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EXTREME FREE-RIDING IN ALL-PAY AUCTIONS★

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Abstract. The paper complements the existing literature on group contests by studying bidding behavior of teams in the all-pay auction setting. We assume the best-shot effort impact, where a team's bid is a maximum of its members' bids. We account for possibly nonlinear bidding cost, imposing very mild assumptions on the bidding cost function. A special case of our model is the basic all-pay auction where the cost of bidding is equal to the bid. It is shown that the free-rider equilibria are the only Nash equilibria regardless of the number of players on each team (as long as it is finite), and whether the valuation of the prize is same or different across teams. Moreover, free-riding is extreme: only one player of each team is active (exerts zero effort with probability less than 1), and everybody else free-rides.

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1. Introduction

Many situations in economics can be modeled using contests. A large variety of real-life examples involve several individuals or groups competing for a reward (prize). The variety of contests is very rich and has been extensively studied in the literature.

A distinct feature of a contest is the rule which determines a winner. Many contests in economics are modeled using a Tullock contest success function, which was introduced by Tullock in his seminal paper [13]. One can also view first-, second-price or all-pay auctions as contests.

We consider two teams each having a finite number of players and competing for a prize in an all-pay auction. A team's bid is the *maximum* of the individual

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bids (effort levels) of its members. The prize is a public good for each member of the team. The valuation of the prize may vary across teams, however it is the same for each player within a team. Players choose their bids simultaneously and independently.

The present paper is related to the works by Baik [2], Chowdhury, Lee, and Sheremeta [4], Kolmar and Rommeswinkel [8], and Lee [10]. All these authors use a Tullock contest success function. Kolmar and Rommeswinkel [8] use a CES effort impact function (best-shot is a specific case of CES). Lee [10] and Chowdhury, Lee, and Sheremeta [4] examine the weakest-link and the best-shot effort impact, respectively, with Tullock contest success function. Baik [2] considers a general contest success function (which depends on the effort levels of all players of all teams) satisfying certain concavity and differentiability properties (see [2]).

The existing literature investigates both cooperative and noncooperative behavior of players within each team. While the above mentioned studies assume players choose their effort levels simultaneously and independently, and so behave purely noncooperatively, Konrad and Kovenock [9] allow for correlation of players' choices within each group.

Surprisingly, we still know very little about the bidding behavior of teams that compete in an auction, particularly in an all-pay auction. We are not aware of any work that examines teams' bidding behavior in all-pay auctions with the best-shot effort impact. While most papers on team contests examine Tullock contests, auctions arise in many practical situations and deserve a detailed investigation.

We investigate teams' bidding behavior with the *best-shot* effort impact in an *all-pay auction* setting. We assume players behave noncooperatively within each team to model the possible lack of communication between the team's members. We account for possibly nonlinear bidding cost, imposing very mild assumptions on the bidding cost function.

How would we expect teams to behave in auctions? Would we expect larger teams to have advantage over small teams (supporting Olson's thesis, see [12])? It turns out that team size does not matter (assuming it is finite)!

We find that regardless of the size of each team and whether the valuation of the prize is the same or differs across teams, the free-riding problem is always present in equilibrium and is extreme: only one player in each team is active (i.e., ever bids more than zero), and everybody else free-rides (always bids zero).

The paper is organized as follows. In the next section we outline our theoretical framework and provide the necessary definitions. Results are presented in Section 3.

2. Notation and Methodology

We consider two teams bidding against each other. Each team has $m_\ell \geq 1$ players, $\ell = 1, 2$. Denote by I_ℓ the **index set** of all players of team ℓ,

$$I_\ell = \{1, \cdots, m_\ell\}$$

A **bid** of player i of team ℓ will be denoted by $x_{\ell,i}$, and the **vector of bids** $(x_{\ell,1}, \cdots, x_{\ell,m_l})$ of team ℓ's members $1, \cdots, m_\ell$ by x_ℓ.

Definition 2.1. *Given a vector of bids of team ℓ's members $x_\ell = (x_{\ell,1}, \cdots, x_{\ell,m_l})$, **team ℓ's bid** is a real number $B(x_\ell)$, where B is a function of m_ℓ variables of the following form:*

$$B(x_\ell) = \max_{i \in I_\ell} x_{\ell,i} \tag{2.1}$$

Definition 2.2. *A **strategy** of player i of team ℓ is a probability distribution over a subset of $\mathbb{R}_+$ (the set of bids of player i). A strategy profile of player i of team ℓ will be denoted $s_{\ell,i}$.*

Definition 2.3. ***Team ℓ's strategy** is an m_ℓ-tuple of its members' strategies $(s_{\ell,1}, \cdots, s_{\ell,m_l})$. Team ℓ's strategy profile will be denoted s_ℓ.*

Team ℓ's strategy s_ℓ generates a probability distribution over the set of team ℓ's bids corresponding to s_ℓ. This probability distribution will be denoted $T(s_\ell)$. Notice that s_ℓ completely determines $T(s_\ell)$, however s_ℓ is not characterized by $T(s_\ell)$.

Given team ℓ's strategy s_ℓ, denote the upper bound of the support of $T(s_\ell)$ (or equivalently, the upper bound of the set of team ℓ's bids corresponding to s_ℓ) by $\bar{s}_\ell$,

$$\bar{s}_\ell = \sup\{x \geq 0 \,:\, x \text{ is in the support of } T(s_\ell)\}$$

Similarly, denote the lower bound of support of $T(s_\ell)$ by $\underline{s}_\ell$,

$$\underline{s}_\ell = \inf\{x \geq 0 \,:\, x \text{ is in the support of } T(s_\ell)\}$$

The prize acquired by the winning team is a public good for each member of the team. That is, the prize is nonrival (consumption of the prize by one member of the team does not reduce the amount of the prize available to other members of the same team) and nonexcludable. All players in each team have the same valuation of the prize. Denote by v_ℓ the **valuation** of team ℓ's member. We allow for possibly different valuations across teams, i.e., $v_1 \neq v_2$.

3. Results

In an all-pay auction setting, the payoff function of a generic player i of team ℓ is given by:

$$u_{\ell,i}(x_\ell, x_{-\ell}) = \begin{cases} v_\ell - c(x_{\ell,i}) & \text{if} \quad B(x_\ell) > B(x_{-\ell}) \\ \frac{v_\ell}{2} - c(x_{\ell,i}) & \text{if} \quad B(x_\ell) = B(x_{-\ell}) \\ -c(x_{\ell,i}) & \text{if} \quad B(x_\ell) < B(x_{-\ell}) \end{cases} \tag{3.1}$$

Here $c(\cdot)$ is the cost function, which represents the cost (or disutility) of bidding.

Assumption 3.1. *The cost function $c(\cdot)$ is the same for all players, increasing, unbounded from above, $c(0) = 0$, and continuous.*

Introduce the following notation for a profile of bids s:

$$\bar{s}_\ell = \max_{j \in I_\ell} \bar{s}_{\ell,j}.$$

3.1. **Equal valuations across teams.**

Assumption 3.2. $v_1 = v_2 = v$

Let r denote $c^{-1}(v)$; it follows by Assumption 3.2 that r is uniquely determined. Notice that for the simplest case when $c(x) = x$, we have $r = v$. The remaining results in this subsection are derived under Assumptions 3.1 and 3.2.

Lemma 3.3. *Let s^* be a Nash equilibrium. Then:*

(1) $\bar{s}^*_\ell = \bar{s}^*_{-\ell} = \bar{s}^*$.
(2) $\bar{s}^* = r$.
(3) *Let k_ℓ be such that $\bar{s}^*_{k_\ell} = r$, then $u_{k_\ell}(s^*) = 0$ for $\ell = 1, 2$.*

Proof. (1) Suppose WLOG $\bar{s}^*_1 < \bar{s}^*_2$, then the player k_2 from team 2 is better-off shifting mass from $(\bar{s}^*_1, \bar{s}^*_2]$ to $\bar{s}^*_1$, which is a contradiction.
(2) Clearly, $\bar{s}^* \leq r$. Suppose $\bar{s}^* < r$, then by bidding slightly above $\bar{s}^*$ player k_1 from team 1 can increase his payoff.
(3) Immediate consequence of (2).
■

Theorem 3.4. *A strategy profile s^* such that one player from each team randomizes according to*

$$F(x) = \frac{c(x)}{v} \quad on \quad [0, r], \tag{3.2}$$

and all the remaining players bid 0 with probability 1, is a Nash equilibrium. Such Nash equilibrium will be called a free-rider equilibrium.

Note: clearly, there exist $m_1 m_2$ distinct free-rider equilibria.

Proof. Let t_1 and t_2 be active players of teams 1 and 2, respectively. Denote the cdf of team ℓ's bids (i.e., cdf of the maximum of the bids of team ℓ's members) by $F_\ell(\cdot)$. Notice that for any $0 \leq x \leq r$:

$$u_{\ell,t_\ell}(x, s^*_{-t_\ell}) = vF_{-\ell}(x) - c(x) = 0 \tag{3.3}$$

Therefore active player of team ℓ is indifferent between all strategies in $[0, r]$. Bidding above r would give him negative payoff, hence active player of team ℓ would not deviate from his strategy described by 3.2.

It remains to show that no inactive player i wants to deviate from 0. Notice that

$$\begin{aligned} u_{1,i}(x, s^*_{-i}) - u_{1,i}(s^*) &= \frac{c(x)}{v}\left[\frac{c(x)}{v} \cdot v - c(x) - v\int_0^x F_\ell(z)dF_\ell(z)\right] \\ &= -c(x)\int_0^x F_\ell(z)df_\ell(z) < 0. \end{aligned}$$

Hence no inactive player wants to deviate from 0, so s^* is a Nash equilibrium. ■

Theorem 3.5. *Free-rider equilibria of Theorem 3.4 are the only Nash equilibria.*

Proof. Suppose, by contradiction, there is a Nash equilibrium s^* and a team ℓ in which more than 1 player are active (i.e., bid 0 with probability strictly less than 1). Let player k of team ℓ be such that $\bar{s}^*_k = r$, where $r = c^{-1}(v)$. But then $u_k(0, s^*_{-k}) > 0$, while $u_{k_\ell}(r, s^*_{-k}) = 0$. By continuity of the payoff function there exists $\delta > 0$ such that $u_k(x, s^*_{-k}) < u_k(0, s^*_{-k})$ for all $x \in (r - \delta, r]$. This implies $\bar{s}_k \leq r - \delta < r$, which contradicts our assumption. Hence free-rider equilibria are the only Nash equilibria. ■

3.2. **Different valuations across teams.** It remains to work out the case $v_1 \neq v_2$. The following assumption takes care of this case without loss of generality.

Assumption 3.6. $v_1 > v_2$.

Let r denote $c^{-1}(v_2)$; it follows by Assumption 3.6 that r is uniquely determined. The remaining results in this subsection are derived under Assumptions 3.1 and 3.6. The following lemma is an analogue of Lemma 3.3.

Lemma 3.7. *Let s^* be a Nash equilibrium. Then:*

(1) $\bar{s}^*_\ell = \bar{s}^*_{-\ell} = \bar{s}^*$.
(2) $\bar{s}^* = r$.
(3) *Let k_2 be such that $\bar{s}^*_{k_2} = r$, then $u_{k_2}(s^*) = 0$.*

Theorem 3.8. *A strategy profile s^* such that one player from team 1 randomizes according to*

$$F(x) = \frac{c(x)}{v_2} \quad on \quad [0, r], \tag{3.4}$$

one player from team 2 randomizes on $[0, v_2]$ according to cdf

$F_2(x) = (1 - \frac{v_2}{v_1}) + \frac{c(x)}{v_1}$, (3.5)

and all the remaining players bid 0 with probability 1, is a Nash equilibrium. Such Nash equilibrium will be called a free-rider equilibrium.

Proof. Let t_1 and t_2 be active players of teams 1 and 2, respectively. Denote the cdf of team ℓ by $F_\ell(\cdot)$. Observe that for any $0 \leq x \leq r$:

$$u_{2,t_2}(x, s^*_{-t_2}) \;=\; v_2 F_1(x) - c(x) = 0,$$

Therefore active player of team 2 is indifferent between all strategies in $[0, r]$. Since bidding above r results in negative payoff, active player of team 2 would not deviate from his strategy described by 3.5.

Similarly, observe that for active player t_1 of team 1, for any $0 \leq x \leq r$:

$$u_{1,t_1}(x, s^*_{-t_1}) \;=\; v_1 F_2(x) - c(x) = v_1 - v_2,$$

hence player t_1 is indifferent between bids in $[0,r]$. Bidding above r would give him a payoff strictly less than $v_1 - v_2$, therefore player t_1 would not deviate from the strategy described in 3.4.

It remains to show that no inactive player of either team would deviate from 0. Let i be a player of team 1 who bids 0 with probability 1. Observe that for every $x \in (0,r]$:

$$\begin{aligned} u_{1,i}(x, s^*_{-i}) - u_{1,i}(0, s^*_{-i}) &= \frac{c(x)}{v_2}\left[v_1\frac{c(x)}{v_1} - \frac{c(x)}{2v_1}v_1\right] - c(x) \\ &= \frac{c(x)}{v_2}\left[c(x) - c(x) - \frac{c(x)}{2}\right] - c(x)(1 - \frac{c(x)}{v_2}) \\ &< 0 \end{aligned}$$

Consequently no inactive player of team 1 wants to deviate from 0. Apply the same argument as in the proof of Theorem 3.4 to show that no inactive player of team 2 wants to deviate from 0. This completes the proof. ■

Lemma 3.9. *There is no Nash equilibrium in which more than one player of team 2 is active.*

Proof. Suppose not, let s^* be an equilibrium where, without loss of generality, players i and j of team 2 are active, and $\bar{s}_{2,i} = r$. Observe that $u_{2,i}(r, s^*_{-i}) = 0$. However since player j is active, $u_{2,i}(0, s^*_{-i}) > 0$. By continuity of the payoff function there is a δ-neighborhood of r such that $u_{2,i}(0, s^*_{-i}) > u_{2,i}(x, s^*_{-i})$ for each x in that neighborhood. But this implies $\bar{s}_{2,i} \leq r - \delta < r$, contradiction. This completes the proof. ■

Lemma 3.10. *Equilibrium payoff of an active player of team 2 is zero.*

Proof. Follows immediately from the fact that the least upper bound of the support of that player is r, and $c(r) = v_2$. ■

Lemma 3.11. *Let s^* be a Nash equilibrium. Then no active player from either team places an atom in $(0,r)$. In other words, for every team ℓ, active player i of ℓ, and $x \in (0,r)$: $F_{l,i}(x) = \lim_{z\to x^-} F_{l,i}(z)$.*

Proof. Suppose by contradiction some player i of team ℓ places an atom at $x \in (0,r)$. Then any active player from the opposite team $(-\ell)$ that randomizes up to r would want to shift mass some very small neighborhood below x to $x+\epsilon$ for ϵ very small. So no active player from team $-\ell$ puts a positive mass in a very small neighborhood below x, but then it is not optimal for player i of team ℓ to place atom at x, contradiction. ■

Recall that we denote the cumulative distribution function of the maximum of the bids of team 1's members, and team 2's members by F_1 and F_2, respectively.

Lemma 3.12. *For each $x \in [0,r]$, $F_1(x) = \frac{c(x)}{v_2}$.*

Proof. For $x \in (0, r)$ this follows directly from Lemma 3.10 and part (4) of Lemma 3.7. By the right continuity of the cdf the result extends to $[0, r]$. ■

Lemma 3.13. *Let s^* be a Nash equilibrium such that at least two players of team 1 are active. Then for each player i of team 1 and $x \in [0, r]$, $F_{1,i}(x) \geq \frac{c(x)}{v_2}$, with strict equality if and only if $x = 0$, or $F_{1,j}(x) = 1$ for each player j of team 1 such that $j \neq i$.*

Proof. Follows directly from Lemma 3.12 and the fact that $F_{1,i}(x) \leq 1$ for each player i and $x \in [0, r]$. ■

Lemma 3.14. *Let s^* be an equilibrium, and F_2 be the corresponding cumulative distribution function of the bids of an active player of team 2. There exists $z \in (0, r)$ such that $F_2(z) \geq \frac{c(z)}{v_1}$.*

Proof. Suppose not, i.e., for every $z \in (0, r)$ we have $F_2(z) < \frac{c(z)}{v_1}$. Notice that for every player i of team 1 and every $z \in (0, r)$

$$u_{1,i}(z, s^*_{-i}) - u_{1,i}(0, s^*_{-i}) \leq v_1 F_2(z) - c(z) < 0.$$

If $\underline{s}_{1,i} > 0$, we get an immediate contradiction, so assume $\underline{s}_{1,i} = 0$. Then every player i of team 1 randomizes between 0 and r, but then an active member of team 2 is better-off shifting mass from ϵ-neighborhood below r to some $\delta > 0$ for ϵ and δ very small, which is a contradiction. This establishes the claim. ■

Theorem 3.15. *Free-rider equilibria of Theorem 3.8 are the only Nash equilibria.*

Proof. From Lemma 3.9 we know that only one player of team 2 is active. It remains to show the same about team 1. Suppose by contradiction there is an equilibrium s^* such that at least two players, say i and j, of team 1 are active. Without loss of generality suppose $\bar{s}_{1,i} \leq \bar{s}_{1,j}$. Let $z \in (0, v_2)$ be such that $F_2(z) \geq \frac{z}{v_1}$. Such z exists by Lemma 3.14. We claim that player i of team 1 is strictly better-off deviating from $\bar{s}_{1,i}$ to z. Notice that

$$\begin{aligned} u_{1,i}(\bar{s}_{1,i}, s^*_{-i}) - u_{1,i}(z, s^*_{-i}) &\leq \frac{v_2 - z}{v_1} \cdot \frac{v_2 - z}{v_2} \cdot v_1 - (v_2 - z) \\ &= (v_2 - z)(\frac{v_2 - z}{v_2} - 1) \\ &< 0 \end{aligned}$$

Therefore by continuity of the payoff function player i of team 1 is better-off deviating from any point in a sufficiently small neighborhood of $\bar{s}_{1,i}$ to $\bar{s}_{1,i} - \epsilon$, so the upper bound of the support of player i's strategy is strictly less than $\bar{s}_{1,i}$, which is a contradiction. This establishes that only one player of team 1 is active. Therefore free-rider equilibria are the only Nash equilibria. ■

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