for any $X_1 > \bar{X}_1$.

Assumption T.4 : $\forall j \in \{0, 1\}$, $F^j(\cdot, \cdot)$ is of class C^3 over $\mathbb{R}_+^* \times \mathbb{R}_+^*$.

Assumption T.5 : $\forall j \in \{0, 1\}$, $\forall (X_{0j}, X_{1j}) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, $\partial F^j(X_{0j}, X_{1j})/\partial X_{0j} > 0$, $\partial F^j(X_{0j}, X_{1j})/\partial X_{1j} > 0$, $\partial^2 F^j(X_{0j}, X_{1j})/\partial (X_{0j})^2 < 0$, $\partial^2 F^j(X_{0j}, X_{1j})/\partial (X_{1j})^2 < 0$.

Firms in industry $j = 0, 1$ take $\{\omega_t^0, \omega_t^1, p_t^j\}$ as given, for ω_t^0, ω_t^1 and p_t^j respectively the labour wage rate, the gross capital rental and the price of good j at $t = 0, 1, \ldots$ They select $\{X_{0j,t}, X_{1j,t}\}$, $j = 0, 1$ in order to maximise their profits :

$$(4) \quad \underset{\{X_{0j,t}, X_{1j,t}\}}{\text{Maximise}} \; p_t^j Y_t^j - \omega_t^0 X_{0j,t} - \omega_t^1 X_{1j,t} \text{ s.t. } Y_t^j \leq F^j(X_{0j,t}, X_{1j,t}), X_{0j,t} \geq 0, X_{1j,t} \geq 0,$$

Capital and labour being freely shiftable from one industry to the other, they move so as to equalise their rental rates and an interior competitive equilibrium of the productive sector is available from:

$$(5a) \quad \omega_t^i / p_t^j = \frac{\partial F^j}{\partial X_{ij}}(1, X_{1j,t}/X_{0j,t}),$$

$$(5b) \quad Y_t^j = (X_{0j,t}/X_{0,t}) X_{0,t} F^j(1, X_{1j,t}/X_{0j,t}),$$

$$(5c) \quad X_{1,t}/X_{0,t} = \sum_{j=0}^{1} (X_{0j,t}/X_{0,t})(X_{1j,t}/X_{0j,t}),$$

$$(5d) \quad 1 = \sum_{j=0}^{1} (X_{ij,t}/X_{i,t}), i = 0, 1.$$

II.2 – A Substitutability-Based «Sectoral Supply Functions» Approach

The consideration of equations (5a) allows for introducing $(\omega^0/\omega^1)^j(X_{1j,t}/X_{0j,t}) \triangleq [\partial F^j(1, X_{1j,t}/X_{0j,t})/\partial X_{0j}]/[\partial F^j(1, X_{1j,t}/X_{0j,t})/\partial X_{1j}]$. From the concavity assumption T.5, these functions are monotonic, they are invertible and in turn open road for expressing the two sectoral input ratios $X_{1j,t}/X_{0j,t}$ as a pair of functions $(X_{1j}/X_{0j})(\cdot)$ of the ratio between the rental rate of the two inputs ω_t^0/ω_t^1 that further satisfy $\partial[(X_{1j}/X_{0j})(\omega_t^0/\omega_t^1)] \ / \ \partial(\omega_t^0/\omega_t^1) > 0$ for $\omega_t^0/\omega_t^1 > 0$. The argument shall then further be simplified by imposing an extra «no factors-intensity reversal» assumption between the two industries :

Assumption T.6 : $\forall (\omega_t^0/\omega_t^1) > 0$, either $(X_{11}/X_{01})(\omega_t^0/\omega_t^1) > (X_{10}/X_{00})(\omega_t^0/\omega_t^1)$ or $(X_{10}/X_{00})(\omega_t^0/\omega_t^1) > (X_{11}/X_{01})(\omega_t^0/\omega_t^1)$.

Under Assumption T.2 and from (5b), $Y_t^j > 0 \iff X_{0j,t}/X_0 > 0$, $j = 0, 1$, namely for ω_t^0/ω_t^1 that satisfies $(X_{11}/X_{01})(\omega_t^0/\omega_t^1) < X_{1,t}/X_{0,t} < (X_{10}/X_{00})(\omega_t^0/\omega_t^1)$ or $(X_{10}/X_{00})(\omega_t^0/\omega_t^1) < X_{1,t}/X_{0,t} < (X_{11}/X_{01})(\omega_t^0/\omega_t^1)$, *i.e.*, for $\omega_t^0/\omega_t^1 \in \,]\underline{(\omega^0/\omega^1)}(X_{1,t}/X_{0,t}), \overline{(\omega^0/\omega^1)}(X_{1,t}/X_{0,t})[$, the two loci $\underline{(\omega^0/\omega^1)}(\cdot)$ and $\overline{(\omega^0/\omega^1)}(\cdot)$ being defined from :

$$\underline{(\omega^0/\omega^1)}(X_{1,t}/X_{0,t}) = (\omega^0/\omega^1)^0(X_{1,t}/X_{0,t})\Big((\omega^0/\omega^1)^1(X_{1,t}/X_{0,t})\Big),$$
$$\overline{(\omega^0/\omega^1)}(X_{1,t}/X_{0,t}) = (\omega^0/\omega^1)^1(X_{1,t}/X_{0,t})\Big((\omega^0/\omega^1)^0(X_{1,t}/X_{0,t})\Big),$$
$$\text{for} \quad (X_{11}/X_{01})(\omega_t^0/\omega_t^1) > (<)(X_{10}/X_{00})(\omega_t^0/\omega_t^1).$$

Then, within the above interval, considering a given rate — be it ω_t^0/p_t^0 or ω_t^1/p_t^0 – in (5a) between the two industries, under Assumption T.6, p_t^1/p_t^0 in its turn emerges as a monotonic function $(p^1/p^0)(\cdot)$ of any $\omega_t^0/\omega_t^1 \in \,]\underline{(\omega^0/\omega^1)}(X_{1,t}/X_{0,t}), \overline{(\omega^0/\omega^1)}(X_{1,t}/X_{0,t})[$, namely $\partial(p^1/p^0)(\omega_t^0/\omega_t^1) \ / \ \partial(\omega_t^0/\omega_t^1) < (>)0$ for $(X_{11}/X_{01})(\omega_t^0/\omega_t^1) > (<)(X_{10}/X_{00})(\omega_t^0/\omega_t^1)$. Letting $(p^1/p^0)(X_{1,t}/X_{0,t}) \triangleq (p^1/p^0)[(\omega^0/\omega^1)(X_{1,t}/X_{0,t})]$, it is finally obtained that no specialisation occurs between the two industries if and only if $p_t^1/p_t^0 \in \,]\underline{(p^1/p^0)}(X_{1,t}/X_{0,t}), \overline{(p^1/p^0)}(X_{1,t}/X_{0,t})[$, where

$$\underline{(p^1/p^0)}(X_{1,t}/X_{0,t}) = (p^1/p^0)[\overline{(\omega^0/\omega^1)}(X_{1,t}/X_{0,t})]\Big((p^1/p^0)[\underline{(\omega^0/\omega^1)}(X_{1,t}/X_{0,t})]\Big),$$
$$\overline{(p^1/p^0)}(X_{1,t}/X_{0,t}) = (p^1/p^0)[\underline{(\omega^0/\omega^1)}(X_{1,t}/X_{0,t})]\Big((p^1/p^0)[\overline{(\omega^0/\omega^1)}(X_{1,t}/X_{0,t})]\Big),$$
$$\text{for} \quad (X_{11}/X_{01})(\omega_t^0/\omega_t^1) > (<)(X_{10}/X_{00})(\omega_t^0/\omega_t^1).$$

Within this interval and from (5c)–(5d), $X_{0j,t}/X_{0,t}$ emerges as

$$(X_{0j}/X_0)[(\omega^0/\omega^1)(p_t^1/p_t^0), X_{1,t}/X_{0,t}] \triangleq \frac{X_{1,t}/X_{0,t} - (X_{1j'}/X_{0j'})[(\omega^1/\omega^0)(p_t^1/p_t^0)]}{(X_{1j}/X_{0j})[(\omega^1/\omega^0)(p_t j^1/p_t^0)], -(X_{1j'}/X_{0j'})[(\omega^1/\omega^0)(p_t^1/p_t^0)]}, j' \neq j.$$

From (5c), this eventually equips the analysis with a pair of structures that shall henceforward be labelled «sectoral supply functions» (SSF) :

(6) $\mathcal{F}^j(X_{0,t}, X_{1,t}, p_t^1/p_t^0)$

$$\triangleq (X_{0j}/X_0)[(\omega^0/\omega^1)(p_t^1/p_t^0), X_{1,t}/X_{0,t}]X_{0,t}F^j\Big(1, (X_{1j}/X_{0j})[(\omega^1/\omega^0)(p_t^1/p_t^0)]\Big),$$

for $j = 0, 1$. From Drugeon [7], any of these SSF $\mathcal{F}^j(\cdot,\cdot;\cdot)$, $j = 0, 1$, is homogeneous of degree one with respect to $X_{0,t}$ and $X_{1,t}$ and satisfies $\mathcal{F}^j\left(X_{0,t}, X_{1,t}, p^1_t/p^0_t\right) > 0 \iff p^1_t/p^0_t \in \left]\underline{(p^1/p^0)}\left(X_{1,t}/X_{0,t}\right), \overline{(p^1/p^0)}\left(X_{1,t}/X_{0,t}\right)\right[$. They relate to competitive prices according to :

$$\omega^i_t/p^0_t = \sum_{j=0}^{1}\left(p^j_t/p^0_t\right)\frac{\partial \mathcal{F}^j}{\partial X_i}\left(X_{0,t}, X_{1,t}, p^1_t/p^0_t\right),$$

$$\sum_{j=0}^{1}\left(p^j_t/p^0_t\right)\frac{\partial \mathcal{F}^1}{\partial\left(p^1/p^0\right)}\left(X_{0,t}, X_{1,t}, p^1_t/p^0_t\right) = 0.$$

Further letting $\pi^j_{X_i} \triangleq \omega^i X_{ij}/\sum_{i=0}^{1}\omega^i X_{ij}$, $i = 0, 1$ feature the shares of total cost devoted to the two inputs in sector j and denoting the direct elasticity of substitution between the two inputs as $\Sigma^j_{X_0X_1} \triangleq d\ln\left(X_{1j}/X_{0j}\right) \;/\; d\ln\left(\omega^1/\omega^0\right)$. Finally letting the respective aggregate shares, *e.g.*, of consumption and profits in national income, be introduced as $\pi_{Y^0} \triangleq p^0Y^0/\left(p^0Y^0 + p^1Y^1\right)$ and $\omega^1X_1/\left(\omega^0X_0 + \omega^1X_1\right) = \sum_{j=0}^{1}\pi_{Y^j}\pi^j_{X_1} \;/\; \left(\sum_{j=0}^{1}\pi_{Y^j}\pi^j_{X_0} + \sum_{j=0}^{1}\pi_{Y^j}\pi^j_{X_1}\right)$, the derivatives of the SSF reltate to the aggregate values of the inputs according to :

$$\partial\mathcal{F}^0/\partial X_0 = \left[\pi^1_{X_1}/\left(\pi^1_{X_1} - \pi^0_{X_1}\right)\right]\left(\partial F^0/\partial X_{00}\right),$$
$$\partial\mathcal{F}^0/\partial X_1 = -\left[\left(1 - \pi^1_{X_1}\right)/\left(\pi^1_{X_1} - \pi^0_{X_1}\right)\right]\left(\partial F^0/\partial X_{10}\right),$$
$$\partial\mathcal{F}^1/\partial X_0 = -\left[\pi^0_{X_1}/\left(\pi^1_{X_1} - \pi^0_{X_1}\right)\right]\left(\partial F^1/\partial X_{01}\right),$$
$$\partial\mathcal{F}^1/\partial X_1 = \left[\left(1 - \pi^0_{X_1}\right)/\left(\pi^1_{X_1} - \pi^0_{X_1}\right)\right]\left(\partial F^1/\partial X_{11}\right).$$

The SSF also relate to the relative price of the investment good according to :

$$\partial\mathcal{F}^0/\partial\left(p^1/p^0\right) = -\sum_{j=0}^{1}\pi^j_{X_0}\pi^j_{X_1}\pi_{Y^j}\Sigma^j_{X_0X_1}\Big/\left(\pi^1_{X_1} - \pi^0_{X_1}\right)^2\left(p^1/p^0\right);$$

$$\partial\mathcal{F}^1/\partial\left(p^1/p^0\right) = \sum_{j=0}^{1}\pi^j_{X_0}\pi^j_{X_1}\pi_{Y^j}\Sigma^j_{X_0X_1}\Big/\left(\pi^1_{X_1} - \pi^0_{X_1}\right)^2\left(p^1/p^0\right)^2.$$

It is also noticed that the previous sectoral profits maximisation (4) can then proficiently be supplemented by seeking the triples $\{X_{0,t}, X_{1,t}, p^1_t/p^0_t\}$ that maximise at each date $t \geq 0$ $p^0_t\mathcal{F}^0\left(X_{0,t}, X_{1,t}, p^1_t/p^0_t\right) + p^1_t\mathcal{F}^0\left(X_{0,t}, X_{1,t}, p^1_t/p^0_t\right) - \omega^0_tX_{0,t} - \omega^1_tX_{1,t}$ subject to $X_{0,t} \geq 0$, $X_{1,t} \geq 0$, $p^1_t/p^0_t \in \left]\underline{(p^1/p^0)}\left(X_{1,t}/X_{0,t}\right), \overline{(p^1/p^0)}\left(X_{1,t}/X_{0,t}\right)\right[$, the alternative representation of prices being then immediate. An alternative representation of the competitive equilibrium of the productive sphere (6) is then available from

$$\text{(7a)}\quad \omega^i_t/p^0_t = \sum_{j=0}^{1}\left(p^j_t/p^0_t\right)\frac{\partial\mathcal{F}^j}{\partial X_i}\left(X_{0,t}, X_{1,t}, p^1_t/p^0_t\right),$$

$$\text{(7b)}\quad Y^j_t = \mathcal{F}^j\left(X_{0,t}, X_{1,t}, p^1_t/p^0_t\right), j = 0, 1.$$

Considering the aggregate elasticity of substitution between the two factors costs ω^0 and ω^1 respectively associated with the inputs X_0 and X_1 and for given levels of the outputs. It is given by $\sum_{j=0}^{1}\pi^j_{X_0}\pi^j_{X_1}\pi_{Y^j}\Sigma^j_{X_0X_j} \;/\; \sum_{j=0}^{1}\pi_{Y^j}\pi^j_{X_0}\sum_{j=0}^{1}\pi_{Y^j}\pi^j_{X_1}$. Similarly, considering the aggregate elasticity of substitution between the two outputs Y^0 and Y^1 respectively associated to the prices p^0 and p^1, namely $\Sigma_{Y^0Y^1}$, for given aggregate values of the inputs. It derives as $\sum_{j=0}^{1}\pi_{X_{0j}}\pi_{X_{1j}}\pi_{Y^j}\Sigma_{X_{0j}X_{1j}}/\pi_{Y^0}\pi_{Y^1}\left(\pi^1_{X_1} - \pi^0_{X_1}\right)^2$.

III – An Argument with Consumption–Leisure Arbitrages

III.1 – A Generalised Class of Preferences

The preferences of an agent of generation t are now featured by an intertemporal utility function $u(c_t^t, \ell_t^t, c_{t+1}^t)$, the properties of which list, omitting arguments for concision, as :

Assumption P.1 : $u \in C^k(\mathbb{R}_+^* \times]0, \bar{l}\,[\times \mathbb{R}_+^*, \mathbb{R})$, $k \geq 4$ and is strictly quasi-concave over $\mathbb{R}_+^*$.

Assumption P.2 : $\forall (c_t^t, \ell_t^t, c_{t+1}^t) \in \mathbb{R}_+^* \times]0, \bar{\ell}[\times \mathbb{R}_+^*$, $\nabla u(c_t^t, \ell_t^t, c_{t+1}^t) \gg 0$.

Assumption P.3 : $\forall (c_t^t, \ell_t^t, c_{t+1}^t) \in \mathbb{R}_+^* \times]0, 1[\times \mathbb{R}_+^*$, $\lim_{c_t^t \to 0} \dfrac{\partial u}{\partial c_t^t}(c_t^t, \ell_t^t, c_{t+1}^t) = \lim_{\ell_t^t \to 0} \dfrac{\partial u}{\partial \ell_t^t}(c_t^t, \ell_t^t, c_{t+1}^t) = \lim_{c_{t+1}^t \to 0} \dfrac{\partial u}{\partial c_{t+1}^t}(c_t^t, \ell_t^t, c_{t+1}^t) = \infty$.

For future references, normality assumptions on the three aguments of the felicity function are achieved through the retainment of the following restrictions :

Assumption P.4 : $\forall (c_t^t, \ell_t^t, c_{t+1}^t) \in \mathbb{R}_+^* \times]0, \bar{\ell}[\times \mathbb{R}_+^*$,

$$\frac{\partial u}{\partial c_t^t}\left[\frac{\partial^2 u}{\partial (\ell_t^t)^2}\frac{\partial^2 u}{\partial (c_{t+1}^t)^2} - \frac{\partial^2 u}{\partial \ell_t^t \partial (c_{t+1}^t)}\frac{\partial^2 u}{\partial (c_{t+1}^t)\partial \ell_t^t}\right] + \frac{\partial u}{\partial \ell_t^t}\left[\frac{\partial^2 u}{\partial \ell_t^t \partial (c_{t+1}^t)}\frac{\partial^2 u}{\partial (c_{t+1}^t)\partial c_t^t} - \frac{\partial^2 u}{\partial c_t^t \partial \ell_t^t}\frac{\partial^2 u}{\partial (c_{t+1}^t)^2}\right] + \frac{\partial u}{\partial (c_{t+1}^t)}\left[\frac{\partial^2 u}{\partial c_t^t \partial \ell_t^t}\frac{\partial^2 u}{\partial \ell_t^t \partial (c_{t+1}^t)} - \frac{\partial^2 u}{\partial \ell_t^t \partial \ell_t^t}\frac{\partial^2 u}{\partial (c_{t+1}^t)\partial c_t^t}\right] > 0 \text{ for any } c_t^t > 0,\ \ell_t^t > 0,\ (c_{t+1}^t) > 0.$$

Assumption P.5 : $\forall (c_t^t, \ell_t^t, c_{t+1}^t) \in \mathbb{R}_+^* \times]0, \bar{\ell}[\times \mathbb{R}_+^*$,

$$\frac{\partial u}{\partial c_t^t}\left[\frac{\partial^2 u}{\partial c_t^t \partial \ell_t^t}\frac{\partial^2 u}{\partial (c_{t+1}^t)^2} - \frac{\partial^2 u}{\partial c_t^t \partial (c_{t+1}^t)}\frac{\partial^2 u}{\partial \ell_t^t \partial (c_{t+1}^t)}\right] + \frac{\partial u}{\partial \ell_t^t}\left[\frac{\partial^2 u}{\partial c_t^t \partial (c_{t+1}^t)}\frac{\partial^2 u}{\partial (c_{t+1}^t)\partial c_t^t} - \frac{\partial^2 u}{\partial (c_t^t)^2}\frac{\partial^2 u}{\partial (c_{t+1}^t)^2}\right] + \frac{\partial u}{\partial (c_{t+1}^t)}\left[\frac{\partial^2 u}{\partial (c_t^t)^2}\frac{\partial^2 u}{\partial \ell_t^t \partial (c_{t+1}^t)} - \frac{\partial^2 u}{\partial \ell_t^t \partial c_t^t}\frac{\partial^2 u}{\partial c_t^t \partial (c_{t+1}^t)}\right] > 0.$$

Assumption P.6 : $\forall (c_t^t, \ell_t^t, c_{t+1}^t) \in \mathbb{R}_+^* \times]0, 1[\times \mathbb{R}_+^*$,

$$\frac{\partial u}{\partial c_t^t}\left[\frac{\partial^2 u}{\partial \ell_t^t \partial c_t^t}\frac{\partial^2 u}{\partial (c_{t+1}^t)\partial \ell_t^t} - \frac{\partial^2 u}{\partial (c_{t+1}^t)\partial c_t^t}\frac{\partial^2 u}{\partial (\ell_t^t)^2}\right] + \frac{\partial u}{\partial \ell_t^t}\left[\frac{\partial^2 u}{\partial c_t^t \partial \ell_t^t}\frac{\partial^2 u}{\partial (c_{t+1}^t)\partial c_t^t} - \frac{\partial^2 u}{\partial (c_{t+1}^t)\partial \ell_t^t}\frac{\partial^2 u}{\partial (c_t^t)^2}\right] + \frac{\partial u}{\partial (c_{t+1}^t)}\left[\frac{\partial^2 u}{\partial (c_t^t)^2}\frac{\partial^2 u}{\partial (\ell_t^t)^2} - \frac{\partial^2 u}{\partial c_t^t \partial \ell_t^t}\frac{\partial^2 u}{\partial \ell_t^t \partial c_t^t}\right] > 0.$$

A representative agent of generation $t \geq 1$ thus selects his consumption span and his first-period leisure choice in order to solve :

$$\underset{\{c_t^t, \ell_t^t, c_{t+1}^t\}}{\text{Maximise}}\ u(c_t^t, \ell_t^t, c_{t+1}^t) \ \text{ subject to } p_t^0 c_t^t + p_{t+1}^0 c_{t+1}^t / \mathcal{R}_{t+1} \leq \omega_t^0(\bar{-}\ell_t^t),$$

First-order interior solutions list as :

$$\frac{\partial u}{\partial c_t^t}(c_t^t, \ell_t^t, c_{t+1}^t)(p_{t+1}^0/p_t^0) - \mathcal{R}_{t+1}\frac{\partial u}{\partial c_t^t}(c_t^t, \ell_t^t, c_{t+1}^t) = 0,$$

$$\frac{\partial u}{\partial \ell_t^t}(c_t^t, \ell_t^t, c_{t+1}^t)(p_{t+1}^0/p_t^0) - (\omega_t^0/p_t^0)\frac{\partial u}{\partial c_t^t}(c_t^t, \ell_t^t, c_{t+1}^t) = 0,$$

$$c_t^t + (p_{t+1}^0/p_t^0)c_{t+1}^t/\mathcal{R}_{t+1} - (\omega_t^0/p_t^0)X_0 = 0.$$

Letting henceforward $c_t^t = \mathcal{C}[\omega_t^o/p_t^o, \mathcal{R}_{t+1}/(p_{t+1}^o/p_t^o)]$, $c_{t+1}^t = \mathcal{C}'[\omega_t^o/p_t^o, \mathcal{R}_{t+1}/(p_{t+1}^o/p_t^o)]$ and $\ell_t^t = \mathcal{L}[\omega_t^o/p_t^o, \mathcal{R}_{t+1}/(p_{t+1}^o/p_t^o)]$ denote the solution to this system, an extra restriction is imposed on both of its components:

Assumption P.7 : $\forall\Big(\omega_t^o/p_t^o, \mathcal{R}_{t+1}/(p_{t+1}^o/p_t^o)\Big) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, $\dfrac{\partial \mathcal{C}}{\partial \mathcal{R}_{t+1}/(p_{t+1}^o/p_t^o)}[\omega_t^o/p_t^o, \mathcal{R}_{t+1}/(p_{t+1}^o/p_t^o)] \neq 0$, $\dfrac{\partial \mathcal{C}'}{\partial \mathcal{R}_{t+1}/(p_{t+1}^o/p_t^o)}[\omega_t^o/p_t^o, \mathcal{R}_{t+1}/(p_{t+1}^o/p_t^o)] \neq 0$, $\dfrac{\partial \mathcal{L}}{\partial \mathcal{R}_{t+1}/(p_{t+1}^o/p_t^o)}[\omega_t^o/p_t^o, \mathcal{R}_{t+1}/(p_{t+1}^o/p_t^o)] \neq 0$.

He is equivalently to solve

$$\underset{\{c_t^t, c_{t+1}^t\}}{\text{Minimise}} \;\; p_t^o c_t^t + \omega_t^o \ell_t^t + p_{t+1}^o c_{t+1}^t/\mathcal{R}_{t+1} - \omega_t^o \bar{\ell} \;\; \text{subject to } u(c_t^t, \ell_t^t, c_t^{t+1}) \geq \bar{u},\; \bar{u} \;\text{ given},$$

that in turns allows for introducing, letting μ denote the shadow price associated to the constraint, the consumption expense function of the consumer as $E(p_t^o, \omega_t^o, p_{t+1}^o, \bar{u}) := p_t^o c_t^{t,H}(p_t^o, \omega_t^o, p_{t+1}^o, \bar{u}) + \omega_t^o \ell_t^{t,H}(p_t^o, \omega_t^o, p_{t+1}^o, \bar{u}) + p_{t+1}^o c_{t+1}^{t,H}(p_t^o, p_{t+1}^o, \bar{u})/\mathcal{R}_{t+1}$, where $c_t^{t,H}(p_t^o, \omega_t^o, p_{t+1}^o, \bar{u})$, $\ell_t^{t,H}(p_t^o, \omega_t^o, p_{t+1}^o, \bar{u})$ and $c_{t+1}^{t,H}(p_t^o, \omega_t^o, p_{t+1}^o, \bar{u})$ denote the dual Hicksian demand functions Introducing then $\pi_c := p_t^o c_t^t/(p_t^o c_t^t + \omega_t^o \ell_t^t + p_{t+1}^o c_{t+1}^t/\mathcal{R}_{t+1})$, $\pi_\ell := \omega_t^o \ell_t^t/(p_t^o c_t^t + \omega_t^o \ell_t^t + p_{t+1}^o c_{t+1}^t/\mathcal{R}_{t+1})$, $\pi_{c'} := (p_{t+1}^o c_{t+1}^t/\mathcal{R}_{t+1}) / (p_t^o c_t^t + \omega_t^o \ell_t^t + p_{t+1}^o c_{t+1}^t/\mathcal{R}_{t+1})$ as the shares of total consumption devoted to first and second period consumptions and letting $\mathcal{S}$ feature the scale elasticity of $u(\cdot,\cdot,\cdot)$, it is noticeable that the share accruing to first-period consumption c_t^t and the scale elaticity $\mathcal{S}$ relate according to

$$\pi_c = \frac{\partial u}{\partial c_t^t} c_t^t \Big/ \mathcal{S}u, \qquad \mathcal{S} = \left(\frac{\partial u}{\partial c_t^t} c_t^t + \frac{\partial u}{\partial c_{t+1}^t} c_{t+1}^t + \frac{\partial u}{\partial \ell_t^t} \ell_t^t\right) \Big/ u$$

and indexes of the following sort describe the second-order elasticities of the expense function:

$$\Sigma_{cc'}^H := E \frac{\partial^2 E}{\partial p^o \partial p^{o\prime}} \Big/ \frac{\partial E}{\partial p^o}\frac{\partial E}{\partial p^{o\prime}}, \qquad \Sigma_{cY}^H := E \frac{\partial^2 E}{\partial p^o \partial u} \Big/ \frac{\partial E}{\partial p^o}\frac{\partial E}{\partial u},$$

$$\Sigma_{YY}^H := E \frac{\partial^2 E}{\partial (u)^2} \Big/ \frac{\partial E}{\partial u}\frac{\partial E}{\partial u},$$

where it is, *e.g.* recalled that any coefficient $\Sigma_{cc'}^H$, refers to a dual Hicksian elasticity of substitution whilst Σ_{cY}^H, features the income elasticity of first-period consumption compensated demands. Goods c and c' are depicted as *substitutes* when $\Sigma_{cc'}^H$ undergoes positive values — a rise in the price of good c would then translate into an increase in the demand for good c' — and as *complements* when $\Sigma_{cc'}^H$ happens to undergo negative values. Finally, Σ_{YY}^H relates to the scale elasticity of the felicity function : for a felicity function homogenous of degree $\mathcal{S}$, it would indeed be available as $(1-\mathcal{S})/\mathcal{S}$.[1]

Any of the components of the Hessian elasticities matrix of $u(\cdot,\cdot,\cdot)$, *e.g.*,

$$\Xi_{cc} := u \frac{\partial^2 u}{\partial (c_t^t)^2} \Big/ \frac{\partial u}{\partial c_t^t}\frac{\partial u}{\partial c_t^t}$$

[1]While the concavity of the expense function with respect to prices implies that Σ_{cc}^H, $\Sigma_{cc'}^H$ and $\Sigma_{\ell\ell}^H$ are negative, in a three goods world, an increment in the price of a given good needs not to entail an increase in the consumption of the outstanding good.

assumes a representation grounded upon these ordinal coefficients. More explicitly, letting and $\mathcal{D} = \Sigma^H_{cc'}\Sigma^H_{c\ell}\pi_c + \Sigma^H_{cc'}\Sigma^H_{c'\ell}\pi_{c'} + \Sigma^H_{c\ell}\Sigma^H_{c'\ell}\pi_\ell > 0$, the earlier formal assumptions will assume a much more satisfactory understanding through these coefficients.

Lemma 1 [A Parametrical Understanding of the Restrictions on the Utility Function]. *Under Assumptions* P.1-7 *:*

(i) *for* $(\mathcal{A}_{c\ell})' = [\,\pi_{c'}\Sigma^H_{c'Y} \quad \pi_{c'}\Sigma^H_{c'Y} \quad -(1-\pi_{c'}\Sigma^H_{c'Y})\,]$, $(\mathcal{A}_{cc'})' = [\,\pi_\ell\Sigma^H_{\ell Y} \quad -(1- \quad \pi_\ell\Sigma^H_{\ell Y}) \quad \pi_\ell\Sigma^H_{\ell Y})\,]$ *and* $(\mathcal{A}_{\ell c'})' = [\,-(1-\pi_c\Sigma^H_{cY}) \quad \pi_c\Sigma^H_{cY} \quad \pi_c\Sigma^H_{cY}\,]$ *, the components of the Hessian elasticities matrix assume the following representation in terms of the coefficients of the expense function:*

$$\mathcal{SM}^E = -1\Sigma^H_{YY}1' + \mathcal{D}^{-1}\Big[\mathcal{A}_{c\ell}\Sigma_{c\ell}(\mathcal{A}_{c\ell})' + \mathcal{A}_{cc'}\Sigma_{cc'}(\mathcal{A}_{cc'})' + \mathcal{A}_{\ell c'}\Sigma_{\ell c'}(\mathcal{A}_{\ell c'})'\Big]$$

(ii) *the normality restrictions of Assumptions* P.4, P.5 *and* P.6 *are respectively equivalent to the holding of* $\Sigma^H_{cY} > 0$, $\Sigma^H_{\ell Y} > 0$ *and* $\Sigma^H_{c'Y} > 0$ *;*

(iii) *the concavity restrictions of Assumption* P.2 *are fit for* $\Sigma^H_{c\ell} \geq 0$, $\Sigma^H_{cc'} \geq 0$ *and* $\Sigma^H_{\ell c'} \geq 0$*, the negativeness of the third-order principal minor of the Hessian of the utility function implying* $\Sigma^H_{YY} \geq 0$ *;*

(iv) *the components of the vector of the Hicksian elasticities of substitution respectively satisfy:*

a/ $\Sigma^H_{c\ell} \gtrless 0 \Longleftrightarrow \Xi_{cc'} + \Xi_{c\ell} - \Xi_{c'c'} - \Xi_{c\ell} \gtrless 0$;

b/ $\Sigma^H_{cc'} \gtrless 0 \Longleftrightarrow \Xi_{\ell c'} + \Xi_{c\ell} - \Xi_{cc'} - \Xi_{\ell\ell} \gtrless 0$;

c/ $\Sigma^H_{\ell c'} \gtrless 0 \Longleftrightarrow \Xi_{cc'} + \Xi_{\ell c} - \Xi_{\ell c'} - \Xi_{cc} \gtrless 0$.

Proof : This builds from a forthright adapation of the proof of Lemma 1 in Drugeon [7] . △

III.2 – An Augmented Equilibrium Argument

Definition 1. Under Assumptions P.1-7, T.1-5, an intertemporal competitive equilibrium is a sequence $\Psi_t := \{X_{0,t}, X_{1,t}, Y^0_t, Y^1_t, \omega^0_t/p^0_t, \omega^1_t/p^0_t, p^1_t/p^0_t, B_t/p^0_t, c^{t-1}_t, \ell^t_t, c^t_t,\ p^1_t/p^0_t\}^{+\infty}_{t=-\infty}$, $\Psi_t \in \mathbb{R}^{12}_+$ such that :

(i) $c^t_t = \mathcal{C}[\omega^0_t/p^0_t, \mathcal{R}_{t+1}/(p^0_{t+1}/p^0_t)]$, $c^t_{t+1} = \mathcal{C}'[\omega^0_t/p^0_t, \mathcal{R}_{t+1}/(p^0_{t+1}/p^0_t)]$ and $\ell^t_t = \mathcal{L}[\omega^0_t/p^0_t, \mathcal{R}_{t+1}/(p^0_{t+1}/p^0_t)]$;

(ii) $\{X_{0,t}, X_{1,t}, p^1_t/p^0_t\}$ maximises $p^0_t\mathcal{F}^0(X_{0,t}, X_{1,t}, p^1_t/p^0_t) + p^1_t\mathcal{F}^1(X_{0,t}, X_{1,t}, p^1_t/p^0_t) - \omega^0_t X_{0,t} - \omega^1_t X_{1,t}$ at any $t \geq 0$;

(iii) $(p^1_t/p^0_t)X_{1,t+1} = (\omega^0_t/p^0_t)X_{0,t} - c^t_t + (B_t/p^0_t)M$;

(iv) $X_{1,t+1} = Y^1_t + (1-\eta)X_{1,t} = \mathcal{F}^1(X_{0,t}, X_{1,t}, p^1_t/p^0_t) + (1-\eta)X_{1,t}$;

(v) $X_{0,t} = \bar{\ell} - \ell^t_t$;

(vi) $c^t_t + c^{t-1}_t = Y^0_t = \mathcal{F}^1(X_{0,t}, X_{1,t}, p^1_t/p^0_t)$;

(vii) $\mathcal{R}_{t+1}/(p^0_{t+1}/p^0_t) = [(\omega^1_{t+1}/p^0_{t+1}) + (1-\eta)(p^1_{t+1}/p^0_{t+1})]/(p^1_t/p^0_t)$:

(viii) $m_t = M$.

To sum up, an intertemporal competitive equilibrium restates as :

$$X_{1,t+1} = Y^1_t + (1-\eta)X_{1,t};$$
$$(\omega^0_t/p^0_t)(\bar{\ell} - \ell^t_t) - c^t_t = (p^1_t/p^0_t)[Y^1_t + (1-\eta)X_{1,t}],$$
$$(B_{t+1}/p^0_{t+1}) - \mathcal{R}_{t+1}(B_t/p^0_t) = 0.$$

or, more explicitly:

$$
\begin{aligned}
&X_{1,t+1} - (1-\eta)X_{1,t} \\
&\quad - \mathcal{F}^1\big(\bar{\ell} - \mathcal{L}\big[(\omega^0/p^0)(p_t^1/p_t^0), \big[(\omega^1/p^0)(p_{t+1}^1/p_{t+1}^0) + (p_{t+1}^1/p_{t+1}^0)\big]/(p_t^1/p_t^0)\big], X_{1,t}, p_t^1/p_t^0\big) = 0; \\
&(\omega^0/p^0)(p_t^1/p_t^0)\big(1 - \mathcal{L}\big[(\omega^0/p^0)(p_t^1/p_t^0), \big[(\omega^1/p^0)(p_{t+1}^1/p_{t+1}^0) + (p_{t+1}^1/p_{t+1}^0)\big]/(p_t^1/p_t^0)\big]\big) \\
&\quad - \mathcal{C}\big[(\omega^0/p^0)(p_t^1/p_t^0), \big[(\omega^1/p^0)(p_{t+1}^1/p_{t+1}^0) + (p_{t+1}^1/p_{t+1}^0)\big]/(p_t^1/p_t^0)\big] \\
&\quad - (p_t^1/p_t^0)\big[\mathcal{F}^1\big(1 - \mathcal{L}\big[(\omega^0/p^0)(p_t^1/p_t^0), \big[(\omega^1/p^0)(p_{t+1}^1/p_{t+1}^0) + (p_{t+1}^1/p_{t+1}^0)\big]/(p_t^1/p_t^0)\big], X_{1,t}, p_t^1/p_t^0\big) \\
&\qquad\qquad + (1-\eta)X_{1,t}\big] = 0, \\
&(B_{t+1}/p_{t+1}^0) - \big[(\omega^1/p^0)(p_{t+1}^1/p_{t+1}^0) + (p_{t+1}^1/p_{t+1}^0)\big](B_t/p_t^0)/(p_t^1/p_t^0) = 0.
\end{aligned}
$$

IV – Wealth-Capital Equilibria

An interior wealth-capital steady state equilibrium is defined by a pair $\{X_1^\star, (p^1/p^0)^\star\} \in]0, \overline{X_1/X_0}[\times]\underline{(p^1/p^0)}(X_1^\star/X_0), \overline{(p^1/p^0)}(X_1^\star/X_0)[$ which solves :

$$
\begin{aligned}
&\eta X_1^\star - \mathcal{F}^1\big(\bar{\ell} - \mathcal{L}\big[(\omega^0/p^0)\big[(p^1/p^0)^\star\big], (\omega^1/p^0)\big((p^1/p^0)^\star\big)/(p^1/p^0)^\star + (1-\eta)\big], X_1^\star, (p^1/p^0)^\star\big) = 0, \\
&(\omega^0/p^0)\big[(p^1/p^0)^\star\big]\big(1 - \mathcal{L}\big[(\omega^0/p^0)\big[(p^1/p^0)^\star\big], (\omega^1/p^0)\big((p^1/p^0)^\star\big)/(p^1/p^0)^\star + (1-\eta)\big]\big) \\
&\quad - \mathcal{C}\big[(\omega^0/p^0)\big[(p^1/p^0)^\star\big], (\omega^1/p^0)\big((p^1/p^0)^\star\big)/(p^1/p^0)^\star + (1-\eta)\big] \\
&\quad - (p^1/p^0)^\star\big[\mathcal{F}^1\big(1 - \mathcal{L}\big[(\omega^0/p^0)\big[(p^1/p^0)^\star\big], (\omega^1/p^0)\big((p^1/p^0)^\star\big)/(p^1/p^0)^\star + (1-\eta)\big], \\
&\qquad\qquad X_1^\star, (p^1/p^0)^\star\big) + (1-\eta)X_1^\star\big] = 0.
\end{aligned}
$$

The key aggregate parameters therein assume a local disaggregated formulation:

Lemma 2. *Under Assumptions* T.1-5, P.1-7, *an interior wealth-capital steady state will be characterised by:*

(i) *aggregate factors shares that satisfy* $\sum_{j=0}^{1} \pi_{X_0}^j \pi_{Y^j} = [\eta^{-1}(1-\pi_\ell)(1-\pi_{X_1}^0)/\pi_{c'}]/[\eta^{-1}(1-\pi_\ell)/\pi_{c'} - \pi_{X_1}^0 + \pi_{X_1}^1]$, $\sum_{j=0}^{1} \pi_{X_1}^j \pi_{Y^j} = [\eta^{-1}(1-\pi_\ell)\pi_{X_1}^0/\pi_{c'} - \pi_{X_1}^0 + \pi_{X_1}^1]/[\eta^{-1}(1-\pi_\ell)/\pi_{c'} - \pi_{X_1}^0 + \pi_{X_1}^1]$;

(ii) *aggregate outputs shares that satisfy:* $\pi_{Y^0} = [\eta^{-1}(1-\pi_\ell)/\pi_{c'} - (1-\pi_{X_1}^1)]/[\eta^{-1}(1-\pi_\ell)/\pi_{c'} - \pi_{X_1}^0 + \pi_{X_1}^1]$ *and* $\pi_{Y^1} = (1-\pi_{X_1}^0)/[\eta^{-1}(1-\pi_\ell)/\pi_{c'} - \pi_{X_1}^0 + \pi_{X_1}^1]$;

(ii) *a steady state expression for the aggregate elasticity of substitution between the inputs that is given by:* $\sum_{j=0}^{1} \pi_{Y^j}\pi_{X_0}^j\pi_{X_1}^j\Sigma_{X_0X_1}^j = \{[\eta^{-1}(1-\pi_\ell)/\pi_{c'} - (1-\pi_{X_1}^1)]\pi_{X_0}^0\pi_{X_1}^0\Sigma_{X_0X_1}^0 + (1-\pi_{X_1}^0)\pi_{X_0}^1\pi_{X_1}^1\Sigma_{X_0X_1}^1\} \;/\; [\eta^{-1}(1-\pi_\ell)/\pi_{c'} - \pi_{X_1}^0 + \pi_{X_1}^1]$.

Proof . The steady state is defined by the holding of $(\omega^0/p^0)^*(\bar{\ell} - \ell^*) - c^\star = (p^1/p^0)^* X_1^*$, rearranging, this reformulates to: $\sum_{j=0}^{1} \pi_{Y^j}\pi_{X_0}^j[1 - \pi_c/(1-\pi_\ell)] = \eta^{-1}\pi_{Y^1}$. Solving in π_{Y^1}, the steady state expression of the latter is obtained as $\pi_{Y^1} = (1-\pi_{X_1}^0)/[\eta^{-1}(1-\pi_\ell)/\pi_{c'} - \pi_{X_1}^0 + \pi_{X_1}^1 igr]$, the obtention of the expressions of π_{Y^0}, $\sum_{j=0}^{1} \pi_{Y^j}\pi_{X_0}^j$, $\sum_{j=0}^{1} \pi_{Y^j}\pi_{X_1}^j$ and $\sum_{j=0}^{1} \pi_{Y^j}\pi_{X_0}^j\pi_{X_1}^j\Sigma_{X_0X_1}^j$ being then immediate. △

Upon existence, the stability properties of a wealth-capital steady state are then listed in the following statement:

Proposition 1. *Under Assumptions* T.1-5, P.1-7, *let* $\Sigma^H_{cc'} > \Sigma^H_{cR}$ *and* $\Sigma^H_{c\ell} > \Sigma^H_{\ell R}$ *and consider an interior wealth-capital steady state position:*

(i) *for* $\Sigma^0_{X_0 X_1} \to +\infty$ *and* $\Sigma^1_{X_0 X_1} \to +\infty$ *, it is uniformly a saddlepoint equilibrium ;*

(ii) *for* $\pi^1_{X_1} > \pi^0_{X_1}$*, it is a saddlepoint equilibrium if* $\Sigma^0_{X_0 X_1} > \pi^1_{X_1}/\pi^0_{X_1}\left[1 + (\eta^{-1} - 1)/\pi^1_{X_1}\right]$ *and* $\Sigma^1_{X_0 X_1} > \pi^1_{X_1}/(1 - \pi^1_{X_1})$ *simultaneously hold;*

(iii) *for* $\pi^1_{X_1} > \pi^0_{X_1}$*, it is locally unstable when either* $\Sigma^0_{X_0 X_1} < \pi^1_{X_1}/\pi^0_{X_1}\left[1 + (\eta^{-1} - 1)/\pi^1_{X_1}\right]$ *or* $\Sigma^1_{X_0 X_1} < \pi^1_{X_1}/(1 - \pi^1_{X_1})$ *holds.*

(iv) *the area for two-period cycles is unequivocally associated with the occurrence of* $\pi^0_{X_1} > \pi^1_{X_1}$*.*

Proof : *Vide* Appendix 1. △

Remark : In a one good environment where $F^0 = F^1$, the coefficients of the characteristic polynomial list as:

$$\begin{aligned}
\mathrm{tr}(\mathcal{J}) = 1 &+ \frac{\pi_{X_1}\eta/\pi_{Y^1} + (1-\eta)}{1 + \pi_c/\pi_{Y^1}\eta^{-1}}\left[1 + \frac{\pi_\ell\left(\Sigma^H_{\ell c} - \Sigma^H_{\ell Y}\right)}{\pi_{c'}\left(\Sigma^H_{\ell c'} - \Sigma^H_{\ell Y}\right)}\right] \\
&+ \frac{\pi_c\left(\Sigma^H_{cc'} - \Sigma^H_{cY}\right)}{\pi_{c'}\left(\Sigma^H_{\ell c'} - \Sigma^H_{\ell Y}\right)}\left(1 + \pi_c/\pi_{Y^1}\eta^{-1}\right) + \frac{\Sigma_{X_0 X_1}\left[\pi_{X_1}\eta/\pi_{Y^1} + (1-\eta)\right]}{\left(\pi_{X_1}\eta/\pi_{Y^1}\right)\pi_\ell\pi_{c'}\left(\Sigma^H_{\ell c'} - \Sigma^H_{\ell Y}\right)} \\
\det(\mathcal{J}) = &\left[\pi_{X_1}\eta/\pi_{Y^1} + (1-\eta)\right]\left[1 + \frac{\Sigma^H_{\ell Y}\pi_\ell + \Sigma^H_{c'Y}\pi_{X_0}}{\pi_\ell\left(\Sigma^H_{\ell c'} - \Sigma^H_{\ell Y}\right)}\right].
\end{aligned}$$

Otherwise stated, and in contradistinction with Proposition 1, the sole area for two-period cycles is to rest upon a relaxation of the gross substitutability assumption on preferences, *i.e.*, either $\Sigma^H_{cc'} < \Sigma^H_{cR}$ and $\Sigma^H_{c\ell} < \Sigma^H_{\ell R}$. As for local unstability and an area for a Poincaré-Hopf bifurcation, under a gross substitutabiliity assumption on preferences and thus for $\Sigma^H_{cc'} > \Sigma^H_{cR}$ and $\Sigma^H_{c\ell} > \Sigma^H_{\ell R}$, it happens to be uniformly associated with $\Sigma_{X_0 X_1} \to 0$ and thus with a complementarity assumption between the productive factors. ◇

V – Golden Rule Monetary Equilibria

An interior golden rule steady state equilibrium is defined by a triple $\{X^\star_1, (p^1/p^0)^\star, (B/p^0)^\star\} \in \left]0, \overline{X_1/X_0}\right[\times \left]\underline{(p^1/p^0)}(X^\star_1/X_0), \overline{(p^1/p^0)}(X^\star_1/X_0)\right[\times]0,+\infty)$ which solves :

$$\begin{aligned}
&\eta X^\star_1 - \mathcal{F}^1\big(1 - \mathcal{L}\left[(\omega^0/p^0)\left[(p^1/p^0)^\star\right], (\omega^1/p^0)\left((p^1/p^0)^\star\right)/(p^1/p^0)^\star + (1-\eta)\right] -, X^\star_1, (p^1/p^0)^\star\big) = 0, \\
&(\omega^0/p^0)\left[(p^1/p^0)^\star\right]\left(1 - \mathcal{L}\left[(\omega^0/p^0)\left[(p^1/p^0)^\star\right], (\omega^1/p^0)\left((p^1/p^0)^\star\right)/(p^1/p^0)^\star + (1-\eta)\right]\right) \\
&\quad - \mathcal{C}\left[(\omega^0/p^0)\left[(p^1/p^0)^\star\right], (\omega^1/p^0)\left((p^1/p^0)^\star\right)/(p^1/p^0)^\star + (1-\eta)\right] - (B/p^0)^\star M - (p^1/p^0)^\star X^\star_1 = 0, \\
&(\omega^1/p^0)\left[(p^1/p^0)^\star\right]/(p^1/p^0)^\star + (1-\eta) - 1 = 0.
\end{aligned}$$

Rearranging, a monetary steady state will be characterised by the holding of

$$\begin{aligned}
1 + \frac{\sum_{j=0}^{1}\pi_{Y^j}\pi^j_{X_0}}{\pi_{Y^1}\eta^{-1}}\frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} &+ \left[\frac{1 - \pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\eta + (1-\eta)\right]\frac{M(B/p^0)^\star}{(p^1/p^0)^\star X^\star_1} \\
&= \frac{\sum_{j=0}^{1}\pi_{Y^j}\pi^j_{X_0}}{\pi_{Y^1}\eta^{-1}}\left(1 - \frac{\pi_c}{1 - \pi_\ell}\right)\left[\frac{1 - \pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\eta + (1-\eta)\right],
\end{aligned}$$

whence a local disaggregated representation for the aggregate parameters:

Lemma 3. —— *Under Assumptions* T.1-5, P.1-7, *an interior golden rule monetary steady state will be characterised by:*

(i) *aggregate factors shares that satisfy* $\sum_{j=0}^{1} \pi_{X_0}^{j} \pi_{Y^j} = (1 - \pi_{X_1}^{1})/[1 - (\pi_{X_1}^{1} - \pi_{X_1}^{0})]$, $\sum_{j=0}^{1} \pi_{X_1}^{j} \pi_{Y^j} = \pi_{X_1}^{0}/[1 - (\pi_{X_1}^{1} - \pi_{X_1}^{0})]$;

(ii) *aggregate outputs shares that satisfy* $\pi_{Y^0} = (1 - \pi_{X_1}^{1})/[1 - (\pi_{X_1}^{1} - \pi_{X_1}^{0})]$ *and* $\pi_{Y^1} = \pi_{X_1}^{0}/[1 - (\pi_{X_1}^{1} - \pi_{X_1}^{0})]$;

(iii) *a steady state expression for the aggregate elasticity of substitution between the inputs that is given by* $\sum_{j=0}^{1} \pi_{Y^j} \pi_{X_0}^{j} \pi_{X_1}^{j} \Sigma_{X_0 X_1}^{j} = (1 - \pi_{X_1}^{1}) \pi_{X_1}^{0} \left[(1 - \pi_{X_1}^{0}) \Sigma_{X_0 X_1}^{0} + \pi_{X_1}^{1} \Sigma_{X_0 X_1}^{1} \right] / \left[1 - (\pi_{X_1}^{1} - \pi_{X_1}^{0})\right]$.

Proof . —— This proceeds from a straightforward adaptation of the line of argument developed for Lemma 2 by noticing that the third component in the above definition of a golden rule steady state can be defined as $\sum_{j=0}^{1} \pi_{Y^j} \pi_{X_1}^{j} = \pi_{Y^1}$. Solving in π_{Y^1} and generalising the approach to the other parameters, the results follow. △

Lemma 4. —— *Under Assumptions* T.1-5, P.1-7, *further let:*

(i) *for* $\Sigma_{\ell c^0{}'}^{H} > \Sigma_{\ell Y}^{H}$, *the set of Classical (Samuelsonian) Economies is associated with* $(1 - \pi_{X_1}^{0})\eta + (1 - \eta)(\pi_{X_1}^{1} - \pi_{X_1}^{0}) > (<) \pi_{X_1}^{1}$;

(i) *for* $\Sigma_{\ell c^0{}'}^{H} < \Sigma_{\ell Y}^{H}$, *the set of Classical (Samuelsonian) Economies is associated with* $(1 - \pi_{X_1}^{0})\eta + (1 - \eta)(\pi_{X_1}^{1} - \pi_{X_1}^{0}) < (>) \pi_{X_1}^{1}$.

Proof. —— *Vide* Appendix 4. △

Proposition 2. *Under Assumptions* T.1-5, P.1-7, *let* $\Sigma_{cc'}^{H} > \Sigma_{cR}^{H}$ *and consider an interior golden rule steady state position:*

(i) *for* $\Sigma_{X_0 X_1} \to \infty$ *and* $\Sigma_{X_0 X_1} \to \infty$, *it is uniformly a saddlepoint equilibrium ;*

(ii) *for Samuelsonian Economies, the saddlepoint property is uniformly available;*

(iii) *for Classical Economies, the saddlepoint property is available for:*

a/ *for* $\pi_{X_1}^{1} > \pi_{X_1}^{0}$, *it is a saddlepoint when* $\Sigma_{cY}^{H} < \Sigma_{cc'}^{H} < \pi_{X_0}^{0} \pi_{\bar{X}_1} \Sigma_{c'Y}^{H} / \pi_C \pi_{X_0}^{1} + \Sigma_{cY}^{H}$;

b/ *for* $\pi_{X_1}^{0} > \pi_{X_1}^{1}$, *it is a saddlepoint when* $\pi_{X_1}^{1} - \pi_{X_1}^{0} < -(1 - \pi_{X_1}^{0}) [\pi_{X_1} \eta / \pi_{Y^1} + (1 - \eta)]$.

Proof : *Vide* Appendix 2. △

Remark : In a one good environment where $F^0 = F^1$, the coefficients of the characteristic polynomial list as:

$$\mathrm{tr}(\mathcal{J}) = 1 + \frac{\pi_{X_0}\eta}{\pi_{X_1}} \left[1 + \frac{\pi_c (\Sigma_{cc'}^{H} - \Sigma_{cY}^{H})}{\pi_{X_1} (\Sigma_{\ell c'}^{H} - \Sigma_{\ell Y}^{H})} \right] + \frac{\pi_{X_1}}{\pi_{X_0}\eta} \left[1 + \frac{\pi_c (\Sigma_{\ell c}^{H} - \Sigma_{\ell Y}^{H})}{\pi_{c'} (\Sigma_{\ell c'}^{H} - \Sigma_{\ell Y}^{H})} \right] + \frac{\Sigma_{X_0 X_1}}{\pi_{X_1} \pi_{c'} (\Sigma_{\ell c'}^{H} - \Sigma_{\ell Y}^{H})},$$

$$\begin{aligned}\mathrm{spm}(\mathcal{J}) = 1 + \frac{\pi_{X_0}\eta}{\pi_{X_1}}\left[1 + \frac{\pi_c\left(\Sigma^H_{cc'} - \Sigma^H_{cY}\right)}{\pi_{X_1}\left(\Sigma^H_{\ell c'} - \Sigma^H_{\ell Y}\right)}\right] + \frac{\pi_{X_1}}{\pi_{X_0}\eta}\left[1 + \frac{\pi_c\left(\Sigma^H_{\ell c} - \Sigma^H_{\ell Y}\right)}{\pi_{c'}\left(\Sigma^H_{\ell c'} - \Sigma^H_{\ell Y}\right)}\right] \\ + \frac{\Sigma_{X_0 X_1}}{\pi_{X_1}\pi_{c'}\left(\Sigma^H_{\ell c'} - \Sigma^H_{\ell Y}\right)} + \frac{\Sigma^H_{\ell Y}\pi_\ell + \Sigma^H_{c'Y}\pi_{X_0}}{\pi_\ell\left(\Sigma^H_{\ell c'} - \Sigma^H_{\ell Y}\right)} \\ + \frac{\pi_{X_0}\eta}{\pi_{X_1}}\frac{\pi_{Y^1}\eta^{-1}\left[\pi_{X_0}\eta/\pi_{Y^1} - \pi_c\eta/\pi_{Y^1} - 1\right]}{\pi_\ell\pi_{c'}\left(\Sigma^H_{\ell c'} - \Sigma^H_{\ell Y}\right)},\end{aligned}$$

$$\det(\mathcal{J}) = 1 + \frac{\Sigma^H_{\ell Y}\pi_\ell + \Sigma^H_{c'Y}\pi_{X_0}}{\pi_\ell\left(\Sigma^H_{\ell c'} - \Sigma^H_{\ell Y}\right)}.$$

Otherwise stated, for samuelsonian economies most of the conclusions of Proposition 2 appear as straightforward generalisations of a homogeneous good environment. Things differ, however, for classical economies where the sign of $\pi^1_{X_1} - \pi^0_{X_1}$ plays a role and a range of configurations appear that were not available with a homogeneous good setting. ◇

VI – References

[1]. Benhabib, J. & G. Laroque «On Competitive Cycles in Productive Economies.» *Journal of Economic Theory* 45 : 145-70, 1988.

[2]. Benhabib, J. & K. Nishimura «The Hopf Bifurcation and the Existence and Stability of closed Orbits in Multisector Models of Optimal Economic Growth.» *Journal of Economic Theory* 21 : 421–44, 1979.

[3]. —— & —— «Competitive Equilibrium Cycles.» *Journal of Economic Theory* 35 : 284–306, 1985.

[4]. Cass, D., M. Okuno & I. Zilcha. «The Role of Money in Supporting the Pareto Optimality of competitive Equilibrium in consumption-Loan TYpe Models.» *Journal of Economic Theory* 20 : 41-80, 1979.

[5]. Cazzavillan, G. & P. Pintus. «Multiple Long-Run Equilibria and competitive Endogenous Fluctuations in an Overlapping Generations Economy with Increasing Returns», presentation at Esem 1997, Toulouse.

[6]. Drugeon, J.-P. «A Capital Theory Approach to the Model of Overlapping Generations.». Manuscript, C.n.r.s., 2005.

[7]. Drugeon, J.-P. «On Sectoral Supply Functions & some Critical Roles for the Consumptions & Leisure Arbitrages in the Stability Properties of a Competitive Equilibrium with Heterogeneous Goods.» *Journal of Mathematical Economics*, 2010.

[8]. Galor, O. «A Two-Sector Overlapping Generations Model : A Global characterisation of the Dynamical System.» *Econometrica* 60 : 1351-86, 1992.

[9]. Grandmont, J.-M., «On Endogenous Competitive Business cycles», *Econometrica* 53 : 995-1046.

[10]. ——, P. Pintus & R. de Vilder. «capital-Labour Substitution and competitive Endogenous Business cycles.» *Journal of Economic Theory* 80, 14-59.

[11]. Guckenheimer, J. & P. Holmes. *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*. New York : Springer-Verlag, 1986.

[12]. Reichlin, P. «Equilibrium cycles and Stabilization Policies in an Overlapping-Generations Economy with Production.» *Journal of Economic Theory* 40 : 89-103, 1986.

[13]. ——, «Endogenous Fluctuations in a Two-Sector Overlapping Generations Economy.» *in* J. Benhabib, Ed., *Cycles and Chaos in Economic Equilibrium*, Princeton University Press, Princeton, N.J., 1992.

VII – Proofs

VII.1 – Proof of Proposition 1.

First-order conditions list as :

$$\frac{\partial u}{\partial \ell_t^t}\big(c_t^t,\ell_t^t,c_{t+1}^t\big)-\big(\omega_t^{o}/p_t^{o}\big)\frac{\partial u}{\partial c_t^t}\big(c_t^t,\ell_t^t,c_{t+1}^t\big)=0,$$

$$\frac{\partial u}{\partial c_t^t}\big(c_t^t,\ell_t^t,c_{t+1}^t\big)\big(p_{t+1}^{o}/p_t^{o}\big)-\mathcal{R}_{t+1}\frac{\partial u}{\partial c_{t+1}^t}\big(c_t^t,\ell_t^t,c_{t+1}^t\big)=0,$$

$$c_t^t+\big(p_{t+1}^{o}/p_t^{o}\big)c_{t+1}^t/\mathcal{R}_{t+1}-\big(\omega_t^{o}/p_t^{o}\big)\big(1-\ell_t^t\big)=0.$$

Linearising :

$$\begin{bmatrix} -\big(\Xi_{cc}-\Xi_{\ell c}\big)\pi_c\mathcal{S} & \big(\Xi_{\ell\ell}-\Xi_{c\ell}\big)\pi_\ell\mathcal{S} & \big(\Xi_{\ell c'}-\Xi_{cc'}\big)\pi_{c'}\mathcal{S}\\ \big(\Xi_{cc}-\Xi_{c'c}\big)\pi_c\mathcal{S} & \big(\Xi_{c\ell}-\Xi_{c'\ell}\big)\pi_\ell\mathcal{S} & -\big(\Xi_{c'c'}-\Xi_{cc'}\big)\pi_{c'}\mathcal{S}\\ \pi_c & \pi_\ell & \pi_{c'}\end{bmatrix}\begin{bmatrix}\mathcal{C}\\ \mathcal{L}\\ \mathcal{C}'\end{bmatrix}$$
$$-\begin{bmatrix}1 & 0\\ 0 & 1\\ 1-\pi_\ell & \pi_{c'}\end{bmatrix}\begin{bmatrix}\Omega_t^{o}-\mathcal{P}_t^{o}\\ \mathcal{R}_{t+1}-\big(\mathcal{P}_{t+1}^{o}-\mathcal{P}_t^{o}\big)\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

The system of demand equations is gathered in the following matrix :

$$\begin{bmatrix}\Delta_{c(\omega^{o}/p^{o})} & \Delta_{c[\mathcal{R}'/(p^{o\prime}/p^{o})]}\\ \Delta_{\ell(\omega^{o}/p^{o})} & \Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^{o})]}\\ \Delta_{c'(\omega^{o}/p^{o})} & \Delta_{c'[\mathcal{R}'/(p^{o\prime}/p^{o})]}\end{bmatrix},$$

whose coefficients, for

$$\begin{aligned}\mathcal{D}_{c\ell c'}=\pi_c\pi_\ell\pi_{c'}\mathcal{S}^2\Big\{&\Xi_{cc}\Xi_{\ell\ell}-\big(\Xi_{c\ell}\big)^2-\Xi_{c'c}\big(\Xi_{\ell\ell}-\Xi_{c\ell}\big)-\Xi_{c'\ell}\big(\Xi_{cc}-\Xi_{\ell c}\big)\\ &+\Xi_{cc}\Xi_{c'c'}-\big(\Xi_{cc'}\big)^2-\Xi_{\ell c}\big(\Xi_{c'c'}-\Xi_{cc'}\big)-\Xi_{\ell c'}\big(\Xi_{cc}-\Xi_{c'c}\big)\\ &+\Xi_{\ell\ell}\Xi_{c'c'}-\big(\Xi_{\ell c'}\big)^2-\Xi_{c\ell}\big(\Xi_{c'c'}-\Xi_{\ell c'}\big)-\Xi_{cc'}\big(\Xi_{\ell\ell}-\Xi_{c'\ell}\big)\Big\},\end{aligned}$$

state as :

$$\begin{aligned}\Delta_{c(\omega^{o}/p^{o})}=\big(\mathcal{D}_{c\ell c'}\big)^{-1}\Big\{&\big[-\big(\Xi_{c'c'}-\Xi_{cc'}\big)-\big(\Xi_{c\ell}-\Xi_{c'\ell}\big)\big]+\big[\Xi_{\ell\ell}\Xi_{c'c'}-\big(\Xi_{\ell c'}\big)^2\\ &-\Xi_{c\ell}\big(\Xi_{c'c'}-\Xi_{\ell c'}\big)-\Xi_{cc'}\big(\Xi_{\ell\ell}-\Xi_{c'\ell}\big)\big]\big(1-\pi_\ell\big)\Big\}\pi_{c'}\pi_\ell\mathcal{S},\end{aligned}$$

$$\begin{aligned}\Delta_{c[\mathcal{R}'/(p^{o\prime}/p^{o})]}=\big(\mathcal{D}_{c\ell c'}\big)^{-1}\Big\{&\big[\big(\Xi_{cc'}-\Xi_{\ell c'}\big)+\big(\Xi_{\ell\ell}-\Xi_{c\ell}\big)\big]+\big[\Xi_{\ell\ell}\Xi_{c'c'}-\big(\Xi_{\ell c'}\big)^2\\ &-\Xi_{c\ell}\big(\Xi_{c'c'}-\Xi_{\ell c'}\big)-\Xi_{cc'}\big(\Xi_{\ell\ell}-\Xi_{c'\ell}\big)\big]\pi_{c'}\Big\}\pi_{c'}\pi_\ell\mathcal{S},\end{aligned}$$

$$\begin{aligned}\Delta_{\ell(\omega^{o}/p^{o})}=\big(\mathcal{D}_{c\ell c'}\big)^{-1}\Big\{&\big[\big(\Xi_{cc}-\Xi_{c'c}\big)+\big(\Xi_{c'c'}-\Xi_{cc'}\big)\big]+\big[\Xi_{cc}\Xi_{c'c'}-\big(\Xi_{cc'}\big)^2\\ &-\Xi_{\ell c}\big(\Xi_{c'c'}-\Xi_{cc'}\big)-\Xi_{\ell c'}\big(\Xi_{cc}-\Xi_{c'c}\big)\big]\big(1-\pi_\ell\big)\Big\}\pi_c\pi_{c'}\mathcal{S},\end{aligned}$$

$$\begin{aligned}\Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^{o})]}=\big(\mathcal{D}_{c\ell c'}\big)^{-1}\Big\{&\big[\big(\Xi_{cc}-\Xi_{\ell c}\big)-\big(\Xi_{cc'}-\Xi_{\ell c'}\big)\big]+\big[\Xi_{cc}\Xi_{c'c'}-\big(\Xi_{cc'}\big)^2\\ &-\Xi_{\ell c}\big(\Xi_{c'c'}-\Xi_{cc'}\big)-\Xi_{\ell c'}\big(\Xi_{cc}-\Xi_{c'c}\big)\big]\pi_{c'}\Big\}\pi_c\pi_{c'}\mathcal{S},\end{aligned}$$

$$\Delta_{c'(\omega^0/p^0)} = \left(\mathcal{D}_{c\ell c'}\right)^{-1}\Big\{-\left[\left(\Xi_{cc} - \Xi_{c'c}\right) - \left(\Xi_{c\ell} - \Xi_{c'\ell}\right)\right] + \big[\Xi_{cc}\Xi_{\ell\ell} - \left(\Xi_{c\ell}\right)^2$$

$$- \Xi_{c'c}\left(\Xi_{\ell\ell} - \Xi_{c\ell}\right) - \Xi_{c'\ell}\left(\Xi_{cc} - \Xi_{\ell c}\right)\big]\left(1 - \pi_\ell\right)\Big\}\pi_c\pi_\ell\mathcal{S},$$

$$\Delta'_{c[\mathcal{R}'/(p^{0'}/p^0)]} = \left(\mathcal{D}_{c\ell c'}\right)^{-1}\Big\{-\left[\left(\Xi_{cc} - \Xi_{\ell c}\right) + \left(\Xi_{\ell\ell} - \Xi_{c\ell}\right)\right] + \big[\Xi_{cc}\Xi_{\ell\ell} - \left(\Xi_{c\ell}\right)^2$$

$$- \Xi_{c'c}\left(\Xi_{\ell\ell} - \Xi_{c\ell}\right) - \Xi_{c'\ell}\left(\Xi_{cc} - \Xi_{\ell c}\right)\big]\pi_{c'}\Big\}\pi_c\pi_\ell\mathcal{S}.$$

The raw form of the linearised version of this dynamical system in a neighbourhood of the steady state is available as :

$$\begin{bmatrix} 1 & -\dfrac{\partial\mathcal{F}^1}{\partial X_0}\left[-\dfrac{\partial\ell}{\partial\left[\mathcal{R}'/(p^{0'}/p^0)\right]}\left[\dfrac{\partial\left(\omega^1/p^0\right)}{\partial\left(p^1/p^0\right)}\dfrac{1}{p^1/p^0} + (1-\eta)\right]\right] \\ 0 & A'_{22} \end{bmatrix}\begin{bmatrix} \Delta X_{1,t+1} \\ \Delta\left(p^1_{t+1}/p^0_{t+1}\right) \end{bmatrix}$$

$$+ \begin{bmatrix} -\dfrac{\partial\mathcal{F}^1}{\partial X_1} - (1-\eta) & A_{12} \\ -\left(p^1/p^0\right)\left[\dfrac{\partial\mathcal{F}^1}{\partial X_1} + (1-\eta)\right] & A_{22} \end{bmatrix}\begin{bmatrix} \Delta X_{1,t} \\ \Delta\left(p^1_t/p^0_t\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$A'_{22} = \left(\omega^0/p^0\right)\left[-\frac{\partial\ell}{\partial\left[\mathcal{R}'/(p^{0'}/p^0)\right]}\left[\frac{\partial\left(\omega^1/p^0\right)}{\partial\left(p^1/p^0\right)}\frac{1}{p^1/p^0} + (1-\eta)\right]\right] - \frac{\partial c}{\partial\left[\mathcal{R}'/(p^{0'}/p^0)\right]}\left[\frac{\partial\left(\omega^1/p^0\right)}{\partial\left(p^1/p^0\right)}\frac{1}{p^1/p^0} + (1 -$$

$$- \left(p^1/p^0\right)\frac{\partial\mathcal{F}^1}{\partial X_0}\left[-\frac{\partial\ell}{\partial\left(\omega^0/p^0\right)}\left[\frac{\partial\left(\omega^1/p^0\right)}{\partial\left(p^1/p^0\right)}\frac{1}{p^1/p^0} + (1-\eta)\right]\right],$$

$$A_{12} = -\frac{\partial\mathcal{F}^1}{\partial\left(p^1/p^0\right)} - \frac{\partial\mathcal{F}^1}{\partial X_0}\left[-\frac{\partial\ell}{\partial\left(\omega^0/p^0\right)}\frac{\partial\left(\omega^0/p^0\right)}{\partial\left(p^1/p^0\right)} - \frac{\partial\ell}{\partial\left[\mathcal{R}'/(p^{0'}/p^0)\right]}\left[-\left(\frac{\partial\left(\omega^1/p^0\right)}{\partial\left(p^1/p^0\right)}\frac{1}{p^1/p^0} + (1-\eta)\right)\right]\frac{1}{p^1/p}\right.$$

$$A_{22} = \left(X_0 - \frac{\partial c}{\partial\left(\omega^0/p^0\right)}\right)\frac{\partial\left(\omega^0/p^0\right)}{\partial\left(p^1/p^0\right)} - \frac{\partial c}{\partial\left[\mathcal{R}'/(p^{0'}/p^0)\right]}\left[-\left(\frac{\partial\left(\omega^1/p^0\right)}{\partial\left(p^1/p^0\right)}\frac{1}{p^1/p^0} + (1-\eta)\right)\right]\frac{1}{p^1/p^0} - \left[\mathcal{F}^1 + (1\right.$$

$$+ \left(\omega^0/p^0\right)\left(-\frac{\partial\ell}{\partial\omega^0}\right)\frac{\partial\left(\omega^0/p^0\right)}{\partial\left(p^1/p^0\right)} + \left(\omega^0/p^0\right)\left(-\frac{\partial\ell}{\partial\left[\mathcal{R}'/(p^{0'}/p^0)\right]}\right)\left[-\left(\frac{\partial\left(\omega^1/p^0\right)}{\partial\left(p^1/p^0\right)}\frac{1}{p^1/p^0} + (1-\eta)\right)\right]$$

$$- \left(p^1/p^0\right)\frac{\partial\mathcal{F}^1}{\partial X_0}\left[-\frac{\partial\ell}{\partial\omega^0}\frac{\partial\left(\omega^0/p^0\right)}{\partial\left(p^1/p^0\right)} - \frac{\partial\ell}{\partial\left[\mathcal{R}'/(p^{0'}/p^0)\right]}\left[-\left(\frac{\partial\left(\omega^1/p^0\right)}{\partial\left(p^1/p^0\right)}\frac{1}{p^1/p^0} + (1-\eta)\right)\right]\frac{1}{p^1/p^0}\right].$$

Let $\pi_{\bar{X}_0} := \sum_{j=0}^{1}\pi^j_{X_0}\pi_{Y^j}$, $\pi_{\bar{X}_1} := \sum_{j=0}^{1}\pi^j_{X_1}\pi_{Y^j}$ and $\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1} := \sum_{j=0}^{1}\pi_{Y^j}\pi^j_{X_0}\pi^j_{X_1}\Sigma^j_{X_0X_1}$.

This linearisation of the raw form of the dynamical system delivers, in elasticities form :

$$
\begin{bmatrix} 1 & \mathcal{A}'_{12} \\ 0 & \mathcal{A}'_{22} \end{bmatrix} \begin{bmatrix} \mathcal{X}_{1,t+1} \\ \mathcal{P}^1_{t+1} - \mathcal{P}^0_{t+1} \end{bmatrix} - \begin{bmatrix} \Delta_{Y^1 X_1}\eta + (1-\eta) & \mathcal{A}_{12} \\ \Delta_{Y^1 X_1}\eta + (1-\eta) & \mathcal{A}_{22} \end{bmatrix} \begin{bmatrix} \mathcal{X}_{1,t} \\ \mathcal{P}^1_t - \mathcal{P}^0_t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
$$

$$
\mathcal{A}'_{12} = \Delta_{Y^1 X_0}\eta \frac{\pi_\ell \Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}/\pi_{X_0}}{\pi_{\bar{X}_1}\eta/\pi_{Y^1} + (1-\eta)} \left\{ \left[\Delta_{(\omega^1/p^0)(p^1/p^0)} \frac{\pi_{\bar{X}_1}\eta}{\pi_{Y^1}} + (1-\eta) \right] \right\},
$$

$$
\begin{aligned}
\mathcal{A}'_{22} = & -\frac{1}{\pi_{Y^1}\eta^{-1}} \left[\frac{\pi_c \Delta_{c[\mathcal{R}'/(p^{0\prime}/p^0)]} + \pi_\ell \Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}}{\pi_{X_1}\eta/\pi_{Y^1} + (1-\eta)} \right] \left[\Delta_{(\omega^1/p^0)(p^1/p^0)} \frac{\pi_{X_1}\eta}{\pi_{Y^1}} + (1-\eta) \right] \\
& + \Delta_{Y^1 X_0}\eta \frac{\pi_\ell \Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}/\pi_{X_0}}{\pi_{\bar{X}_1}\eta/\pi_{Y^1} + (1-\eta)} \left\{ \left[\Delta_{\omega^1(p^1/p^0)} \frac{\pi_{\bar{X}_1}\eta}{\pi_{Y^1}} + (1-\eta) \right] \right\},
\end{aligned}
$$

$$
\mathcal{A}_{12} = \Delta_{Y^1 X_0}\eta \frac{\pi_\ell}{\pi_{X_0}} \left[-\Delta_{\ell(\omega^0/p^0)} \Delta_{(\omega^0/p^0)(p^1/p^0)} + \Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]} \right] + \Delta_{Y^1(p^1/p^0)}\eta,
$$

$$
\begin{aligned}
\mathcal{A}_{22} = & \Delta_{Y^1 X_0}\eta \frac{\pi_\ell}{\pi_{X_0}} \left[-\Delta_{\ell(\omega^0/p^0)} \Delta_{(\omega^0/p^0)(p^1/p^0)} + \Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]} \right] + \Delta_{Y^1(p^1/p^0)}\eta \\
& - \frac{\pi_{\bar{X}_0}}{\pi_{Y^1}\eta^{-1}} \Delta_{(\omega^0/p^0)(p^1/p^0)} - \frac{\pi_\ell}{\pi_{Y^1}\eta^{-1}} \left[-\Delta_{\ell(\omega^0/p^0)} \Delta_{(\omega^0/p^0)(p^1/p^0)} + \Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]} \right] \\
& - \frac{\pi_c}{\pi_{Y^1}\eta^{-1}} \left[-\Delta_{c(\omega^0/p^0)} \Delta_{(\omega^0/p^0)(p^1/p^0)} + \Delta_{c[\mathcal{R}'/(p^{0\prime}/p^0)]} \right] + 1.
\end{aligned}
$$

Or:

$$
\begin{bmatrix} 1 & \mathcal{A}'_{12} \\ 0 & \mathcal{A}'_{22} \end{bmatrix} \begin{bmatrix} \mathcal{X}_{1,t+1} \\ \mathcal{P}^1_{t+1} - \mathcal{P}^0_{t+1} \end{bmatrix} - \begin{bmatrix} \frac{1-\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \frac{\pi_{X_1}}{\pi_{Y^1}}\eta + (1-\eta) & \mathcal{A}_{12} \\ \frac{1-\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \frac{\pi_{X_1}}{\pi_{Y^1}}\eta + (1-\eta) & \mathcal{A}_{22} \end{bmatrix} \begin{bmatrix} \mathcal{X}_{1,t} \\ \mathcal{P}^1_t - \mathcal{P}^0_t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
$$

$$
\mathcal{A}'_{12} = \Delta_{Y^1 X_0}\eta \frac{\pi_\ell \Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}/\pi_{X_0}}{\pi_{\bar{X}_1}\eta/\pi_{Y^1} + (1-\eta)} \left\{ \left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \frac{\pi_{X_1}}{\pi_{Y^1}}\eta + (1-\eta) \right] \right\},
$$

$$
\begin{aligned}
\mathcal{A}'_{22} = & -\frac{1}{\pi_{Y^1}\eta^{-1}} \left[\frac{\pi_c \Delta_{c[\mathcal{R}'/(p^{0\prime}/p^0)]} + \pi_\ell \Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}}{\pi_{X_1}\eta/\pi_{Y^1} + (1-\eta)} \right] \left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \frac{\pi_{X_1}}{\pi_{Y^1}}\eta + (1-\eta) \right] \\
& - \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \frac{\pi_{X_0}}{\pi_{Y^1}}\eta \frac{\pi_\ell \Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}/\pi_{X_0}}{\pi_{\bar{X}_1}\eta/\pi_{Y^1} + (1-\eta)} \left\{ \left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \frac{\pi_{X_1}}{\pi_{Y^1}}\eta + (1-\eta) \right] \right\},
\end{aligned}
$$

$$
\mathcal{A}_{12} = -\frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \frac{\pi_{X_0}}{\pi_{Y^1}}\eta \frac{\pi_\ell}{\pi_{X_0}} \left[\Delta_{\ell(\omega^0/p^0)} \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} + \Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]} \right] + \frac{\Sigma_{\bar{X}_0 \bar{X}_1} \pi_{\bar{X}_0} \pi_{\bar{X}_1}}{\left(\pi^1_{X_1} - \pi^0_{X_1}\right)^2} \frac{\eta}{\pi_{Y^1}},
$$

…3

$$\mathcal{A}_{22} = -\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\pi_{X_0}}{\pi_{Y^1}}\eta\frac{\pi_\ell}{\pi_{X_0}}\left[\Delta_{\ell(\omega^0/p^0)}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}+\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\right]+\frac{\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}}{\left(\pi^1_{X_1}-\pi^0_{X_1}\right)^2}\frac{\eta}{\pi_{Y^1}}$$
$$+\frac{\pi_{\bar{X}_0}}{\pi_{Y^1}\eta^{-1}}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}-\frac{\pi_\ell}{\pi_{Y^1}\eta^{-1}}\left[\Delta_{\ell(\omega^0/p^0)}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}+\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\right]$$
$$-\frac{\pi_c}{\pi_{Y^1}\eta^{-1}}\left[\Delta_{c(\omega^0/p^0)}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}+\Delta_{c[\mathcal{R}'/(p^{0\prime}/p^0)]}\right]+1.$$

Letting

$$\mathcal{D}_{A'} = -\frac{1}{\pi_{Y^1}\eta^{-1}}\left[\frac{\pi_c\Delta_{c[\mathcal{R}'/(p^{0\prime}/p^0)]}+\pi_\ell\left[1+\pi^0_{X_1}/\left(\pi^1_{X_1}-\pi^0_{X_1}\right)\right]\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}}{\pi_{X_1}\eta/\pi_{Y^1}+(1-\eta)}\right]$$
$$\times\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\pi_{X_1}\eta}{\pi_{Y^1}}+(1-\eta)\right],$$

the components of the Jacobian Matrix emerge as :

$$\mathcal{J}_{11} = \left(\mathcal{D}_{A'}\right)^{-1}\Bigg\{-\frac{1}{\pi_{Y^1}\eta^{-1}}\left[\frac{\pi_c\Delta_{c[\mathcal{R}'/(p^{0\prime}/p^0)]}+\left[1+\pi^0_{X_1}/\left(\pi^1_{X_1}-\pi^0_{X_1}\right)\right]\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}}{\pi_{X_1}\eta/\pi_{Y^1}+(1-\eta)}\right]$$
$$\times\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\pi_{X_1}\eta}{\pi_{Y^1}}+(1-\eta)\right]^2$$
$$+\frac{1}{\pi_{Y^1}\eta^{-1}}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\frac{\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\pi_{X_1}\eta}{\pi_{Y^1}}+(1-\eta)\right]^2}{\pi_{X_1}\eta/\pi_{Y^1}+(1-\eta)}\Bigg\};$$

$$\mathcal{J}_{12} = \left(\mathcal{D}_{A'}\right)^{-1}\Bigg\{-\frac{1}{\pi_{Y^1}\eta^{-1}}\left[\frac{\pi_c\Delta_{c[\mathcal{R}'/(p^{0\prime}/p^0)]}+\left[1+\pi^0_{X_1}/\left(\pi^1_{X_1}-\pi^0_{X_1}\right)\right]\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}}{\pi_{X_1}\eta/\pi_{Y^1}+(1-\eta)}\right]$$
$$\times\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\pi_{X_1}\eta}{\pi_{Y^1}}+(1-\eta)\right]\Bigg[\frac{\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}}{\left(\pi^1_{X_1}-\pi^0_{X_1}\right)^2}\frac{1}{\pi_{Y^1}\eta^{-1}}$$
$$-\frac{1}{\pi_{Y^1}\eta^{-1}}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\pi_\ell\left(\Delta_{\ell(\omega^0/p^0)}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}+\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\right)\Bigg]$$
$$+\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}/\pi_{Y^1}\eta^{-1}}{\pi_{\bar{X}_1}\eta/\pi_{Y^1}+(1-\eta)}\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\pi_{\bar{X}_1}\eta}{\pi_{Y^1}}+(1-\eta)\right]$$
$$\times\Bigg\{-\frac{1}{\pi_{Y^1}\eta^{-1}}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\pi_\ell\left(\Delta_{\ell(\omega^0/p^0)}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}+\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\right)+\frac{\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}}{\left(\pi^1_{X_1}-\pi^0_{X_1}\right)^2}$$
$$+\frac{\pi_{\bar{X}_0}}{\pi_{Y^1}\eta^{-1}}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}-\frac{\pi_\ell}{\pi_{Y^1}\eta^{-1}}\left(\Delta_{\ell(\omega^0/p^0)}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}+\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\right)$$
$$-\frac{\pi_c}{\pi_{Y^1}\eta^{-1}}\left(\Delta_{c(\omega^0/p^0)}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}+\Delta_{c[\mathcal{R}'/(p^{0\prime}/p^0)]}\right)+1\Bigg\}\Bigg\}$$

$$\mathcal{J}_{21} = \left(\mathcal{D}_{A'}\right)^{-1}\left\{\left[\frac{\pi_{\bar{X}_1}}{\pi_{Y^1}\eta^{-1}}\frac{\pi^0_{X_0}}{\pi^1_{X_1}-\pi^0_{X_1}}+(1-\eta)\right]\right\};$$

$$\mathcal{J}_{22} = \left(\mathcal{D}_{A'}\right)^{-1}\Bigg\{-\frac{1}{\pi_{Y^1}\eta^{-1}}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\pi_\ell\Bigg(\Delta_{\ell(\omega^0/p^0)}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}+\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\Bigg)+\frac{\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}}{\left(\pi^1_{X_1}-\pi^0_{X_1}\right)^2}\frac{1}{\pi_{Y^1}\eta^{-}}$$
$$+\frac{\pi_{\bar{X}_0}}{\pi_{Y^1}\eta^{-1}}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}-\frac{\pi_\ell}{\pi_{Y^1}\eta^{-1}}\Bigg(\Delta_{\ell(\omega^0/p^0)}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}+\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\Bigg)$$
$$-\frac{\pi_c}{\pi_{Y^1}\eta^{-1}}\Bigg(\Delta_{c(\omega^0/p^0)}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}+\Delta_{c[\mathcal{R}'/(p^{0\prime}/p^0)]}\Bigg)+1\Bigg\}.$$

The determinant of the Jacobian Matrix follows from

$$\begin{aligned}
\det([\mathcal{J}]) &= \mathcal{J}_{11}\mathcal{J}_{22}-\mathcal{J}_{12}\mathcal{J}_{21}\\
&= \left(\mathcal{D}_{A'}\right)^{-1}\Bigg\{\Bigg[\frac{\pi^0_{X_0}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\pi_{\bar{X}_1}\eta}{\pi_{Y^1}}+(1-\eta)\Bigg]\Bigg\}\\
&\qquad\times\Bigg\{\frac{\pi_{\bar{X}_0}}{\pi_{Y^1}\eta^{-1}}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}-\frac{\pi_\ell}{\pi_{Y^1}\eta^{-1}}\Bigg[\Delta_{\ell(\omega^0/p^0)}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}+\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\Bigg]\\
&\qquad\qquad-\frac{\pi_c}{\pi_{Y^1}\eta^{-1}}\Bigg[\Delta_{c(\omega^0/p^0)}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}+\Delta_{c[\mathcal{R}'/(p^{0\prime}/p^0)]}\Bigg]+1\Bigg\}\\
&= \Bigg[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\pi_{\bar{X}_1}\eta}{\pi_{Y^1}}+(1-\eta)\Bigg]\\
&\qquad\times\Bigg\{\frac{\pi_{\bar{X}_0}}{\pi_{Y^1}\eta^{-1}}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}-\frac{\pi_\ell}{\pi_{Y^1}\eta^{-1}}\Bigg(\Delta_{\ell(\omega^0/p^0)}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}+\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\Bigg)\\
&\qquad\qquad-\frac{\pi_c}{\pi_{Y^1}\eta^{-1}}\Bigg(\Delta_{c(\omega^0/p^0)}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}+\Delta_{c[\mathcal{R}'/(p^{0\prime}/p^0)]}\Bigg)+1\Bigg\}\\
&\quad\times\Bigg\{-\frac{1}{\pi_{Y^1}\eta^{-1}}\Bigg[\Delta_{c[\mathcal{R}'/(p^{0\prime}/p^0)]}+\Bigg(1+\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\Bigg)\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\Bigg]\\
&\qquad\qquad\times\frac{\Bigg[\dfrac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\dfrac{\pi_{X_1}\eta}{\pi_{Y^1}}+(1-\eta)\Bigg]}{\left[\pi_{X_1}\eta/\pi_{Y^1}+(1-\eta)\right]}\Bigg\}^{-1}.
\end{aligned}$$

$$
\begin{aligned}
\det([\mathcal{J}]) = &\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\pi_{\bar{X}_1}\eta}{\pi_{Y^1}}+(1-\eta)\right] \\
&\times\left\{\frac{1}{\pi_{Y^1}\eta^{-1}}\left[\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\left(\pi_{\bar{X}_0}-\pi_\ell\Delta_{\ell(\omega^0/p^0)}-\pi_c\Delta_{c(\omega^0/p^0)}\right)\right.\right. \\
&\left.\left.-\left(\pi_c\Delta_{c[\mathcal{R}'/(p^{0\prime}/p^0)]}+\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\right)\right]+1\right\} \\
&\times\left\{-\frac{1}{\pi_{Y^1}\eta^{-1}}\left[\frac{\pi_c\Delta_{c[\mathcal{R}'/(p^{0\prime}/p^0)]}+\left[1+\pi^0_{X_1}/\left(\pi^1_{X_1}-\pi^0_{X_1}\right)\right]\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}}{\pi_{X_1}\eta/\pi_{Y^1}+(1-\eta)}\right]\right. \\
&\left.\times\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\pi_{X_1}\eta}{\pi_{Y^1}}+(1-\eta)\right]\right\}^{-1} \\
= &\frac{\pi_{X_1}\eta}{\pi_{Y^1}}+(1-\eta)+\left\{\frac{1}{\pi_{Y^1}\eta^{-1}}\left[\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\left(\pi_{\bar{X}_0}-\pi_\ell\left(\Delta_{\ell(\omega^0/p^0)}\right.\right.\right.\right. \\
&\left.\left.\left.\left.-\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\right)-\pi_c\Delta_{c(\omega^0/p^0)}\right)\right]+1\right\} \\
&\times\left\{-\frac{1}{\pi_{Y^1}\eta^{-1}}\left[\frac{\pi_c\Delta_{c[\mathcal{R}'/(p^{0\prime}/p^0)]}+\left[1+\pi^0_{X_1}/\left(\pi^1_{X_1}-\pi^0_{X_1}\right)\right]\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}}{\pi_{X_1}\eta/\pi_{Y^1}+(1-\eta)}\right]\right\}^{-1}.
\end{aligned}
$$

As for the trace, this is obtained as :

$$
\begin{aligned}
\operatorname{tr}([\mathcal{J}]) &= \mathcal{J}_{11} + \mathcal{J}_{22} \\
&= \Bigg\{ -\frac{1}{\pi_{Y^1}\eta^{-1}} \left[\frac{\pi_c \Delta_{c[\mathcal{R}'/(p^{o\prime}/p^o)]} + \big[1 + \pi^0_{X_1}/\big(\pi^1_{X_1} - \pi^0_{X_1}\big)\big]\pi_\ell \Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^o)]}}{\pi_{X_1}\eta/\pi_{Y^1} + (1-\eta)} \right] \\
&\qquad\qquad \times \left[\frac{1 - \pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \frac{\pi_{X_1}\eta}{\pi_{Y^1}} + (1-\eta) \right] \Bigg\}^{-1} \\
&\times \Bigg\{ -\frac{1}{\pi_{Y^1}\eta^{-1}} \left[\frac{\pi_c \Delta_{c[\mathcal{R}'/(p^{o\prime}/p^o)]} + \big[1 + \pi^0_{X_1}/\big(\pi^1_{X_1} - \pi^0_{X_1}\big)\big]\pi_\ell \Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^o)]}}{\pi_{X_1}\eta/\pi_{Y^1} + (1-\eta)} \right] \\
&\qquad\qquad \times \left[\frac{1 - \pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \frac{\pi_{X_1}\eta}{\pi_{Y^1}} + (1-\eta) \right]^2 \\
&\qquad + \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \frac{\pi_\ell \Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^o)]}/\pi_{Y^1}\eta^{-1}}{\pi_{X_1}\eta/\pi_{Y^1} + (1-\eta)} \left[\frac{1 - \pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \frac{\pi_{X_1}\eta}{\pi_{Y^1}} + (1-\eta) \right]^2 \\
&\qquad - \frac{1}{\pi_{Y^1}\eta^{-1}} \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \pi_\ell \left(\Delta_{\ell(\omega^o/p^o)} \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} + \Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^o)]} \right) + \frac{\Sigma_{\bar{X}_0 \bar{X}_1} \pi_{\bar{X}_0} \pi_{\bar{X}_1}}{\big(\pi^1_{X_1} - \pi^0_{X_1}\big)^2} \frac{1}{\pi_{Y^1}} \\
&\qquad + \frac{\pi_{\bar{X}_0}}{\pi_{Y^1}\eta^{-1}} \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} - \frac{\pi_\ell}{\pi_{Y^1}\eta^{-1}} \left(\Delta_{\ell(\omega^o/p^o)} \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} + \Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^o)]} \right) \\
&\qquad\qquad - \frac{\pi_c}{\pi_{Y^1}\eta^{-1}} \left(\Delta_{c(\omega^o/p^o)} \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} + \Delta_{c[\mathcal{R}'/(p^{o\prime}/p^o)]} \right) + 1 \Bigg\} \\
&= \left[\frac{1 - \pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \frac{\pi_{\bar{X}_1}\eta}{\pi_{Y^1}} + (1-\eta) \right] + \big[\pi_{X_1}\eta/\pi_{Y^1} + (1-\eta)\big] \left[\frac{1 - \pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \frac{\pi_{\bar{X}_1}\eta}{\pi_{Y^1}} + (1-\eta) \right]^{-1} \\
&\quad + \Bigg\{ \frac{\Sigma_{\bar{X}_0 \bar{X}_1} \pi_{\bar{X}_0} \pi_{\bar{X}_1}}{\big(\pi^1_{X_1} - \pi^0_{X_1}\big)^2} \frac{1}{\pi_{Y^1}\eta^{-1}} + \frac{\pi_{\bar{X}_0}}{\pi_{Y^1}\eta^{-1}} \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \\
&\qquad\qquad - \frac{\pi_\ell}{\pi_{Y^1}\eta^{-1}} \Delta_{\ell(\omega^o/p^o)} \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} - \frac{\pi_c}{\pi_{Y^1}\eta^{-1}} \Delta_{c(\omega^o/p^o)} \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} + 1 \Bigg\} \\
&\times \Bigg\{ -\frac{1}{\pi_{Y^1}\eta^{-1}} \left[\frac{\pi_c \Delta_{c[\mathcal{R}'/(p^{o\prime}/p^o)]} + \big[1 + \pi^0_{X_1}/\big(\pi^1_{X_1} - \pi^0_{X_1}\big)\big]\pi_\ell \Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^o)]}}{\pi_{X_1}\eta/\pi_{Y^1} + (1-\eta)} \right] \\
&\qquad\qquad \times \left[\frac{1 - \pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \frac{\pi_{X_1}\eta}{\pi_{Y^1}} + (1-\eta) \right] \Bigg\}^{-1} .
\end{aligned}
$$

Useful at that stage to recall that the components of the Hessian of the utility function list, for

$\mathcal{A} := \big(\Sigma^H_{c\ell}\Sigma^H_{cc'}\pi_c + \Sigma^H_{\ell c}\Sigma^H_{\ell c'}\pi_\ell + \Sigma^H_{c'c}\Sigma^H_{c'\ell}\pi_{c'}\big)$, as:

$$\mathcal{S}\Xi_{cc} = -\Sigma^H_{YY} - \Big[\Sigma^H_{c\ell}\pi_{c'}\big(\Sigma^H_{c'Y}\big)^2 + \Sigma^H_{cc'}\pi_\ell\big(\Sigma^H_{\ell Y}\big)^2 + \Sigma^H_{\ell c'}\big(\pi_\ell\Sigma^H_{\ell Y} + \pi_{c'}\Sigma^H_{c'Y}\big)^2/\pi_\ell\Big] \Big/ \mathcal{A};$$

$$\mathcal{S}\Xi_{c\ell} = -\Sigma^H_{YY} + \Big[-\Sigma^H_{c\ell}\pi_{c'}\big(\Sigma^H_{c'Y}\big)^2 + \Sigma^H_{cc'}\Sigma^H_{\ell Y}\big(\pi_c\Sigma^H_{cY} + \pi_{c'}\Sigma^H_{c'Y}\big) + \Sigma^H_{\ell c'}\Sigma^H_{cY}\big(\pi_\ell\Sigma^H_{\ell Y} + \pi_{c'}\Sigma^H_{c'Y}\big)\Big] \Big/ \mathcal{A};$$

$$\mathcal{S}\Xi_{cc'} = -\Sigma^H_{YY} + \Big[\Sigma^H_{c\ell}\Sigma^H_{c'Y}\big(\pi_c\Sigma^H_{cY} + \pi_\ell\Sigma^H_{\ell Y}\big) - \Sigma^H_{cc'}\pi_\ell\big(\Sigma^H_{\ell Y}\big)^2 + \Sigma^H_{\ell c'}\Sigma^H_{cY}\big(\pi_\ell\Sigma^H_{\ell Y} + \pi_{c'}\Sigma^H_{c'Y}\big)\Big] \Big/ \mathcal{A};$$

$$\mathcal{S}\Xi_{\ell\ell} = -\Sigma^H_{YY} - \Big[\Sigma^H_{c\ell}\pi_{c'}\big(\Sigma^H_{c'Y}\big)^2 + \Sigma^H_{cc'}\big(\pi_c\Sigma^H_{cY} + \pi_{c'}\Sigma^H_{c'Y}\big)^2/\pi_\ell + \Sigma^H_{\ell c'}\pi_c\big(\Sigma^H_{cY}\big)^2\Big] \Big/ \mathcal{A};$$

$$\mathcal{S}\Xi_{\ell c'} = -\Sigma^H_{YY} + \Big[\Sigma^H_{c\ell}\Sigma^H_{c'Y}\big(\pi_c\Sigma^H_{cY} + \pi_\ell\Sigma^H_{\ell Y}\big) + \Sigma^H_{cc'}\Sigma^H_{\ell Y}\big(\pi_c\Sigma^H_{cY} + \pi_{c'}\Sigma^H_{c'Y}\big) - \Sigma^H_{\ell c'}\pi_c\big(\Sigma^H_{cY}\big)^2\Big] \Big/ \mathcal{A};$$

$$\mathcal{S}\Xi_{c'c'} = -\Sigma^H_{YY} - \Big[\Sigma^H_{c\ell}\big(\pi_c\Sigma^H_{cY} + \pi_\ell\Sigma^H_{\ell Y}\big)^2/\pi_{c'} + \Sigma^H_{cc'}\pi_\ell\big(\Sigma^H_{\ell Y}\big)^2 + \Sigma^H_{\ell c'}\pi_c\big(\Sigma^H_{cY}\big)^2\Big] \Big/ \mathcal{A}.$$

Scrutinising the expressions of the demand functions, some early computations are useful :

$$\mathcal{S}\big[\big(\Xi_{c'c'} - \Xi_{cc'}\big) + \big(\Xi_{c\ell} - \Xi_{c'\ell}\big)\big] = -\frac{\Sigma^H_{c\ell}/\pi_{c'}}{\Sigma^H_{c\ell}\Sigma^H_{cc'}\pi_c + \Sigma^H_{\ell c}\Sigma^H_{\ell c'}\pi_\ell + \Sigma^H_{c'c}\Sigma^H_{c'\ell}\pi_{c'}},$$

$$\mathcal{S}\big[\big(\Xi_{\ell\ell} - \Xi_{c\ell}\big) + \big(\Xi_{cc'} - \Xi_{\ell c'}\big)\big] = -\frac{\Sigma^H_{cc'}/\pi_\ell}{\Sigma^H_{c\ell}\Sigma^H_{cc'}\pi_c + \Sigma^H_{\ell c}\Sigma^H_{\ell c'}\pi_\ell + \Sigma^H_{c'c}\Sigma^H_{c'\ell}\pi_{c'}},$$

$$\mathcal{S}\big[\big(\Xi_{cc} - \Xi_{c'c}\big) - \big(\Xi_{c\ell} - \Xi_{c'\ell}\big)\big] = -\frac{\Sigma^H_{c'\ell}/\pi_c}{\Sigma^H_{c\ell}\Sigma^H_{cc'}\pi_c + \Sigma^H_{\ell c}\Sigma^H_{\ell c'}\pi_\ell + \Sigma^H_{c'c}\Sigma^H_{c'\ell}\pi_{c'}},$$

$$\mathcal{S}\big[\big(\Xi_{cc} - \Xi_{\ell c}\big) + \big(\Xi_{\ell\ell} - \Xi_{c\ell}\big)\big] = -\frac{\Sigma^H_{cc'}/\pi_\ell - \Sigma_{\ell c'}/\pi_c}{\Sigma^H_{c\ell}\Sigma^H_{cc'}\pi_c + \Sigma^H_{\ell c}\Sigma^H_{\ell c'}\pi_\ell + \Sigma^H_{c'c}\Sigma^H_{c'\ell}\pi_{c'}},$$

$$\mathcal{S}\big[\big(\Xi_{cc} - \Xi_{c'c}\big) + \big(\Xi_{c'c'} - \Xi_{cc'}\big)\big] = -\frac{\Sigma^H_{cc'}/\pi_\ell - \Sigma_{\ell c'}/\pi_c}{\Sigma^H_{c\ell}\Sigma^H_{cc'}\pi_c + \Sigma^H_{\ell c}\Sigma^H_{\ell c'}\pi_\ell + \Sigma^H_{c'c}\Sigma^H_{c'\ell}\pi_{c'}},$$

$$\mathcal{S}\big[\big(\Xi_{cc} - \Xi_{c'c}\big) - \big(\Xi_{c\ell} - \Xi_{c'\ell}\big)\big] = -\frac{\Sigma^H_{c'\ell}/\pi_c}{\Sigma^H_{c\ell}\Sigma^H_{cc'}\pi_c + \Sigma^H_{\ell c}\Sigma^H_{\ell c'}\pi_\ell + \Sigma^H_{c'c}\Sigma^H_{c'\ell}\pi_{c'}}.$$

On a more canonical basis :

$$\mathcal{S}^2\big[\Xi_{cc}\Xi_{\ell\ell} - \big(\Xi_{c\ell}\big)^2 - \Xi_{c'c}\big(\Xi_{\ell\ell} - \Xi_{c\ell}\big) - \Xi_{c'\ell}\big(\Xi_{cc} - \Xi_{\ell c}\big)\big] = \frac{\Sigma^H_{c'Y}\pi_{c'}}{\pi_c\pi_{c'}\pi_\ell}\frac{1}{\Sigma^H_{c\ell}\Sigma^H_{cc'}\pi_c + \Sigma^H_{\ell c}\Sigma^H_{\ell c'}\pi_\ell + \Sigma^H_{c'c}\Sigma^H_{c'\ell}\pi_{c'}},$$

$$\mathcal{S}^2\big[\Xi_{cc}\Xi_{c'c'} - \big(\Xi_{cc'}\big)^2 - \Xi_{\ell c}\big(\Xi_{c'c'} - \Xi_{cc'}\big) - \Xi_{\ell c'}\big(\Xi_{cc} - \Xi_{c'c}\big)\big] = \frac{\Sigma^H_{\ell Y}\pi_\ell}{\pi_c\pi_{c'}\pi_\ell}\frac{1}{\Sigma^H_{c\ell}\Sigma^H_{cc'}\pi_c + \Sigma^H_{\ell c}\Sigma^H_{\ell c'}\pi_\ell + \Sigma^H_{c'c}\Sigma^H_{c'\ell}\pi_{c'}},$$

$$\mathcal{S}^2\big[\Xi_{\ell\ell}\Xi_{c'c'} - \big(\Xi_{\ell c'}\big)^2 - \Xi_{c\ell}\big(\Xi_{c'c'} - \Xi_{\ell c'}\big) - \Xi_{cc'}\big(\Xi_{\ell\ell} - \Xi_{c'\ell}\big)\big] = \frac{\Sigma^H_{cY}\pi_c}{\pi_c\pi_{c'}\pi_\ell}\frac{1}{\Sigma^H_{c\ell}\Sigma^H_{cc'}\pi_c + \Sigma^H_{\ell c}\Sigma^H_{\ell c'}\pi_\ell + \Sigma^H_{c'c}\Sigma^H_{c'\ell}\pi_{c'}},$$

that in turn implies, replacing :

$$\mathcal{D}_{c\ell c'} = \frac{\pi_c\pi_\ell\pi_{c'}\mathcal{S}}{\pi_c\pi_\ell\pi_{c'}\mathcal{S}}\frac{1}{\Sigma^H_{c\ell}\Sigma^H_{cc'}\pi_c + \Sigma^H_{\ell c}\Sigma^H_{\ell c'}\pi_\ell + \Sigma^H_{c'c}\Sigma^H_{c'\ell}\pi_{c'}}.$$

This finally allows for deriving a more satisfactory picture of the demand functions :

$$\begin{aligned}
\Delta_{c(\omega^o/p^o)} &= \left(\frac{1}{\mathcal{A}}\right)^{-1}\left(\frac{\Sigma^H_{c\ell}\pi_c\pi_\ell}{\pi_c\pi_\ell\pi_{c'}\mathcal{AS}} + \frac{\Sigma^H_{cY}\pi_c}{\pi_c\pi_\ell\pi_{c'}\mathcal{S}}\frac{\pi_{\bar{X}_o}}{\mathcal{A}}\right)\pi_{c'}\pi_\ell\mathcal{S}\\
&= \left(\Sigma^H_{c\ell}\pi_\ell + \Sigma^H_{cY}\pi_{\bar{X}_o}\right),\\
\Delta_{c[\mathcal{R}'/(p^{o\prime}/p^o)]} &= \left(\frac{1}{\mathcal{A}}\right)^{-1}\left(-\frac{\Sigma^H_{cc'}\pi_c\pi_{c'}}{\pi_c\pi_\ell\pi_{c'}\mathcal{AS}} + \frac{\Sigma^H_{cY}\pi_c}{\pi_c\pi_\ell\pi_{c'}\mathcal{S}}\frac{\pi_{c'}}{\mathcal{A}}\right)\pi_{c'}\pi_\ell\mathcal{S}\\
&= \left(\Sigma^H_{cY} - \Sigma^H_{cc'}\right)\pi_{c'},\\
\Delta_{\ell(\omega^o/p^o)} &= \left(\frac{1}{\mathcal{A}}\right)^{-1}\left(-\frac{\Sigma^H_{c\ell}\pi_\ell\pi_{c'} + \Sigma^H_{c'\ell}\pi_\ell\pi_{c'}}{\pi_c\pi_\ell\pi_{c'}\mathcal{AS}} + \frac{\Sigma^H_{\ell Y}\pi_\ell}{\pi_c\pi_\ell\pi_{c'}\mathcal{S}}\frac{\pi_{\bar{X}_o}}{\mathcal{A}}\right)\pi_c\pi_{c'}\mathcal{S}\\
&= \left(-\Sigma^H_{c\ell}\pi_c - \Sigma^H_{\ell c'}\pi_{c'} + \Sigma^H_{\ell Y}\pi_{\bar{X}_o}\right),\\
\Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^o)]} &= \left(\frac{1}{\mathcal{A}}\right)^{-1}\left(-\frac{\Sigma^H_{\ell c'}\pi_\ell\pi_{c'}}{\pi_c\pi_\ell\pi_{c'}\mathcal{AS}} + \frac{\Sigma^H_{\ell Y}\pi_\ell}{\pi_c\pi_\ell\pi_{c'}\mathcal{S}}\frac{\pi_{c'}}{\mathcal{A}}\right)\pi_c\pi_{c'}\mathcal{S}\\
&= \left(\Sigma^H_{\ell Y} - \Sigma^H_{c'\ell}\right)\pi_{c'},\\
\Delta_{c'(\omega^o/p^o)} &= \left(\frac{1}{\mathcal{A}}\right)^{-1}\left(\frac{\Sigma^H_{\ell c'}\pi_\ell\pi_{c'}}{\pi_c\pi_\ell\pi_{c'}\mathcal{AS}} + \frac{\Sigma^H_{c'Y}\pi_{c'}}{\pi_c\pi_\ell\pi_{c'}\mathcal{S}}\frac{\pi_{\bar{X}_o}}{\mathcal{A}}\right)\pi_c\pi_\ell\mathcal{S}\\
&= \left(\Sigma^H_{cc'}\pi_\ell + \Sigma^H_{c'Y}\pi_{\bar{X}_o}\right),\\
\Delta_{c'[\mathcal{R}'/(p^{o\prime}/p^o)]} &= \left(\frac{1}{\mathcal{A}}\right)^{-1}\left(-\frac{\Sigma^H_{cc'}\pi_c\pi_{c'} + \Sigma^H_{c'\ell}\pi_\ell\pi_{c'}}{\pi_c\pi_\ell\pi_{c'}\mathcal{AS}} + \frac{\Sigma^H_{c'Y}\pi_{c'}}{\pi_c\pi_\ell\pi_{c'}\mathcal{S}}\frac{\pi_{c'}}{\mathcal{A}}\right)\pi_c\pi_\ell\mathcal{S}\\
&= \Sigma^H_{cY}\pi_{c'} - \left(\Sigma^H_{cc'}\pi_c + \Sigma^H_{c\ell}\pi_\ell\right).
\end{aligned}$$

Then reconsidering some coefficients that appeared in the determinant and the trace of the Jacobian Matrix :

$$\begin{aligned}
\pi_{\bar{X}_o} - \pi_c\Delta_{c(\omega^o/p^o)} - \pi_\ell\Delta_{\ell(\omega^o/p^o)} &= \pi_{\bar{X}_o} - \pi_c\left(\Sigma^H_{c\ell}\pi_\ell + \Sigma^H_{cY}\pi_{\bar{X}_o}\right)\\
&\qquad - \pi_\ell\left(-\Sigma^H_{c\ell}\pi_c - \Sigma^H_{\ell c'}\pi_{c'} + \Sigma^H_{\ell Y}\pi_{\bar{X}_o}\right)\\
&= \pi_{\bar{X}_o}\left(1 - \Sigma^H_{cY}\pi_c - \Sigma^H_{\ell Y}\pi_\ell\right) - \Sigma^H_{\ell c'}\pi_\ell\pi_{c'}\\
&= \pi_{c'}\left(\Sigma^H_{c'Y}\pi_{\bar{X}_o} + \Sigma_{\ell c'}\pi_\ell\right),\\
\Delta_{c[\mathcal{R}'/(p^{o\prime}/p^o)]}\pi_c + \Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^o)]}\pi_\ell &= \pi_c\left(\Sigma^H_{cY} - \Sigma^H_{cc'}\right)\pi_{c'} + \pi_\ell\left(\Sigma^H_{\ell Y} - \Sigma^H_{c'\ell}\right)\pi_{c'}\\
&= \pi_{c'}\left[\left(\Sigma^H_{cY} - \Sigma^H_{cc'}\right)\pi_c + \left(\Sigma^H_{\ell Y} - \Sigma^H_{c'\ell}\right)\pi_\ell\right].
\end{aligned}$$

The determinant of the Jacobian Matrix hence reformulates to :

$$\begin{aligned}
\det(\mathcal{J}) = &\left[\frac{1 - \pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\frac{\pi_{\bar{X}_1}\eta}{\pi_{Y^1}} + (1-\eta)\right]\\
&\times\left\{\frac{\pi_{c'}}{\pi_{Y^1}\eta^{-1}}\left[\frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\left(\Sigma^H_{c'Y}\pi_{\bar{X}_o} + \Sigma_{\ell c'}\pi_\ell\right)\right.\right.\\
&\qquad\left.\left. - \left[\left(\Sigma^H_{cY} - \Sigma^H_{cc'}\right)\pi_c + \left(\Sigma^H_{\ell Y} - \Sigma^H_{c'\ell}\right)\pi_\ell\right]\right] + 1\right\}
\end{aligned}$$

$$\times \left\{ -\frac{\pi_{c'}}{\pi_{Y^1}\eta^{-1}} \left[\left[\left(\Sigma^H_{cY} - \Sigma^H_{cc'} \right) \pi_c + \left(\Sigma^H_{\ell Y} - \Sigma^H_{c'\ell} \right) \pi_\ell \right] \right. \right.$$

$$\left. + \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \left(\Sigma^H_{\ell Y} - \Sigma^H_{c'\ell} \right) \pi_\ell \right] \cdot$$

$$\left. \times \left[\frac{1 - \pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \frac{\pi_{X_1}\eta}{\pi_{Y^1}} + (1-\eta) \right] \Big/ \left[\pi_{X_1}\eta/\pi_{Y^1} + (1-\eta) \right] \right\}^{-1}$$

$$= \left\{ \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \left(\Sigma_{\ell c'}\pi_\ell + \Sigma^H_{c'Y}\pi_{\bar{X}_0} \right) + \left(\Sigma^H_{cc'} - \Sigma^H_{cY} \right)\pi_c + \left(\Sigma^H_{c'\ell} - \Sigma^H_{\ell Y} \right)\pi_\ell + 1 \right\}$$

$$\times \left\{ \left[\left(\Sigma^H_{cc'} - \Sigma^H_{cY} \right)\pi_c + \left(1 + \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \right) \left(\Sigma^H_{c'\ell} - \Sigma^H_{\ell Y} \right)\pi_\ell \right] \right.$$

$$\left. \Big/ \left[\pi_{X_1}\eta/\pi_{Y^1} + (1-\eta) \right] \right\}^{-1}$$

$$= \pi_{X_1}\eta/\pi_{Y^1} + (1-\eta) + \left\{ \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \left(\Sigma^H_{c'Y}\pi_{X_0} + \Sigma^H_{\ell Y}\pi_\ell \right) + 1 \right\}$$

$$\times \left\{ \left[\left(\Sigma^H_{cc'} - \Sigma^H_{cY} \right)\pi_c + \left(1 + \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \right) \left(\Sigma^H_{c'\ell} - \Sigma^H_{\ell Y} \right)\pi_\ell \right] \right.$$

$$\left. \Big/ \left[\pi_{X_1}\eta/\pi_{Y^1} + (1-\eta) \right] \right\}^{-1}.$$

As for the trace of the Jacobian Matrix, it is available as :

$$\operatorname{tr}(\mathcal{J}) = \left[\frac{1 - \pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \frac{\pi_{\bar{X}_1}\eta}{\pi_{Y^1}} + (1-\eta) \right]$$

$$+ \left\{ \frac{\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}}{\left(\pi^1_{X_1} - \pi^0_{X_1} \right)^2} \frac{1}{\pi_{Y^1}\eta^{-1}} + \frac{\pi_{c'}}{\pi_{Y^1}\eta^{-1}} \left[\frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \left(\Sigma^H_{c'Y}\pi_{\bar{X}_0} + \Sigma_{\ell c'}\pi_\ell \right) \right. \right.$$

$$\left. \left. - \left[\left(\Sigma^H_{cY} - \Sigma^H_{cc'} \right)\pi_c + \left(\Sigma^H_{\ell Y} - \Sigma^H_{c'\ell} \right)\pi_\ell \right] \right] + 1 \right\}$$

$$\times \left\{ -\frac{\pi_{c'}}{\pi_{Y^1}\eta^{-1}} \left[\left[\left(\Sigma^H_{cY} - \Sigma^H_{cc'} \right)\pi_c + \left(\Sigma^H_{\ell Y} - \Sigma^H_{c'\ell} \right)\pi_\ell \right] \right. \right.$$

$$\left. + \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \left(\Sigma^H_{\ell Y} - \Sigma^H_{c'\ell} \right)\pi_\ell \right]$$

$$\left. \times \left[\frac{1 - \pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \frac{\pi_{X_1}\eta}{\pi_{Y^1}} + (1-\eta) \right] \Big/ \left[\pi_{X_1}\eta/\pi_{Y^1} + (1-\eta) \right] \right\}^{-1}$$

$$= \left[\frac{1 - \pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \frac{\pi_{\bar{X}_1}\eta}{\pi_{Y^1}} + (1-\eta) \right]$$

$$+ \left\{ \frac{\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}}{\left(\pi^1_{X_1} - \pi^0_{X_1} \right)^2} \frac{1}{\pi_{Y^1}\eta^{-1}} + \left[\frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \left(\Sigma_{\ell c'}\pi_\ell + \Sigma^H_{c'Y}\pi_{\bar{X}_0} \right) \right. \right.$$

$$\left. \left. + \left(\Sigma^H_{cc'} - \Sigma^H_{cY} \right)\pi_c + \left(\Sigma^H_{c'\ell} - \Sigma^H_{\ell Y} \right)\pi_\ell \right] \left[\frac{\pi_{\bar{X}_1}\eta}{\pi_{Y^1}} + (1-\eta) \right] + 1 \right\}$$

$$\times \left\{ \left[\left(\Sigma^H_{cc'} - \Sigma^H_{cY} \right)\pi_c + \left(\Sigma^H_{c'\ell} - \Sigma^H_{\ell Y} \right)\pi_\ell + \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \left(\Sigma^H_{c'\ell} - \Sigma^H_{\ell Y} \right)\pi_\ell \right] \right.$$

$$\left. \times \left[\frac{1 - \pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \frac{\pi_{X_1}\eta}{\pi_{Y^1}} + (1-\eta) \right] \Big/ \left[\pi_{\bar{X}_1}\eta/\pi_{Y^1} + (1-\eta) \right] \right\}^{-1}$$

It hence reformulates to :

$$
\begin{aligned}
\operatorname{tr}(\mathcal{J}) = {} & \pi_{\bar{X}_1}\eta/\pi_{Y^1} + (1-\eta) + \frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\pi_{\bar{X}_1}\eta}{\pi_{Y^1}} + (1-\eta) \\
& + \left\{ \frac{\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}}{\left(\pi^1_{X_1}-\pi^0_{X_1}\right)^2}\frac{1}{\pi_{Y^1}\eta^{-1}} + \left[\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\left(\Sigma_{\ell Y}\pi_\ell + \Sigma^H_{c'Y}\pi_{\bar{X}_0}\right)\right] + 1 \right\} \\
& \times \left\{ \left[\left(\Sigma^H_{cc'} - \Sigma^H_{cY}\right)\pi_c + \left(1 + \frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\right)\left(\Sigma^H_{c'\ell} - \Sigma^H_{\ell Y}\right)\pi_\ell \right] \right. \\
& \qquad \left. \times \left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\pi_{X_1}\eta}{\pi_{Y^1}} + (1-\eta)\right] \Big/ \left[\pi_{\bar{X}_1}\eta/\pi_{Y^1} + (1-\eta)\right] \right\}^{-1}.
\end{aligned}
$$

Then computing $\mathcal{Z}(-1) = 1 + \operatorname{tr}(\mathcal{J}) + \det(\mathcal{J})$ and $\mathcal{Z}(+1) = 1 - \operatorname{tr}(\mathcal{J}) + \det(\mathcal{J})$:

$$
\begin{aligned}
\mathcal{Z}(-1) = {} & \left[1 + \frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\pi_{X_1}}{\pi_{Y^1}}\eta + (1-\eta)\right] \\
& \times \left[1 + \left\{ \left[\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\left(\Sigma_{\ell c'}\pi_\ell + \Sigma^H_{c'Y}\pi_{\bar{X}_0}\right) \right.\right.\right. \\
& \qquad \left.\left. + \left(\Sigma^H_{cc'} - \Sigma^H_{cY}\right)\pi_c + \left(\Sigma^H_{c'\ell} - \Sigma^H_{\ell Y}\right)\pi_\ell \right]\left[\frac{\pi_{\bar{X}_1}\eta}{\pi_{Y^1}} + (1-\eta)\right] + 1 \right\} \\
& \qquad \times \left\{ \left[\left(\Sigma^H_{cc'} - \Sigma^H_{cY}\right)\pi_c + \left(\Sigma^H_{c'\ell} - \Sigma^H_{\ell Y}\right)\pi_\ell \right.\right. \\
& \qquad \left.\left.\left. + \frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\left(\Sigma^H_{c'\ell} - \Sigma^H_{\ell Y}\right)\pi_\ell \right]\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\pi_{X_1}\eta}{\pi_{Y^1}} + (1-\eta)\right] \right\}^{-1} \right] \\
& + \frac{\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}}{\left(\pi^1_{X_1}-\pi^0_{X_1}\right)^2}\frac{1}{\pi_{Y^1}\eta^{-1}} \left\{ \left[\left(\Sigma^H_{cc'} - \Sigma^H_{cY}\right)\pi_c + \left(\Sigma^H_{c'\ell} - \Sigma^H_{\ell Y}\right)\pi_\ell \right.\right. \\
& \qquad \left.\left. + \frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\left(\Sigma^H_{c'\ell} - \Sigma^H_{\ell Y}\right)\pi_\ell \right]\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\pi_{X_1}\eta}{\pi_{Y^1}} + (1-\eta)\right] \right\}^{-1}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{Z}(+1) = {} & \left[1 - \frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\pi_{X_1}}{\pi_{Y^1}}\eta - (1-\eta)\right] \\
& \times \left[1 - \left\{ \left[\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\left(\Sigma_{\ell c'}\pi_\ell + \Sigma^H_{c'Y}\pi_{\bar{X}_0}\right) \right.\right.\right. \\
& \qquad \left.\left. + \left(\Sigma^H_{cc'} - \Sigma^H_{cY}\right)\pi_c + \left(\Sigma^H_{c'\ell} - \Sigma^H_{\ell Y}\right)\pi_\ell \right]\left[\frac{\pi_{\bar{X}_1}\eta}{\pi_{Y^1}} + (1-\eta)\right] + 1 \right\} \\
& \qquad \times \left\{ \left[\left(\Sigma^H_{cc'} - \Sigma^H_{cY}\right)\pi_c + \left(\Sigma^H_{c'\ell} - \Sigma^H_{\ell Y}\right)\pi_\ell \right.\right. \qquad\qquad v \\
& \qquad \left.\left.\left. + \frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\left(\Sigma^H_{c'\ell} - \Sigma^H_{\ell Y}\right)\pi_\ell \right]\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\pi_{X_1}\eta}{\pi_{Y^1}} + (1-\eta)\right] \right\}^{-1} \right] \\
& - \frac{\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}}{\left(\pi^1_{X_1}-\pi^0_{X_1}\right)^2}\frac{1}{\pi_{Y^1}\eta^{-1}} \left\{ \left[\left(\Sigma^H_{cc'} - \Sigma^H_{cY}\right)\pi_c + \left(\Sigma^H_{c'\ell} - \Sigma^H_{\ell Y}\right)\pi_\ell \right.\right. \\
& \qquad \left.\left. + \frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\left(\Sigma^H_{c'\ell} - \Sigma^H_{\ell Y}\right)\pi_\ell \right]\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\pi_{X_1}\eta}{\pi_{Y^1}} + (1-\eta)\right] \right\}^{-1}
\end{aligned}
$$

The expression of $\mathcal{Z}(+1)$ reformulates to:

$$
\begin{aligned}
\mathcal{Z}(+1) = {} & \left[1 - \frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\pi_{X_1}}{\pi_{Y^1}}\eta - (1-\eta)\right] \\
& \times \Bigg[\Bigg\{\Big[\big(\Sigma^H_{cc'} - \Sigma^H_{cY}\big)\pi_c + \big(\Sigma^H_{c'\ell} - \Sigma^H_{\ell Y}\big)\pi_\ell\Big]\frac{1-\pi^1_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\pi_{X_1}\eta}{\pi_{Y^1}} \\
& \qquad + \frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\Bigg\{\big(\Sigma^H_{c'\ell} - \Sigma^H_{\ell Y}\big)\pi_\ell\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\pi_{X_1}\eta}{\pi_{Y^1}} + (1-\eta)\right] \\
& \qquad\qquad - \big(\Sigma_{\ell c'}\pi_\ell + \Sigma^H_{c'Y}\pi_{\bar{X}_0}\big)\left[\frac{\pi_{X_1}\eta}{\pi_{Y^1}} + (1-\eta)\right]\Bigg\} - 1\Bigg\} \\
& \qquad \times \Bigg\{\Bigg[\big(\Sigma^H_{cc'} - \Sigma^H_{cY}\big)\pi_c + \big(\Sigma^H_{c'\ell} - \Sigma^H_{\ell Y}\big)\pi_\ell \\
& \qquad\quad + \frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\big(\Sigma^H_{c'\ell} - \Sigma^H_{\ell Y}\big)\pi_\ell\Bigg]\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\pi_{X_1}\eta}{\pi_{Y^1}} + (1-\eta)\right]\Bigg\}^{-1}\Bigg] \\
& - \frac{\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}}{\big(\pi^1_{X_1}-\pi^0_{X_1}\big)^2}\frac{1}{\pi_{Y^1}\eta^{-1}}\Bigg\{\Bigg[\big(\Sigma^H_{cc'} - \Sigma^H_{cY}\big)\pi_c + \big(\Sigma^H_{c'\ell} - \Sigma^H_{\ell Y}\big)\pi_\ell \\
& \qquad\quad + \frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\big(\Sigma^H_{c'\ell} - \Sigma^H_{\ell Y}\big)\pi_\ell\Bigg]\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\pi_{X_1}\eta}{\pi_{Y^1}} + (1-\eta)\right]\Bigg\}^{-1}
\end{aligned}
$$

The details of Proposition 1 are immediate from a careful parameter examination of the scopes for $\det(\mathcal{J}) \lessgtr 1$, $\mathcal{Z}(-1) \lessgtr 0$ and $\mathcal{Z}(+1) \lessgtr 0$. △

VIII.2 – Proof of Proposition 2.

The raw form of the linearised version of this dynamical system in a neighbourhood of the steady state is available as :

$$
\begin{bmatrix}
1 & -\dfrac{\partial\mathcal{F}^1}{\partial X_0}\left[-\dfrac{\partial\ell}{\partial\big[\mathcal{R}'/(p^{0\prime}/p^0)\big]}\left[\dfrac{\partial\big(\omega^1/p^0\big)}{\partial\big(p^1/p^0\big)}\dfrac{1}{p^1/p^0} + (1-\eta)\right]\right] & 0 \\
0 & A'_{22} & 0 \\
0 & -\left[\dfrac{\partial\big(\omega^1/p^0\big)}{\partial\big(p^1/p^0\big)}\dfrac{1}{p^1/p^0} + (1-\eta)\right]\big(B/p^0\big) & 1
\end{bmatrix}
\begin{bmatrix}
\Delta X_{1,t+1} \\
\Delta\big(p^1_{t+1}/p^0_{t+1}\big) \\
\Delta\big(B_{t+1}/p^0_{t+1}\big)
\end{bmatrix}
$$

$$
+ \begin{bmatrix}
-\dfrac{\partial\mathcal{F}^1}{\partial X_1} - (1-\eta) & A_{12} & 0 \\
-\big(p^1/p^0\big)\left[\dfrac{\partial\mathcal{F}^1}{\partial X_1} + (1-\eta)\right] & A_{22} & -M \\
0 & \left[\dfrac{\omega^1/p^0}{p^1/p^0} + (1-\eta)\right]\dfrac{B/p^0}{p^1/p^0} & -\left[\dfrac{\omega^1/p^0}{p^1/p^0} + (1-\eta)\right]
\end{bmatrix}
\begin{bmatrix}
\Delta X_{1,t} \\
\Delta\big(p^1_t/p^0_t\big) \\
\Delta\big(B_t/p^0_t\big)
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\begin{aligned}
A'_{22} = {} & \left(\omega^0/p^0\right)\left[-\frac{\partial \ell}{\partial\left[\mathcal{R}'/(p^{0\prime}/p^0)\right]}\left[\frac{\partial\left(\omega^1/p^0\right)}{\partial\left(p^1/p^0\right)}\frac{1}{p^1/p^0}+(1-\eta)\right]\right] - \frac{\partial c}{\partial\left[\mathcal{R}'/(p^{0\prime}/p^0)\right]}\left[\frac{\partial\left(\omega^1/p^0\right)}{\partial\left(p^1/p^0\right)}\frac{1}{p^1/p^0}+(1-\right. \\
& - \left(p^1/p^0\right)\frac{\partial \mathcal{F}^1}{\partial X_0}\left[-\frac{\partial \ell}{\partial\left(\omega^0/p^0\right)}\left[\frac{\partial\left(\omega^1/p^0\right)}{\partial\left(p^1/p^0\right)}\frac{1}{p^1/p^0}+(1-\eta)\right]\right], \\
A_{12} = {} & -\frac{\partial \mathcal{F}^1}{\partial\left(p^1/p^0\right)} - \frac{\partial \mathcal{F}^1}{\partial X_0}\left[-\frac{\partial \ell}{\partial\left(\omega^0/p^0\right)}\frac{\partial\left(\omega^0/p^0\right)}{\partial\left(p^1/p^0\right)} - \frac{\partial \ell}{\partial\left[\mathcal{R}'/(p^{0\prime}/p^0)\right]}\left[-\left(\frac{\partial\left(\omega^1/p^0\right)}{\partial\left(p^1/p^0\right)}\frac{1}{p^1/p^0}+(1-\eta)\right)\right]\frac{1}{p^1/p\ldots}\right. \\
A_{22} = {} & \left(X_0 - \frac{\partial c}{\partial\left(\omega^0/p^0\right)}\right)\frac{\partial\left(\omega^0/p^0\right)}{\partial\left(p^1/p^0\right)} - \frac{\partial c}{\partial\left[\mathcal{R}'/(p^{0\prime}/p^0)\right]}\left[-\left(\frac{\partial\left(\omega^1/p^0\right)}{\partial\left(p^1/p^0\right)}\frac{1}{p^1/p^0}+(1-\eta)\right)\right]\frac{1}{p^1/p^0} - \left[\mathcal{F}^1 + (1\ldots\right. \\
& + \left(\omega^0/p^0\right)\left(-\frac{\partial \ell}{\partial \omega^0}\right)\frac{\partial\left(\omega^0/p^0\right)}{\partial\left(p^1/p^0\right)} + \left(\omega^0/p^0\right)\left(-\frac{\partial \ell}{\partial\left[\mathcal{R}'/(p^{0\prime}/p^0)\right]}\right)\left[-\left(\frac{\partial\left(\omega^1/p^0\right)}{\partial\left(p^1/p^0\right)}\frac{1}{p^1/p^0}+(1-\eta)\right)\right] \\
& - \left(p^1/p^0\right)\frac{\partial \mathcal{F}^1}{\partial X_0}\left[-\frac{\partial \ell}{\partial \omega^0}\frac{\partial\left(\omega^0/p^0\right)}{\partial\left(p^1/p^0\right)} - \frac{\partial \ell}{\partial\left[\mathcal{R}'/(p^{0\prime}/p^0)\right]}\left[-\left(\frac{\partial\left(\omega^1/p^0\right)}{\partial\left(p^1/p^0\right)}\frac{1}{p^1/p^0}+(1-\eta)\right)\right]\frac{1}{p^1/p^0}\right].
\end{aligned}
$$

This linearisation of the raw form of the dynamical system delivers, in elasticities form :

$$
\begin{bmatrix}
1 & \mathcal{A}'_{12} & 0 \\
0 & \mathcal{A}'_{22} & 0 \\
0 & -\frac{1}{\pi_{X_1}\eta/\pi_{Y^1}+(1-\eta)}\left[\Delta_{(\omega^1/p^0)(p^1/p^0)}\frac{\pi_{X_1}\eta}{\pi_{Y^1}}+(1-\eta)\right] & 1
\end{bmatrix}
\begin{bmatrix}
\mathcal{X}_{1,t+1} \\
\mathcal{P}^1_{t+1}-\mathcal{P}^0_{t+1} \\
\mathcal{B}_{t+1}-\mathcal{P}^0_{t+1}
\end{bmatrix}
-
\begin{bmatrix}
\Delta_{Y^1X_1}\eta+(1-\eta) & \mathcal{A}_{12} & 0 \\
\Delta_{Y^1X_1}\eta+(1-\eta) & \mathcal{A}_{22} & \frac{M\left(B/p^0\right)^\star}{\pi_{Y^1}\eta^{-1}} \\
0 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
\mathcal{X}_{1,t} \\
\mathcal{P}^1_{t}-\mathcal{P}^0_{t} \\
\mathcal{B}_{t}-\mathcal{P}^0_{t}
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0
\end{bmatrix},
$$

$$
\begin{aligned}
\mathcal{A}'_{12} = {} & \Delta_{Y^1X_0}\eta\frac{\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}/\pi_{X_0}}{\pi_{\bar{X}_1}\eta/\pi_{Y^1}+(1-\eta)}\left\{\left[\Delta_{(\omega^1/p^0)(p^1/p^0)}\frac{\pi_{\bar{X}_1}\eta}{\pi_{Y^1}}+(1-\eta)\right]\right\}, \\
\mathcal{A}'_{22} = {} & -\frac{1}{\pi_{Y^1}\eta^{-1}}\left[\frac{\pi_c\Delta_{c[\mathcal{R}'/(p^{0\prime}/p^0)]}+\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}}{\pi_{X_1}\eta/\pi_{Y^1}+(1-\eta)}\right]\left[\Delta_{(\omega^1/p^0)(p^1/p^0)}\frac{\pi_{X_1}\eta}{\pi_{Y^1}}+(1-\eta)\right] \\
& + \Delta_{Y^1X_0}\eta\frac{\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}/\pi_{X_0}}{\pi_{\bar{X}_1}\eta/\pi_{Y^1}+(1-\eta)}\left\{\left[\Delta_{\omega^1(p^1/p^0)}\frac{\pi_{\bar{X}_1}\eta}{\pi_{Y^1}}+(1-\eta)\right]\right\}, \\
\mathcal{A}_{12} = {} & \Delta_{Y^1X_0}\eta\frac{\pi_\ell}{\pi_{X_0}}\left[-\Delta_{\ell(\omega^0/p^0)}\Delta_{(\omega^0/p^0)(p^1/p^0)}+\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\right]+\Delta_{Y^1(p^1/p^0)}\eta, \\
\mathcal{A}_{22} = {} & \Delta_{Y^1X_0}\eta\frac{\pi_\ell}{\pi_{X_0}}\left[-\Delta_{\ell(\omega^0/p^0)}\Delta_{(\omega^0/p^0)(p^1/p^0)}+\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\right]+\Delta_{Y^1(p^1/p^0)}\eta \\
& - \frac{\pi_{\bar{X}_0}}{\pi_{Y^1}\eta^{-1}}\Delta_{(\omega^0/p^0)(p^1/p^0)} - \frac{\pi_\ell}{\pi_{Y^1}\eta^{-1}}\left[-\Delta_{\ell(\omega^0/p^0)}\Delta_{(\omega^0/p^0)(p^1/p^0)}+\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\right] \\
& - \frac{\pi_c}{\pi_{Y^1}\eta^{-1}}\left[-\Delta_{c(\omega^0/p^0)}\Delta_{(\omega^0/p^0)(p^1/p^0)}+\Delta_{c[\mathcal{R}'/(p^{0\prime}/p^0)]}\right]+1.
\end{aligned}
$$

More explicitly, integrating the holding of $\pi_{X_1}\eta/\pi_{Y^1}+(1-\eta)=1$ at a golden rule steady state, it derives :

$$\begin{bmatrix} 1 & \mathcal{A}'_{12} & 0 \\ 0 & \mathcal{A}'_{22} & 0 \\ 0 & -\left[\dfrac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\eta+(1-\eta)\right] & 1 \end{bmatrix} \begin{bmatrix} \mathcal{X}_{1,t+1} \\ \mathcal{P}^1_{t+1}-\mathcal{P}^0_{t+1} \\ \mathcal{B}_{t+1}-\mathcal{P}^0_{t+1} \end{bmatrix}$$
$$-\begin{bmatrix} \dfrac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\eta+(1-\eta) & \mathcal{A}_{12} & 0 \\ \dfrac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\eta+(1-\eta) & \mathcal{A}_{22} & \dfrac{M\left(B/p^0\right)^\star}{\pi_{Y^1}\eta^{-1}} \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \mathcal{X}_{1,t} \\ \mathcal{P}^1_t-\mathcal{P}^0_t \\ \mathcal{B}_t-\mathcal{P}^0_t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\mathcal{A}'_{12} = -\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\pi_{X_0}}{\pi_{Y^1}}\eta\frac{\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}/\pi_{X_0}}{\pi_{\bar{X}_1}\eta/\pi_{Y^1}+(1-\eta)}\left\{\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\eta+(1-\eta)\right]\right\},$$

$$\mathcal{A}'_{22} = -\frac{1}{\pi_{Y^1}\eta^{-1}}\left[\frac{\pi_c\Delta_{c[\mathcal{R}'/(p^{0\prime}/p^0)]}+\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}}{\pi_{X_1}\eta/\pi_{Y^1}+(1-\eta)}\right]\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\eta+(1-\eta)\right]$$
$$-\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\pi_{X_0}}{\pi_{Y^1}}\eta\frac{\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}/\pi_{X_0}}{\pi_{\bar{X}_1}\eta/\pi_{Y^1}+(1-\eta)}\left\{\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\eta+(1-\eta)\right]\right\},$$

$$\mathcal{A}_{12} = -\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\pi_{X_0}}{\pi_{Y^1}}\eta\frac{\pi_\ell}{\pi_{X_0}}\left[\Delta_{\ell(\omega^0/p^0)}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}+\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\right]+\frac{\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}}{\left(\pi^1_{X_1}-\pi^0_{X_1}\right)^2}\frac{1}{\pi_{Y^1}\eta^{-1}},$$

$$\mathcal{A}_{22} = -\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\pi_{X_0}}{\pi_{Y^1}}\eta\frac{\pi_\ell}{\pi_{X_0}}\left[\Delta_{\ell(\omega^0/p^0)}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}+\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\right]+\frac{\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}}{\left(\pi^1_{X_1}-\pi^0_{X_1}\right)^2}\frac{1}{\pi_{Y^1}\eta^{-1}}$$
$$-\frac{\pi_{\bar{X}_0}}{\pi_{Y^1}\eta^{-1}}\Delta_{(\omega^0/p^0)(p^1/p^0)}-\frac{\pi_\ell}{\pi_{Y^1}\eta^{-1}}\left[\Delta_{\ell(\omega^0/p^0)}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}+\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\right]$$
$$-\frac{\pi_c}{\pi_{Y^1}\eta^{-1}}\left[\Delta_{c(\omega^0/p^0)}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}+\Delta_{c[\mathcal{R}'/(p^{0\prime}/p^0)]}\right]+1.$$

Letting

$$\mathcal{D}_{A'} = -\frac{1}{\pi_{Y^1}\eta^{-1}}\left[\pi_c\Delta_{c[\mathcal{R}'/(p^{0\prime}/p^0)]}+\left(1+\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\right)\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\right]\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\eta+(1-\eta)\right],$$

the transposed form of the co-matrix that pre-multiplies $\left[\, \mathcal{X}_{1,t+1} \quad \mathcal{Q}_{t+1} \quad \mathcal{B}_{t+1} \,\right]'$ is available as :

$$\begin{bmatrix} \mathcal{A}'_{22} & -\mathcal{A}'_{12} & 0 \\ 0 & 1 & 0 \\ 0 & \dfrac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\eta+(1-\eta) & \mathcal{A}'_{22} \end{bmatrix},$$

whence the components of the Jacobian Matrix :

$$\mathcal{J}_{11} = \left(\mathcal{D}_{A'}\right)^{-1}\left\{-\frac{1}{\pi_{Y^1}\eta^{-1}}\left(\pi_c\Delta_{c[\mathcal{R}'/(p^{o\prime}/p^{o})]} + \pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^{o})]}\right)\left[\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta + (1-\eta)\right]^2\right\},$$

$$\begin{aligned}\mathcal{J}_{12} = \left(\mathcal{D}_{A'}\right)^{-1}\Bigg\{&-\frac{1}{\pi_{Y^1}\eta^{-1}}\left(\pi_c\Delta_{c[\mathcal{R}'/(p^{o\prime}/p^{o})]} + \pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^{o})]}\right)\left[\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta + (1-\eta)\right]\\
&\times\Bigg[-\frac{1}{\pi_{Y^1}\eta^{-1}}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\pi_\ell\left(\Delta_{\ell(\omega^{o}/p^{o})}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}} + \Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^{o})]}\right) + \frac{\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}}{\left(\pi^{1}_{X_1}-\pi^{0}_{X_1}\right)^2}\frac{1}{\pi_{Y^1}}\\
&+\frac{1}{\pi_{Y^1}\eta^{-1}}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^{o})]}\left[\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta + (1-\eta)\right]\\
&\times\Bigg[\frac{\pi_{\bar{X}_0}}{\pi_{Y^1}\eta^{-1}}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}} - \frac{\pi_\ell}{\pi_{Y^1}\eta^{-1}}\left(\Delta_{\ell(\omega^{o}/p^{o})}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}} + \Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^{o})]}\right)\\
&\qquad - \frac{\pi_c}{\pi_{Y^1}\eta^{-1}}\left(\Delta_{c(\omega^{o}/p^{o})}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}} + \Delta_{c[\mathcal{R}'/(p^{o\prime}/p^{o})]}\right) + 1\Bigg]\Bigg\},\end{aligned}$$

$$\mathcal{J}_{13} = \left(\mathcal{D}_{A'}\right)^{-1}\left\{\frac{1}{\pi_{Y^1}\eta^{-1}}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^{o})]}\left[\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta + (1-\eta)\right]\frac{M\left(B/p^{o}\right)^{\star}}{\pi_{Y^1}\eta^{-1}}\right\},$$

$$\mathcal{J}_{21} = \left(\mathcal{D}_{A'}\right)^{-1}\left\{\left[\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta + (1-\eta)\right]\right\},$$

$$\begin{aligned}\mathcal{J}_{22} = \left(\mathcal{D}_{A'}\right)^{-1}\Bigg\{&-\frac{1}{\pi_{Y^1}\eta^{-1}}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\pi_\ell\left(\Delta_{\ell(\omega^{o}/p^{o})}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}} + \Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^{o})]}\right) + \frac{\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}}{\left(\pi^{1}_{X_1}-\pi^{0}_{X_1}\right)^2}\frac{1}{\pi_{Y^1}\eta^{-}}\\
&+\frac{\pi_{\bar{X}_0}}{\pi_{Y^1}\eta^{-1}}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}} - \frac{\pi_\ell}{\pi_{Y^1}\eta^{-1}}\left(\Delta_{\ell(\omega^{o}/p^{o})}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}} + \Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^{o})]}\right)\\
&\qquad - \frac{\pi_c}{\pi_{Y^1}\eta^{-1}}\left(\Delta_{c(\omega^{o}/p^{o})}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}} + \Delta_{c[\mathcal{R}'/(p^{o\prime}/p^{o})]}\right) + 1\Bigg\},\end{aligned}$$

$$\mathcal{J}_{23} = \left(\mathcal{D}_{A'}\right)^{-1}\left\{\frac{M\left(B/p^{o}\right)^{\star}}{\pi_{Y^1}\eta^{-1}}\right\},$$

$$\mathcal{J}_{31} = \left(\mathcal{D}_{A'}\right)^{-1}\left\{\left[\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta + (1-\eta)\right]^2\right\},$$

$$\begin{aligned}\mathcal{J}_{32} = \left(\mathcal{D}_{A'}\right)^{-1}\Bigg\{&\left[\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta + (1-\eta)\right]\\
&\times\Bigg[-\frac{1}{\pi_{Y^1}\eta^{-1}}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\pi_\ell\left(\Delta_{\ell(\omega^{o}/p^{o})}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}} + \Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^{o})]}\right) + \frac{\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}}{\left(\pi^{1}_{X_1}-\pi^{0}_{X_1}\right)^2}\frac{}{\pi_{Y}}\\
&\quad+\frac{\pi_{\bar{X}_0}}{\pi_{Y^1}\eta^{-1}}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}} - \frac{\pi_\ell}{\pi_{Y^1}\eta^{-1}}\left(\Delta_{\ell(\omega^{o}/p^{o})}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}} + \Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^{o})]}\right)\\
&\qquad - \frac{\pi_c}{\pi_{Y^1}\eta^{-1}}\left(\Delta_{c(\omega^{o}/p^{o})}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}} + \Delta_{c[\mathcal{R}'/(p^{o\prime}/p^{o})]}\right) + 1\Bigg]\\
&+\frac{1}{\pi_{Y^1}\eta^{-1}}\left[\pi_c\Delta_{c[\mathcal{R}'/(p^{o\prime}/p^{o})]} + \left(1+\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\right)\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^{o})]}\right]\left[\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta + (1-\eta)\right.\end{aligned}$$

$$\begin{aligned}\mathcal{J}_{33} = \left(\mathcal{D}_{A'}\right)^{-1}\Bigg\{&\left[\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta + (1-\eta)\right]\frac{M\left(B/p^{o}\right)^{\star}}{\pi_{Y^1}\eta^{-1}}\\
&-\frac{1}{\pi_{Y^1}\eta^{-1}}\left[\pi_c\Delta_{c[\mathcal{R}'/(p^{o\prime}/p^{o})]} + \left(1+\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\right)\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^{o})]}\right]\left[\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta + (1-\eta)\right.\end{aligned}$$

The trace of the Jacobian Matrix hence emerges as :

$$
\begin{aligned}
\operatorname{tr}(\mathcal{J}) &= \mathcal{J}_{11}+\mathcal{J}_{22}+\mathcal{J}_{33}\\
&= \left(\mathcal{D}_{A'}\right)^{-1}\Bigg\{-\frac{1}{\pi_{Y^1}\eta^{-1}}\left(\pi_c\Delta_{c[\mathcal{R}'/(p^{o\prime}/p^{o})]}+\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^{o})]}\right)\left[\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta+(1-\eta)\right]^2\\
&\qquad -\frac{1}{\pi_{Y^1}\eta^{-1}}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\pi_\ell\left(\Delta_{\ell(\omega^{o}/p^{o})}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}+\Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^{o})]}\right)+\frac{\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}}{\left(\pi^{1}_{X_1}-\pi^{0}_{X_1}\right)^2}\,\frac{\cdots}{\pi_{Y}\cdots}\\
&\qquad +\frac{\pi_{\bar{X}_0}}{\pi_{Y^1}\eta^{-1}}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}-\frac{\pi_\ell}{\pi_{Y^1}\eta^{-1}}\left(\Delta_{\ell(\omega^{o}/p^{o})}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}+\Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^{o})]}\right)\\
&\qquad\qquad -\frac{\pi_c}{\pi_{Y^1}\eta^{-1}}\left(\Delta_{c(\omega^{o}/p^{o})}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}+\Delta_{c[\mathcal{R}'/(p^{o\prime}/p^{o})]}\right)+1\\
&\qquad +\left[\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta+(1-\eta)\right]\frac{M\left(B/p^{o}\right)^\star}{\pi_{Y^1}\eta^{-1}}\\
&\quad -\frac{1}{\pi_{Y^1}\eta^{-1}}\left[\pi_c\Delta_{c[\mathcal{R}'/(p^{o\prime}/p^{o})]}+\left(1+\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\right)\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^{o})]}\right]\left[\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta+(1-\cdots\right.\\
&= 1+\left(\mathcal{D}_{A'}\right)^{-1}\Bigg\{-\frac{1}{\pi_{Y^1}\eta^{-1}}\left(\pi_c\Delta_{c[\mathcal{R}'/(p^{o\prime}/p^{o})]}+\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^{o})]}\right)\left[\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta+(1-\eta)\right]^2\\
&\qquad -\frac{1}{\pi_{Y^1}\eta^{-1}}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\pi_\ell\left(\Delta_{\ell(\omega^{o}/p^{o})}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}+\Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^{o})]}\right)+\frac{\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}}{\left(\pi^{1}_{X_1}-\pi^{0}_{X_1}\right)^2}\,\frac{\cdots}{\pi_{Y}\cdots}\\
&\qquad +\frac{\pi_{\bar{X}_0}}{\pi_{Y^1}\eta^{-1}}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}-\frac{\pi_\ell}{\pi_{Y^1}\eta^{-1}}\left(\Delta_{\ell(\omega^{o}/p^{o})}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}+\Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^{o})]}\right)\\
&\qquad\qquad -\frac{\pi_c}{\pi_{Y^1}\eta^{-1}}\left(\Delta_{c(\omega^{o}/p^{o})}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}+\Delta_{c[\mathcal{R}'/(p^{o\prime}/p^{o})]}\right)+1\\
&\qquad +\left[\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta+(1-\eta)\right]\frac{M\left(B/p^{o}\right)^\star}{\pi_{Y^1}\eta^{-1}}\Bigg\}\\
&= 1+\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta+(1-\eta)\\
&\quad +\left(\mathcal{D}_{A'}\right)^{-1}\Bigg\{-\frac{1}{\pi_{Y^1}\eta^{-1}}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\pi_\ell\left(\Delta_{\ell(\omega^{o}/p^{o})}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}+\Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^{o})]}\right)+\frac{\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}}{\left(\pi^{1}_{X_1}-\pi^{0}_{X_1}\right)^2}\,\frac{\cdots}{\pi_{Y}\cdots}\\
&\qquad +\frac{\pi_{\bar{X}_0}}{\pi_{Y^1}\eta^{-1}}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}-\frac{\pi_\ell}{\pi_{Y^1}\eta^{-1}}\left(\Delta_{\ell(\omega^{o}/p^{o})}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}+\Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^{o})]}\right)\\
&\qquad\qquad -\frac{\pi_c}{\pi_{Y^1}\eta^{-1}}\left(\Delta_{c(\omega^{o}/p^{o})}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}+\Delta_{c[\mathcal{R}'/(p^{o\prime}/p^{o})]}\right)+1\\
&\quad +\left[\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta+(1-\eta)\right]\left[\frac{M\left(B/p^{o}\right)^\star}{\pi_{Y^1}\eta^{-1}}+\frac{\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^{o})]}}{\pi_{Y^1}\eta^{-1}}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\left[\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta+(\cdots\right.\right.\\
&= 1+\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta+(1-\eta)\\
&+\left(\mathcal{D}_{A'}\right)^{-1}\Bigg\{\frac{\pi_{\bar{X}_0}}{\pi_{Y^1}\eta^{-1}}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}+\frac{\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}}{\left(\pi^{1}_{X_1}-\pi^{0}_{X_1}\right)^2}\frac{1}{\pi_{Y^1}\eta^{-1}}-\frac{\pi_c}{\pi_{Y^1}\eta^{-1}}\left(\Delta_{c(\omega^{o}/p^{o})}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}+\Delta_{c[\mathcal{R}'\cdots}\right.\\
&\qquad -\frac{\pi_\ell}{\pi_{Y^1}\eta^{-1}}\left(1+\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\right)\left(\Delta_{\ell(\omega^{o}/p^{o})}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}+\Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^{o})]}\right)+1\\
&\quad +\left[\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta+(1-\eta)\right]\left[\frac{M\left(B/p^{o}\right)^\star}{\pi_{Y^1}\eta^{-1}}+\frac{\pi_\ell\Delta_{\ell\mathcal{R}'}}{\pi_{Y^1}\eta^{-1}}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\left[\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta+(1-\eta)\right]\right]
\end{aligned}
$$

where extra simplifications should build from the holding of

$$\begin{aligned}\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta+(1-\eta) &\\ &= \left[\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\pi_{\bar{X}_0}\right]\frac{\eta}{\pi_{Y^1}}+1\\ &= \frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}+1+\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\frac{M\big(B/p^{0}\big)^{\star}}{\pi_{Y^1}\eta^{-1}}+\frac{\pi_c}{\pi_{Y^1}\eta^{-1}},\end{aligned}$$

that also implies the one of

$$\begin{aligned}\frac{\pi_{X_0}}{\pi_{Y^1}\eta^{-1}}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}+1+\left[\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta+(1-\eta)\right]\frac{M\big(B/p^{0}\big)^{\star}}{\pi_{Y^1}\eta^{-1}} &\\ &= \left[1+\frac{M\big(B/p^{0}\big)^{\star}}{\pi_{Y^1}\eta^{-1}}\right]\left[\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta+(1-\eta)\right]\\ &= \left(\frac{\pi_{X_0}}{\pi_{Y^1}\eta^{-1}}-\frac{\pi_c}{\pi_{Y^1}\eta^{-1}}\right)\left[\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta+(1-\eta)\right]\\ &= \frac{\pi_{c'}}{\pi_{Y^1}\eta^{-1}}\left[\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta+(1-\eta)\right]\end{aligned}$$

The determinant of the Jacobian Matrix derives as :

$$\begin{aligned}\det(\mathcal{J}) = \mathcal{J}_{11}\big(\mathcal{J}_{22}\mathcal{J}_{33}-\mathcal{J}_{23}\mathcal{J}_{32}\big)+\mathcal{J}_{12}\big(\mathcal{J}_{23}\mathcal{J}_{31}-\mathcal{J}_{21}\mathcal{J}_{33}\big)&\\ +\mathcal{J}_{13}\big(\mathcal{J}_{21}\mathcal{J}_{32}-\mathcal{J}_{31}\mathcal{J}_{22}\big)&\end{aligned}$$

for

$$\begin{aligned}
\mathcal{J}_{11}\mathcal{J}_{22}-\mathcal{J}_{12}\mathcal{J}_{21} = \big(\mathcal{D}_{A'}\big)^{-1}\Bigg\{ & \left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\eta+(1-\eta)\right]\left[\frac{\pi_{\bar{X}_0}}{\pi_{Y^1}\eta^{-1}}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\right.\\
& -\frac{\pi_\ell}{\pi_{Y^1}\eta^{-1}}\left(\Delta_{\ell(\omega^0/p^0)}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}+\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\right)\\
& \left.-\frac{\pi_c}{\pi_{Y^1}\eta^{-1}}\left(\Delta_{c(\omega^0/p^0)}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}+\Delta_{c[\mathcal{R}'/(p^{0\prime}/p^0)]}\right)+1\right]\Bigg\};\\
\mathcal{J}_{11}\mathcal{J}_{33}-\mathcal{J}_{13}\mathcal{J}_{31} = \big(\mathcal{D}_{A'}\big)^{-1}\Bigg\{ & \left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\eta+(1-\eta)\right]^2\\
& \times\left[-\frac{1}{\pi_{Y^1}\eta^{-1}}\big(\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}+\pi_c\Delta_{c[\mathcal{R}'/(p^{0\prime}/p^0)]}\big)+\frac{M\big(B/p^0\big)^\star}{\pi_{Y^1}\eta^{-1}}\right]\Bigg\};\\
\mathcal{J}_{22}\mathcal{J}_{33}-\mathcal{J}_{23}\mathcal{J}_{32} = \big(\mathcal{D}_{A'}\big)^{-1}\Bigg\{ & \Delta_{Y^1(p^1/p^0)}+\frac{\pi_{\bar{X}_0}}{\pi_{Y^1}\eta^{-1}}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\\
& -\frac{\pi_\ell}{\pi_{Y^1}\eta^{-1}}\left(1+\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\right)\left(\Delta_{\ell(\omega^0/p^0)}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}+\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\right)\\
& -\frac{\pi_c}{\pi_{Y^1}\eta^{-1}}\left(\Delta_{c(\omega^0/p^0)}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}+\Delta_{c[\mathcal{R}'/(p^{0\prime}/p^0)]}\right)+1+\frac{M\big(B/p^0\big)^\star}{\pi_{Y^1}\eta^{-1}}\Bigg\};\\
\mathcal{J}_{23}\mathcal{J}_{31}-\mathcal{J}_{21}\mathcal{J}_{33} = \big(\mathcal{D}_{A'}\big)^{-1}\Bigg\{ & -\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\eta+(1-\eta)\right]\Bigg\};\\
\mathcal{J}_{21}\mathcal{J}_{32}-\mathcal{J}_{22}\mathcal{J}_{31} = \big(\mathcal{D}_{A'}\big)^{-1}\Bigg\{ & -\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\eta+(1-\eta)\right]\Bigg\};
\end{aligned}$$

whence :

$$\begin{aligned}
\det(\mathcal{J}) = \big(\mathcal{D}_{A'}\big)^{-1}\Bigg\{ & \left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\eta+(1-\eta)\right]\left[\frac{\pi_{\bar{X}_0}}{\pi_{Y^1}\eta^{-1}}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\right.\\
& -\frac{\pi_\ell}{\pi_{Y^1}\eta^{-1}}\left(\Delta_{\ell(\omega^0/p^0)}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}+\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\right)\\
& \left.-\frac{\pi_c}{\pi_{Y^1}\eta^{-1}}\left(\Delta_{c(\omega^0/p^0)}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}+\Delta_{c[\mathcal{R}'/(p^{0\prime}/p^0)]}\right)+1+\frac{M\big(B/p^0\big)^\star}{\pi_{Y^1}\eta^{-1}}\right]\Bigg\}.\\
= 1+\big(\mathcal{D}_{A'}\big)^{-1}\Bigg\{ & \left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\eta+(1-\eta)\right]\\
& \times\left[\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\left[\frac{\pi_{\bar{X}_0}}{\pi_{Y^1}\eta^{-1}}-\frac{\pi_c}{\pi_{Y^1}\eta^{-1}}\Delta_{c(\omega^0/p^0)}-\frac{\pi_\ell}{\pi_{Y^1}\eta^{-1}}\Delta_{\ell(\omega^0/p^0)}\right]\right.\\
& \left.+\frac{\pi_\ell}{\pi_{Y^1}\eta^{-1}}\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}+1+\frac{M\big(B/p^0\big)^\star}{\pi_{Y^1}\eta^{-1}}\right]\Bigg\}.
\end{aligned}$$

In ordinal terms, this gives:

$$
\begin{aligned}
\det(\mathcal{J}) = {} & \left[\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta + (1-\eta)\right] \\
& \times\left\{\frac{\pi_{c'}}{\pi_{Y^1}\eta^{-1}}\left[\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\big(\Sigma^{H}_{c'Y}\pi_{\bar{X}_0}+\Sigma_{\ell c'}\pi_{\ell}\big)\right.\right. \\
& \qquad\qquad \left.\left. -\left[\big(\Sigma^{H}_{cY}-\Sigma^{H}_{cc'}\big)\pi_c+\big(\Sigma^{H}_{\ell Y}-\Sigma^{H}_{c'\ell}\big)\pi_{\ell}\right]\right]+1+\frac{M\big(B/p^{0}\big)^{\star}}{\pi_{Y^1}\eta^{-1}}\right\} \\
& \times\left\{-\frac{\pi_{c'}}{\pi_{Y^1}\eta^{-1}}\left[\left[\big(\Sigma^{H}_{cY}-\Sigma^{H}_{cc'}\big)\pi_c+\big(\Sigma^{H}_{\ell Y}-\Sigma^{H}_{c'\ell}\big)\pi_{\ell}\right]\right.\right. \\
& \qquad \left.\left. +\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\big(\Sigma^{H}_{\ell Y}-\Sigma^{H}_{c'\ell}\big)\pi_{\ell}\right]\left[\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta+(1-\eta)\right]\right\}^{-1} \\
= {} & \left\{\frac{\pi_{c'}}{\pi_{Y^1}\eta^{-1}}\left[\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\big(\Sigma_{\ell c'}\pi^{H}_{\ell}+\Sigma^{H}_{c'Y}\pi_{\bar{X}_0}\big)+\big(\Sigma^{H}_{cc'}-\Sigma^{H}_{cY}\big)\pi_c+\big(\Sigma^{H}_{c'\ell}-\Sigma^{H}_{\ell Y}\big)\pi_{\ell}\right]+1+\frac{M\big(B/}{\pi_{Y^1}\eta}\right. \\
& \times\left\{\frac{\pi_{c'}}{\pi_{Y^1}\eta^{-1}}\left[\big(\Sigma^{H}_{cc'}-\Sigma^{H}_{cY}\big)\pi_c+\left(1+\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\right)\big(\Sigma^{H}_{c'\ell}-\Sigma^{H}_{\ell Y}\big)\pi_{\ell}\right]\right\}^{-1} \\
= {} & 1+\left\{\frac{\pi_{c'}}{\pi_{Y^1}\eta^{-1}}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\big(\Sigma^{H}_{c'Y}\pi_{X_0}+\Sigma^{H}_{\ell Y}\pi_{\ell}\big)+1+\frac{M\big(B/p^{0}\big)^{\star}}{\pi_{Y^1}\eta^{-1}}\right\} \\
& \qquad\qquad \times\left\{\frac{\pi_{c'}}{\pi_{Y^1}\eta^{-1}}\left[\big(\Sigma^{H}_{cc'}-\Sigma^{H}_{cY}\big)\pi_c+\left(1+\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\right)\big(\Sigma^{H}_{c'\ell}-\Sigma^{H}_{\ell Y}\big)\pi_{\ell}\right]\right\}^{-1} \\
= {} & 1+\left\{\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\big(\Sigma^{H}_{c'Y}\pi_{X_0}+\Sigma^{H}_{\ell Y}\pi_{\ell}\big)+1\right\} \\
& \qquad\qquad \times\left\{\big(\Sigma^{H}_{cc'}-\Sigma^{H}_{cY}\big)\pi_c+\left(1+\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\right)\big(\Sigma^{H}_{c'\ell}-\Sigma^{H}_{\ell Y}\big)\pi_{\ell}\right\}^{-1}
\end{aligned}
$$

The sum of the principal minors of order two of the Jacobian Matrix is available as :

$$
\begin{aligned}
\operatorname{spm}(\mathcal{J}) &= \mathcal{J}_{11}\mathcal{J}_{22} - \mathcal{J}_{12}\mathcal{J}_{21} + \mathcal{J}_{11}\mathcal{J}_{33} - \mathcal{J}_{13}\mathcal{J}_{31} + \mathcal{J}_{22}\mathcal{J}_{33} - \mathcal{J}_{23}\mathcal{J}_{32} \\
&= \left(\mathcal{D}_{A'}\right)^{-1}\Bigg\{\left[\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta + (1-\eta)\right]\Bigg\{\Bigg[\frac{\pi_{\bar{X}_0}}{\pi_{Y^1}\eta^{-1}}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}} \\
&\qquad - \frac{\pi_\ell}{\pi_{Y^1}\eta^{-1}}\left(\Delta_{\ell(\omega^0/p^0)}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}} + \Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\right) \\
&\qquad - \frac{\pi_c}{\pi_{Y^1}\eta^{-1}}\left(\Delta_{c(\omega^0/p^0)}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}} + \Delta_{c[\mathcal{R}'/(p^{0\prime}/p^0)]}\right) + 1\Bigg] \\
&\qquad + \left[\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta + (1-\eta)\right]\Bigg[-\frac{1}{\pi_{Y^1}\eta^{-1}}\big(\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]} \\
&\qquad\qquad + \pi_c\Delta_{c[\mathcal{R}'/(p^{0\prime}/p^0)]}\big) + \frac{M\big(B/p^0\big)^\star}{\pi_{Y^1}\eta^{-1}}\Bigg]\Bigg\} \\
&\quad + \Delta_{Y^1(p^1/p^0)}\eta + \frac{\pi_{\bar{X}_0}}{\pi_{Y^1}\eta^{-1}}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}} \\
&\qquad - \frac{\pi_\ell}{\pi_{Y^1}\eta^{-1}}\left(1 + \frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\right)\left(\Delta_{\ell(\omega^0/p^0)}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}} + \Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\right) \\
&\qquad - \frac{\pi_c}{\pi_{Y^1}\eta^{-1}}\left(\Delta_{c(\omega^0/p^0)}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}} + \Delta_{c[\mathcal{R}'/(p^{0\prime}/p^0)]}\right) + 1 + \frac{M\big(B/p^0\big)^\star}{\pi_{Y^1}\eta^{-1}}\Bigg\} \\
&= 1 + \frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta + (1-\eta) \\
&\quad + \left(\mathcal{D}_{A'}\right)^{-1}\Bigg\{\left[\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta + (1-\eta)\right]\Bigg\{\Bigg[\frac{\pi_{\bar{X}_0}}{\pi_{Y^1}\eta^{-1}}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}} \\
&\qquad - \frac{\pi_\ell}{\pi_{Y^1}\eta^{-1}}\Delta_{\ell(\omega^0/p^0)}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}} + \frac{\pi_\ell}{\pi_{Y^1}\eta^{-1}}\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\Big) \\
&\qquad - \frac{\pi_c}{\pi_{Y^1}\eta^{-1}}\Delta_{c(\omega^0/p^0)}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}} + 1\Bigg] \\
&\quad + \left[\frac{1-\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\eta + (1-\eta)\right]\left[\frac{1}{\pi_{Y^1}\eta^{-1}}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]} + \frac{M\big(B/p^0\big)^\star}{\pi_{Y^1}\eta^{-1}}\right]\Bigg\} \\
&\quad + \Delta_{Y^1(p^1/p^0)}\eta + \frac{\pi_{\bar{X}_0}}{\pi_{Y^1}\eta^{-1}}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}} \\
&\qquad - \frac{\pi_\ell}{\pi_{Y^1}\eta^{-1}}\left(1 + \frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}}\right)\left(\Delta_{\ell(\omega^0/p^0)}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}} + \Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\right) \\
&\qquad - \frac{\pi_c}{\pi_{Y^1}\eta^{-1}}\left(\Delta_{c(\omega^0/p^0)}\frac{\pi^{0}_{X_1}}{\pi^{1}_{X_1}-\pi^{0}_{X_1}} + \Delta_{c[\mathcal{R}'/(p^{0\prime}/p^0)]}\right) + 1 + \frac{M\big(B/p^0\big)^\star}{\pi_{Y^1}\eta^{-1}}\Bigg\}
\end{aligned}
$$

where extra simplifications should build from the holding of

$$\begin{aligned}
&\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\eta+(1-\eta)\\
&\qquad=\left[\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\pi_{\bar{X}_0}\right]\frac{\eta}{\pi_{Y^1}}+1\\
&\qquad=\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}+1+\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{M\big(B/p^0\big)^\star}{\pi_{Y^1}\eta^{-1}}+\frac{\pi_c}{\pi_{Y^1}\eta^{-1}},
\end{aligned}$$

that also implies the one of

$$\begin{aligned}
&\frac{\pi_{X_0}}{\pi_{Y^1}\eta^{-1}}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}+1+\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\eta+(1-\eta)\right]\frac{M\big(B/p^0\big)^\star}{\pi_{Y^1}\eta^{-1}}\\
&\qquad\qquad=\left[1+\frac{M\big(B/p^0\big)^\star}{\pi_{Y^1}\eta^{-1}}\right]\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\eta+(1-\eta)\right]\\
&\qquad\qquad=\left(\frac{\pi_{X_0}}{\pi_{Y^1}\eta^{-1}}-\frac{\pi_c}{\pi_{Y^1}\eta^{-1}}\right)\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\eta+(1-\eta)\right]\\
&\qquad\qquad=\frac{\pi_{c'}}{\pi_{Y^1}\eta^{-1}}\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\eta+(1-\eta)\right]
\end{aligned}$$

Then computing $\mathcal{Z}(-1)=1+\mathrm{tr}(\mathcal{J})+\mathrm{spm}(\mathcal{J})+\det(\mathcal{J})$ and $\mathcal{Z}(+1)=-1+\mathrm{tr}(\mathcal{J})-\mathrm{spm}(\mathcal{J})+\det(\mathcal{J})$:

$$\begin{aligned}
(\mathcal{Z})(+1)&=-\big(\mathcal{D}_{A'}\big)^{-1}\left\{\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\eta+(1-\eta)\right]-1\right\}^2\\
&=\left\{\frac{1}{\pi_{Y^1}\eta^{-1}}\left[\pi_c\Delta_{c[\mathcal{R}'/(p^{0\prime}/p^0)]}+\left(1+\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\right)\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\right]\right.\\
&\qquad\left.\times\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\eta+(1-\eta)\right]\right\}^{-1}\left\{\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\eta+(1-\eta)\right]-1\right\}^2.
\end{aligned}$$

The details of the statement then follow for a standard characterisation of the parameters configurations that underlie the signs of $\det(\mathcal{J})\lesseqgtr 1$, $\mathcal{Z}(-1)\lesseqgtr 0$ and $\mathcal{Z}(+1)\lesseqgtr 0$. △