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## Measurement Scales and Welfarist Social Choice

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**Keywords.** grading; measurement scales; social welfare functionals; utility aggregation; welfarism

**JEL Classification Numbers.** D60, D71.

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## 1. Introduction

The social welfare functional approach to social choice theory fails to distinguish between a genuine change in individual well-beings from a merely representational change due to the use of different measurement scales. It is possible for a change in well-beings to be exactly compensated by a change in the scales with which they are measured. When this is the case, the social ordering of the alternatives remains unchanged. In effect, the real change in well-beings is rendered invisible by the compensating change in measurement scales. We introduce a generalization of the concept of a social welfare functional that explicitly takes account of the scales that are used to measure well-beings so as to distinguish between these two kinds of changes. This generalization of the standard theoretical framework permits a wide range of responses to the compensating changes described above; invariance of the social ordering is no longer required.

*Welfarism* is the doctrine that social alternatives are evaluated and socially ranked solely in terms of the well-beings of the relevant individuals. We argue that a satisfactory formulation of this doctrine also requires a specification of the scales used to measure well-beings (i.e., welfarism is *scale dependent*), otherwise the numbers with which well-beings are described cannot be interpreted as being measures of well-being. Similarly, to say that the length of an object is the number  $l$  requires a unit of measurement for this statement to be meaningful. We axiomatize scale-dependent welfarism using our new framework. The implications of our approach for characterizing classes of social welfare orderings are also considered.

The Arrovian approach to social choice theory (Arrow, 1951) relies exclusively on information about how individual preferences order the social alternatives when determining how to socially order them. Arrow precludes from the outset taking account of any information about the individual well-beings when making this social choice other than what is embodied in the individual orderings. To overcome this limitation of the Arrovian approach, Sen (1970, 1974) has introduced the concept of a social welfare functional. A *social welfare functional* assigns a social ordering of the alternatives to each profile of individual utility functions in its domain.<sup>1</sup> These utility func-

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<sup>1</sup>A formally equivalent aggregation procedure was introduced by Luce and Raiffa (1957, p. 343). In their formalism, a social ordering is assigned to matrices whose columns are the individuals, whose rows are the alternatives, and whose entry in column  $i$  and row  $j$  is the utility that individual  $i$  obtains with alternative  $j$ .

tions provide the information about the individual well-beings that is used to determine the social ordering. Sen’s framework can accommodate different assumptions about what kinds of utility comparisons are possible both intrapersonally and interpersonally, including the Arrovian special case in which utilities are ordinally measurable and interpersonally noncomparable.

Sen’s framework provides a formal way of considering the merits of social objectives that are only meaningful when some kinds of interpersonal utility comparisons are possible. In particular, social welfare functionals have been used to axiomatize a wide range of methods for aggregating individual utilities for the purpose of socially comparing different alternatives. For example, there are axiomatizations of the utilitarian sum, maximin utility, and the constant elasticity of substitution functions that constitute the Atkinson (1970) class, all of which presuppose that some kind of interpersonal utility comparisons are possible—gains and losses in the case of utilitarianism, levels in the case of maximin, and ratios in the case of the Atkinson class. Social welfare functionals have also been used to address questions in population ethics (see Blackorby, Bossert, and Donaldson, 2005b).<sup>2</sup>

Welfare economics is built on welfarist foundations. In the standard formulation of welfarism, well-beings are identified with utilities, *however measured*. Social alternatives are collectively ordered using a social welfare ordering (or a social welfare function that represents it) on the vectors of individual utilities that are obtained with them (see Jehle and Reny, 2011, Chap. 6). No other information other than these utility values is taken account of when determining the social ordering of the alternatives. One of the most important implications of the conventional understanding of welfarism is that all of the information embodied in a social welfare functional can be equivalently described by a single social welfare ordering of the attainable vectors of individual utilities. Specifically, with  $n$  individuals, for any profile of utility functions  $U = (U_1, \dots, U_n)$  and any pair of alternatives  $x$  and  $y$ , how  $x$  and  $y$  are socially ranked when the profile is  $U$  is the same as how the vectors of individual utilities  $U(x) = (U_1(x), \dots, U_n(x))$  and  $U(y) = (U_1(y), \dots, U_n(y))$  are ranked by the social welfare ordering.

When the domain of a social welfare functional is unrestricted (Unrestricted Domain), the existence of such a social welfare ordering is equivalent

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<sup>2</sup>Surveys of the literature that employs social welfare functionals may be found in Sen (1977), d’Aspremont (1985), d’Aspremont and Gevers (2002), and Bossert and Weymark (2004).

to requiring the social welfare functional to satisfy Pareto Indifference and Independence of Irrelevant Alternatives. Henceforth, we refer to them as the *welfarism axioms*. Pareto Indifference requires two alternatives to be socially indifferent if each individual is equally well off with them. Independence requires the social preference for a pair of alternatives to be the same for two different profiles if these profiles coincide on this pair. This independence condition is the social welfare functional analogue of Arrow's independence axiom. The two welfarism axioms are also jointly equivalent to Strong Neutrality, which requires that the only features of alternatives that are used to socially rank them are their utilities—the alternatives' physical descriptions and the profile of utility functions that generate these utilities are irrelevant.

With a social welfare functional, different assumptions about the measurability and interpersonal comparability of utilities are formalized by requiring the social ordering of the alternatives to be invariant to any transformation of the profile of utility functions that preserves the meaningful utility comparisons, a property of a social welfare functional called *information invariance*. For example, if only levels of utility are interpersonally comparable, then the social ordering of the alternatives must be invariant to a common increasing transform of each person's utility function. Similarly, if both levels and differences in utilities are meaningful (what is known as *cardinal full comparability*), then invariance is only required if each individual's utility function is subjected to a common increasing affine transform. The modeling of utility comparisons in terms of information invariance transforms was developed by Sen (1970, 1974, 1977), d'Aspremont and Gevers (1977), and Roberts (1980), among others.<sup>3</sup> This approach builds on the way that measurement scales have been formalized and classified in the representational theory of measurement. Pioneering contributions to this theory include Stevens (1946) and Luce (1959). See Krantz, Luce, Suppes, and Tversky (1971) for a classic presentation of measurement theory.

Sen (1977, p. 1542) has drawn attention to the importance of distinguishing between real and representational changes in well-beings in the following insightful passage:

with cardinal full comparability (indeed even with ratio-scale full comparability), the invariance property is unable to distinguish between (i) everyone having more welfare (better off) in some

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<sup>3</sup>Information invariance transforms were used by Luce and Raiffa (1957, pp. 343–346), but not in any systematic way.

real sense and (ii) a reduction in the unit of measurement of personal welfares. Invariance in all the cases considered here, including cardinal full comparability (or ratio-scale full comparability), requires that the social ordering should not change at all if everyone's welfare function is, say, doubled. But while this requirement is reasonable enough if interpreted as a halving of the unit of measurement, it is quite a restriction if the interpretation is that of a general increase in personal welfare, since the social welfare ordering need not be accepted to be a mean-independent function of individual welfares. But in all cases of measurability-comparability frameworks discussed here (and in other works), the invariance requirement covers both interpretations since there is no natural "unit" of measurement of personal welfare.

We agree with Sen that information invariance is a reasonable requirement when all that has changed is the measurement scale. The failure of a social welfare functional to distinguish between a real change in individual well-beings from one that is obtained by using different scales to measure them is due to the combination of using the traditional formulation of welfarism (either directly or by assuming Unrestricted Domain, Pareto Indifference, and Independence of Irrelevant Alternatives) and the scale invariance property. Having pointed out the failure of a social welfare functional to properly distinguish between real and representational changes in well-beings, Sen does not propose a way of doing so. That is what we do here.

A fundamental problem with the social welfare functional approach is that it is not possible to interpret what the utility values measure because the scales used to measure them have not been specified. Thus, when somebody's utility for an alternative changes from, say, 1 to 2, it is not known whether this change involves a change in the scale in which these numbers are expressed or not. It is for this reason that a social welfare functional can fail to correctly identify when a welfare change is real. As a consequence, the formalization of welfarism using social welfare functionals is incomplete because an individual's well-being is underspecified if the scale with which it is being measured is not known.

In order to deal with this problem, we generalize Sen's definition of a social welfare functional by including the scales that are used to measure individual utilities in its domain in addition to the profile of utility functions. We allow for the possibility that different individuals use different scales of

measurement, so the profile of utility functions is supplemented with a profile of scales. To distinguish our functionals from those of Sen, ours are called *scale-dependent social welfare functionals*. A *utility scale* consists of the possible utility values linearly ordered from best to worst and an interpretation of what these utility values mean. The specification of a scale is an essential component of the empirical procedure by which numbers are assigned to alternatives and interpreted as being utility values.<sup>4</sup>

By taking account of the scales of measurement, not just the profile of utility functions, when making collective decisions, we can (i) restrict the application of the Pareto, independence, and neutrality axioms to comparisons that employ the same measurement scale and (ii) restrict the application of the information invariance axioms to comparisons in which only changes in scales are involved. Thus, we are able to distinguish real welfare changes from those that are due to changes in the measurement scales, something which is not always possible using Sen's framework.

Our framework allows us to provide a more satisfactory formulation of what constitutes welfarism, what we call *scale-dependent welfarism* so as to distinguish it from the traditional formulation of welfarism. According to scale-dependent welfarism, the only information that is used to evaluate and socially rank alternatives are the utilities of the relevant individuals together with the scales with which these utilities are measured. We believe that this formulation of what welfarism requires is more satisfactory than the standard one because it includes an interpretation of what the utility values mean. Moreover, scale-dependent welfarism permits social rankings to reflect real welfare changes even when accompanied by a change in the measurement scales that would otherwise obscure the fact that there has been any real change in well-beings at all.

We characterize scale-dependent welfarism using the restricted forms of the Pareto, independence, and neutrality axioms described above. There is no longer a single social welfare ordering of utility vectors. Instead, there is one for each profile of measurement scales. Nevertheless, given the ordering for one profile of scales, the ordering for any other profile of scales is uniquely

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<sup>4</sup>Similarly, when assigning lengths to objects, it is necessary to know what units of length are being used in order to interpret these numbers as lengths. Measurement theory has had much to say about the procedures by which numbers are interpreted as measurements. Having noted the importance of this issue, we do not consider it further because nothing depends on the specific procedures used. There is no need for what Sen calls a "natural" unit for measuring utility.

determined once an information invariance condition has been specified. The axiomatic characterizations of various social welfare orderings found in the literature make essential use of the traditional formulation of welfarism that is implied by the standard Pareto and independence axioms for social welfare functionals when the domain is suitably unrestricted. By limiting these axioms so that they only apply for fixed measurement scales, we show that there is much greater freedom in choosing a social welfare ordering that is compatible with the other axioms that one might wish to adopt. Thus, welfarism is a less restrictive doctrine when measurement scales are properly accounted for than when they are not.

Our approach is closely related to the use of grading functions by Balinski and Laraki (2007, 2010) to make collective decisions. It is also related to the way that Blackorby, Bossert, and Donaldson (2005a) have generalized Sen's definition of a social welfare functional so as to take account of non-welfare information. We discuss the relationship between our approach and theirs in later sections.

The rest of this article is organized as follows. Section 2 sets out our formal framework. In Section 3, we review how welfarism is formalized and axiomatically characterized using Sen's social welfare functionals. Then, in Section 4, we reformulate the basic welfarism axioms in scale-dependent terms and provide a scale-dependent analogue of the welfarism theorem established using the traditional approach. In Section 5, we illustrate our theorem with an example of a social aggregation procedure that is not welfarist in the traditional sense, but does satisfy our scale-dependent form of welfarism. In Section 6, we describe how information invariance restricts how social welfare orderings for different scales are related and discuss the implications of our approach for characterizing classes of social welfare orderings. Finally, we provide some concluding remarks in Section 7.

## **2. The Formal Framework**

We are concerned with the problem of determining a social preference ordering for a set of alternatives as a function of the utility functions that a group of individuals have for them. We restrict attention to a fixed set of individuals, but our analysis straightforwardly extends to variable population comparisons. The set of alternatives can be anything for which a social preference might be wanted: feasible allocations in an economy, candidates in an election, etc. Formally, the set of alternatives is  $X$ , with  $X$  assumed to con-

tain at least three alternatives so that all of our assumptions are non-vacuous. The set of individuals is  $N = \{1, \dots, n\}$ , with  $n \geq 2$ . Each individual has a utility function defined on  $X$  and these functions are aggregated to determine the collective ordering.

Utilities are measured using a scale. In this respect, utilities are no different from other measurable quantities such as length and temperature, which could be measured in feet or meters, or in degrees Celsius or Fahrenheit, respectively. Formally, for each  $i \in N$ ,  $i$ 's *utility scale* is a triple  $S_i = (\mathcal{U}_i, >, \mathcal{I}_i)$ , where  $\mathcal{U}_i \subseteq \mathbb{R}$  is the set of *permissible utility values*,  $>$  linearly orders these utilities from best to worst, and  $\mathcal{I}_i$  is the *interpretation procedure* that fixes the meaning of the utilities in  $\mathcal{U}_i$ . We assume that the cardinality of  $\mathcal{U}_i$  is at least three. It is convenient with numerical scales to let  $>$  be the standard “larger than” relation for the real line and to suppress further mention of this component.

For example, when the social alternatives are lotteries, as in the von Neumann and Morgenstern (1944) approach to decision-making under uncertainty, utility values are often normalized to lie in the unit interval  $[0, 1]$ , with 1 interpreted as being the best possible level of well-being. That is one scale. By applying an increasing affine transform that results in the utility values lying in  $[1, 3]$ , 1 is then interpreted as being the worst possible level. Disambiguation of the meaning of the numbers used to express utility values is made possible by the specification of the scale.

The set of *admissible utility scales for individual  $i$*  is  $\mathcal{S}_i$ . We allow for the possibility that different individuals use different scales and that they have different sets of admissible scales. Our analysis also applies if the  $\mathcal{S}_i$  are the same for everybody. A *profile of utility scales* is an  $n$ -tuple  $S = (S_1, \dots, S_n)$ . The set of *admissible profiles of utility scales* is  $\mathcal{S} \subseteq \prod_{i \in N} \mathcal{S}_i$ . We assume that  $\mathcal{S}$  contains at least two profiles. If there is only one profile, then our framework is formally equivalent to the one used by Sen (1970). In principle,  $\mathcal{S}$  could be the set of all logically possible profiles of utility scales. Depending on the particular aggregation problem being considered, it might be natural to restrict  $\mathcal{S}$  in various ways.

For each individual  $i \in N$  and utility scale  $S_i = (\mathcal{U}_i, \mathcal{I}_i) \in \mathcal{S}_i$ ,  $i$ 's *utility function* is a function  $U_i: X \rightarrow \mathcal{U}_i$ . We can think of  $U_i$  as a report that lists, for each  $x \in X$ , which utility value in  $\mathcal{U}_i$  that  $i$  has assigned to  $x$ . A *profile of utility functions* is an  $n$ -tuple  $U = (U_1, \dots, U_n)$ . The set of all logically possible profiles of utility functions for the profile of utility scales  $S$  is  $\mathcal{U}^S$ . For all  $x \in X$  and  $U \in \mathcal{U}^S$ ,  $U(x) = (U_1(x), \dots, U_n(x))$  is the vector of utility

values obtained with  $x$ . A *profile* is a pair  $T = (U, S)$  such that  $U \in \mathfrak{U}^S$  where  $S \in \mathcal{S}$ . Let  $\mathcal{T}$  be the set of all profiles that can be obtained with  $\mathcal{S}$ .

Let  $\mathcal{R}$  denote the set of all orderings of  $X$ .<sup>5</sup> A *scale-dependent social welfare functional*  $F: \mathcal{D} \rightarrow \mathcal{R}$  determines a weak social preference ordering  $F(T)$  (“socially weakly preferred to”) of the alternatives in  $X$  for each profile  $T$  in the set of *admissible profiles*  $\mathcal{D} \subseteq \mathcal{T}$ . This definition of a social welfare functional generalizes that of Sen (1970) by pairing each profile of utility functions with a profile of measurement scales in the domain. For notational simplicity, let  $R_T = F(T)$  denote the social preference ordering that results if the profile is  $T$ . The strict preference and indifference relations corresponding to  $R_T$  are denoted by  $P_T$  and  $I_T$ , respectively.<sup>6</sup>

While we focus attention on the problem of aggregating individual utilities, the framework we employ is flexible enough to accommodate a wide variety of other collective decision-making problems in which the individual inputs take the form of grades assigned to the alternatives, as in Balinski and Laraki (2007, 2010). Examples include the evaluation of research projects by an expert panel, prioritizing job candidates by a hiring committee, and the evaluation of evidence for different medical interventions by a scientific panel. Grades need not be numerical, but if they are not, they can be easily converted to numbers so that our analysis applies.

In general, a scale consists of a set of available grades, a linear order of these grades from best to worst, and an interpretation procedure that establishes the meanings of the grades. For example, a scale might consist of the grades  $\{excellent, very\ good, good, fair, poor\}$  used in a variety of applications or the set of letter grades  $\{A, B, C, D, F\}$  often used in academic evaluation, ordered in the usual way. The interpretation specifies the characteristics of an alternative that merit a particular grade. In the case of course letter grades, the interpretation could fix the meaning of an  $A$  by saying that the mastery of the course material is outstanding. Formally,  $i$ 's *grading scale* is a triple  $S_i = (\mathcal{G}_i, \succ_i, \mathcal{I}_i)$ , where  $\mathcal{G}_i$  is the set of available grades,  $\succ_i$  is the linear order that indicates how these grades are ranked ( $g \succ_i g'$  indicates that  $g$  is a better grade than  $g'$ ), and  $\mathcal{I}_i$  is the interpretation procedure that fixes the meaning of the grades in  $\mathcal{G}_i$ .<sup>7</sup> A *grading function* for individual  $i$  is

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<sup>5</sup>A binary relation  $R$  on  $X$  is (i) *reflexive* if for all  $x \in X$ ,  $xRx$ , (ii) *complete* if for all distinct  $x, y \in X$ ,  $xRy \vee yRx$ , and (iii) *transitive* if for all  $x, y, z \in X$ ,  $xRy \wedge yRz \rightarrow xRz$ . An *ordering* is a reflexive, complete, and transitive binary relation.

<sup>6</sup>For all  $x, y \in X$ , (i)  $xP_Ty \leftrightarrow [xR_Ty \wedge \neg(yR_Tx)]$  and (ii)  $xI_Ty \leftrightarrow [xR_Ty \wedge yR_Tx]$ .

<sup>7</sup>Balinski and Laraki (2010, p. 176) assume that everybody uses the same grading

a function  $G_i: X \rightarrow \mathcal{G}_i$ . A *grading functional* aggregates a profile of grading functions and a profile of grading scales into a collective ranking of the alternatives in  $X$ .

Grades can be a set of real numbers, in which case  $\mathcal{G}_i \subseteq \mathbb{R}$  and the relation  $\succ_i$  is simply the inequality  $>$  for the real line. Utility scales have this numerical structure. Non-numerical grades can be accommodated by encoding them as numbers. For example, self reports of health status (SRHS), which are often used in making decisions about the allocation of resources for health care, typically use the grades  $\{excellent, very\ good, good, fair, poor\}$ . It is common to convert these verbal grades to the numerical grades  $\{5, 4, 3, 2, 1\}$  for statistical analysis. However, the grades only have ordinal significance, so it is equally valid to use a nonlinear set of grades such as  $\{10, 4, 3, 2, 1\}$  in which only the value assigned to an excellent outcome has been changed (see Allison and Foster, 2004). Similarly, the letter grades  $\{A, B, C, D, F\}$  can be described by the numbers  $\{5, 4, 3, 2, 1\}$  or, equivalently, by  $\{4, 3, 2, 1, 0\}$  as in done in the United States when computing grade point averages. Two scales may use the same set of grades but attach meanings to these grades using a different interpretation procedure. For example, self reports of pain and the Mohs hardness scale both use the numbers from 1 to 10 as grades, but differ in their interpretations of these grades.

Henceforth, we restrict attention to the aggregation of utilities and, for simplicity, omit the qualifier “utility” when referring to a utility scale.

### 3. Scale-Independent Welfarism

In this section, we review how welfarism is formalized and axiomatically characterized using the social welfare functionals introduced by Sen (1970, 1974). Sen does not include scales in his definition of a social welfare functional. For him, a utility function for individual  $i$  is a function  $U_i: X \rightarrow \mathbb{R}$  and a

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scale  $(\mathcal{G}, \succ)$ , which they call a *language*. Balinski and Laraki do not explicitly include an interpretation of the grades in their definition of a grading language. Rather, they assume that these grades have commonly understood meanings. They claim that individuals over time will tend to converge on a common understanding of their grading language (Balinski and Laraki, 2010, pp. 166–169). Intuitively, this claim seems quite plausible. However, because their framework contains no formal treatment of interpretation procedures, it is not obvious how to spell out the content of this claim in precise terms. Our own approach lends itself better to studying questions about interpretations of grades and their consequences for collective decision-making.

profile is an  $n$ -tuple  $U$  of such functions. A (Sen) *social welfare functional* is a mapping  $F: \mathcal{D} \rightarrow \mathcal{R}$ , where  $\mathcal{D}$  is now the domain of profiles of utility functions that are to be aggregated. In this section, a profile is simply a profile of utility functions  $U$ , so  $F(U)$  will be denoted by  $R_U$ .

We now formally define the unrestricted domain condition and the two welfarism axioms.

**Unrestricted Domain.** The domain  $\mathcal{D}$  consists of all possible profiles of utility functions on  $X$ .

Unrestricted Domain is appropriate when there are no *a priori* restrictions on the profiles that are to be aggregated.

**Pareto Indifference.** For all  $x, y \in X$  and all  $U \in \mathcal{D}$ , if  $U(x) = U(y)$ , then  $xI_U y$ .

Pareto Indifference requires two alternatives to be socially indifferent when each individual obtains the same utility from them. This axiom ensures that the social preference bears some relationship to the individual utilities, but only in the weak form of respecting universal indifference. It precludes a social preference from being imposed by tradition or by an outside party.

**Independence of Irrelevant Alternatives.** For all  $x, y \in X$  and all  $U, U' \in \mathcal{D}$ , if  $U(x) = U'(x)$  and  $U(y) = U'(y)$ , then  $xR_U y \leftrightarrow xR_{U'} y$ .<sup>8</sup>

Independence of Irrelevant Alternatives requires the social preference for a pair of alternatives to be the same in two profiles if the individual utilities obtained with them in the two profiles coincide on this pair. This axiom implies that the social preference for any pair of alternatives is independent of the utilities obtained with any other alternative.

The welfarist requirement that alternatives be socially evaluated solely in terms of the well-beings (here, utilities) of the individuals is captured by the following neutrality axiom.

**Strong Neutrality.** For all  $w, x, y, z \in X$  and all  $U, U' \in \mathcal{D}$ , if  $U(w) = U'(y)$  and  $U(x) = U'(z)$ , then  $wR_U x \leftrightarrow yR_{U'} z$ .

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<sup>8</sup>In this and similar expressions, the biconditional that is symmetric to the one in the consequence (here,  $yR_U x \leftrightarrow yR_{U'} x$ ) is redundant and so is omitted.

Strong Neutrality precludes taking account of any non-utility information about the alternatives when determining a social preference. In particular, the physical descriptions and names of the alternatives and the profile of utility functions that generate the utilities have no relevance except as mediated through the utilities they generate.

Consider a subset  $\Omega$  of  $\mathbb{R}^n$  whose elements are vectors of utility numbers. A social ordering  $R^*$  of  $\Omega$  is called a *social welfare ordering*. If this ordering can be represented by a function  $W: \Omega \rightarrow \mathbb{R}$ , the representation  $W$  is called a *social welfare function*. For the profile of utility functions  $U$ , the set of *attainable utility vectors* is

$$\Omega_U = \{u \in \mathbb{R}^n \mid u = U(x) \text{ for some } x \in X\}.$$

For the domain  $\mathcal{D}$  of profiles of utility functions, let

$$\Omega_{\mathcal{D}} = \cup_{U \in \mathcal{D}} \Omega_U.$$

When the domain is unrestricted,  $\Omega_{\mathcal{D}}$  is all of  $\mathbb{R}^n$ .

A social welfare ordering  $R^*$  of  $\Omega_{\mathcal{D}}$  is *homologous* to the social welfare functional  $F: \mathcal{D} \rightarrow \mathcal{R}$  if for all  $x, y \in X$  and all  $U \in \mathcal{D}$ ,

$$xR_U y \leftrightarrow U(x)R^*U(y). \tag{1}$$

When the social welfare functional  $F$  satisfies (1), for any profile of utility functions  $U$  in its domain, the social ranking of any pair of alternatives  $x$  and  $y$  can be inferred from how the social welfare ordering  $R^*$  ranks the corresponding utility vectors  $U(x)$  and  $U(y)$ . In effect, all of the information embodied in  $F$  is summarized by the single ordering  $R^*$ . This informational parsimony is a direct consequence of the traditional formulation of welfarism, as we shall now see.

Provided that social preferences are reflexive, Strong Neutrality implies both Pareto Indifference and Independence of Irrelevant Alternatives. If Unrestricted Domain is assumed, these two axioms jointly imply Strong Neutrality, which is why we call them the welfarism axioms. Moreover, on this domain, the two welfarism axioms (or, equivalently, Strong Neutrality) are equivalent to the existence of a social welfare ordering homologous to the social welfare functional.<sup>9</sup>

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<sup>9</sup>The equivalence of the combination of Pareto Indifference and Independence of Ir-

**Theorem 1** (Sen (1977); Hammond (1979)). *For a social social welfare functional  $F: \mathcal{D} \rightarrow \mathcal{R}$  that satisfies Unrestricted Domain, the following conditions are equivalent:*

- (i) *Pareto Indifference and Independence of Irrelevant Alternatives,*
- (ii) *Strong Neutrality, and*
- (iii) *there exists a social welfare ordering  $R^*$  of  $\mathbb{R}^n$  that is homologous to  $F$ .*

The ordering  $R^*$  in this theorem is unique. If utilities were restricted to be nonnegative (resp. positive), then the homologous social welfare ordering would only be defined on the nonnegative (resp. positive) orthant of  $\mathbb{R}^n$ .

Pareto Indifference by itself implies a more limited form of welfarism that is profile dependent. With *profile-dependent welfarism*, the only information that is used to socially rank alternatives is the individual utilities associated with them and the profile of utility functions that generate these utilities. The following neutrality axiom formalizes this property.

**Profile-Dependent Neutrality.** For all  $w, x, y, z \in X$  and all  $U \in \mathcal{D}$ , if  $U(w) = U(y)$  and  $U(x) = U(z)$ , then  $wR_Ux \leftrightarrow yR_Uz$ .

As is the case with Strong Neutrality, Profile-Dependent Neutrality takes no account of the physical descriptions or names of the alternatives when socially ranking them.

For the profile of utility functions  $U$ , the social welfare ordering  $R_U^*$  of  $\Omega_U$  is *homologous* to the social preference ordering  $R_U$  of  $X$  if for all  $x, y \in X$ ,

$$xR_Uy \leftrightarrow U(x)R_U^*U(y). \quad (2)$$

In contrast to (1), the social welfare orderings in (2) are profile dependent.

The analogue of Theorem 1 for profile-dependent welfarism is Theorem 2.<sup>10</sup>

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relevant Alternatives with (i) Strong Neutrality and (ii) the existence of a social welfare ordering that is homologous to the social welfare functional when Unrestricted Domain is assumed are due to Sen (1977) and Hammond (1979), respectively. Sen's Theorem is a Pareto Indifference version of a theorem established by d'Aspremont and Gevers (1977) using a stronger Pareto principle.

<sup>10</sup>Proofs of Theorems 1 and 2 may be found in Bossert and Weymark (2004) and Blackorby, Donaldson, and Weymark (1990), respectively.

**Theorem 2** (Blackorby, Donaldson, and Weymark (1990)). *For a social welfare functional  $F: \mathcal{D} \rightarrow \mathcal{R}$ , the following conditions are equivalent:*

- (i) *Pareto Indifference,*
- (ii) *Profile-Dependent Neutrality, and*
- (iii) *for each  $U \in \mathcal{D}$ , there exists a social welfare ordering  $R_U^*$  of  $\Omega_U$  that is homologous to  $R_U$ .*

Theorem 2 holds for any domain. Pareto Indifference and Profile-Dependent Neutrality place no cross-profile restrictions, so the equivalences in this theorem apply profile by profile. Independence of Irrelevant Alternatives does place cross-profile restrictions and, as we see from Theorem 1, it implies that each social welfare ordering  $R_U^*$  in Theorem 2 is the restriction of a *single* social welfare ordering  $R^*$  of  $\mathbb{R}^n$  to the set of attainable utility vectors  $\Omega_U$  for the profile  $U$  when Unrestricted Domain is satisfied.

#### 4. Scale-Dependent Welfarism

We have argued in Section 1 that a satisfactory formulation of welfarism requires that measurement scales be explicitly considered. We provide a scale-dependent formulation of welfarism in this section and axiomatically characterize it using scale-dependent versions of the axioms presented in Section 3. With *scale-dependent welfarism*, the only information that is used to socially rank alternatives is the individual utilities associated with them and the scales with which these utilities are measured. With scale-dependent welfarism, it is possible to distinguish between real changes in utility from changes due to the use of different measurement scales. The Pareto, independence, and neutrality axioms considered in Section 3 fail to make this distinction.

To fix ideas, we begin with a stylized example.<sup>11</sup> Consider a two-person society. Suppose that with the profile of scales  $S$ , the utility vectors obtained with  $x$  and  $y$  when the profile of utility functions is  $U$  are  $(200, 100)$  and  $(300, 50)$ , respectively. Using the *same* scales, suppose that the utility values associated with every alternative are decreased by a factor of 10 when the profile of utility functions is instead  $U'$ . The utility values for  $x$  and  $y$  are

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<sup>11</sup>There is a completely specified example along similar lines in Section 5.

	$x$	$y$
$(U, S)$	(200, 100)	(300, 50)
$(U', S)$	(20, 10)	(30, 5)
$(U, S')$	(200, 100)	(300, 50)

Table 1: Independence of Irrelevant Alternatives conflates real and scale changes in utilities.

then (20, 10) and (30, 5). If utilities are measured using a common ratio scale, the utility values in the latter case are increased by a factor of 10 if a profile of new scales  $S'$  is used that increases the value of a unit of utility by this factor. The profile of utility functions is now the same as in the first case, but the utility values are measured with a *different* scale. The utilities in these three situations are summarized in Table 1.

Suppose that  $x$  is socially preferred to  $y$  in the first case, perhaps for egalitarian reasons, but that the social preference is reversed in the second case because priority is given to the worst off if utilities are relatively small when measured using the profile of scales  $S$ . It is natural to assume that this preference is invariant to a revaluation of utilities due to a change in the unit of measurement. Hence,  $y$  should also be socially preferred to  $x$  with the profile  $(U, S')$ . But the utility values for  $x$  and  $y$  with this profile are the same as with  $(U, S)$ , so this social preference is inconsistent with Independence of Irrelevant Alternatives. The problem is that this independence axiom takes no account of the scales used and so when scale changes exactly compensate for real (fixed-scales) changes in utilities, it is as if there has been no real change at all. This axiom rules out a perfectly appropriate social response to real changes in utilities when these are countervailed by changes in the scales with which they are measured.

We should thus restrict the scope of the Pareto, independence, and neutrality axioms to comparisons that employ the same scales. This is what we now do using the framework introduced in Section 2. Recall that in this framework, a scale-dependent social welfare functional is a mapping  $F: \mathcal{D} \rightarrow \mathcal{R}$ , where an element of  $\mathcal{D}$  is a profile  $T = (U, S)$  consisting of an  $n$ -tuple of utility functions  $U$  and an  $n$ -tuple of scales  $S$ .

**Scale-Dependent Pareto Indifference.** For all  $x, y \in X$  and all  $T = (U, S) \in \mathcal{D}$ , if  $U(x) = U(y)$ , then  $xI_T y$ .

**Scale-Dependent Independence of Irrelevant Alternatives.** For all  $x, y \in X$  and all  $T = (U, S), T' = (U', S) \in \mathcal{D}$ , if  $U(x) = U'(x)$  and  $U(y) = U'(y)$ , then  $xR_T y \leftrightarrow xR_{T'} y$ .

**Scale-Dependent Strong Neutrality.** For all  $w, x, y, z \in X$  and all  $T = (U, S), T' = (U', S) \in \mathcal{D}$ , if  $U(w) = U'(y)$  and  $U(x) = U'(z)$ , then  $wR_T x \leftrightarrow yR_{T'} z$ .

These axioms differ from their counterparts in Section 3 by explicitly taking into account the measurement scales that give utility values their meanings. In addition, they hold measurement scales fixed. Consider, for example, Scale-Dependent Independence. For this axiom to require that the social ordering of a pair of alternatives be the same for two profiles, it is not enough for the individual utility *values* to be the same in both profiles. In addition, each individual's utility values must be measured using the same scale so that, in fact, he derives the same *utility* from  $w$  and  $y$  and from  $x$  and  $z$ .

Our scale-dependent welfarism theorem employs a scale-dependent analogue of Unrestricted Domain.

**Scale-Dependent Unrestricted Domain.** The domain  $\mathcal{D}$  is  $\mathcal{T}$ .

With this domain assumption, for any fixed profile of scales  $S$  in the set of admissible profiles  $\mathcal{S}$ , any profile of utility functions  $U$  that is in the set of logically possible utility profiles  $\mathcal{U}^S$  for these scales is included in  $\mathcal{T}$ . Thus, this axiom is a natural generalization of Unrestricted Domain when profiles of utility functions are paired with the measurement scales that give utility values their meanings.

Which utility vectors are attainable now depends on both the profile of utility functions and the profile of scales used. Formally, for the profile  $T = (U, S)$ , the set of *attainable utility vectors* is

$$\Omega_T = \{u \in \mathbb{R}^n \mid u = U(x) \text{ for some } x \in X\}.$$

Let

$$\mathcal{D}_S = \{T \in \mathcal{D} \mid \sigma(T) = S\}$$

be the set of all profiles in the domain for which the profile of scales is  $S$ , where  $\sigma(T)$  is the second component of  $T$  ( $T$ 's profile of scales). The set of *attainable utility vectors for the profile of scales  $S$*  is

$$\Omega_S = \cup_{T \in \mathcal{D}_S} \Omega_T.$$

Scale-Dependent Unrestricted Domain implies that  $\Omega_S = \prod_{i \in N} \mathcal{U}_i$  for all  $S \in \mathcal{S}$ , where  $\mathcal{U}_i$  is the set of possible utility values for individual  $i$  with the scale  $S_i$ .

Let  $F|_{\mathcal{D}_S}$  be the restriction of the scale-dependent social welfare functional  $F: \mathcal{D} \rightarrow \mathcal{R}$  to the subdomain  $\mathcal{D}_S$ . A social welfare ordering  $R_S^*$  of  $\Omega_S$  is *homologous* to  $F|_{\mathcal{D}_S}$  if for all  $x, y \in X$  and all  $T = (U, S) \in \mathcal{D}_S$ ,

$$xR_T y \leftrightarrow U(x)R_S^*U(y). \quad (3)$$

The social welfare orderings in (3) are scale dependent.

Theorem 3 is our scale-dependent welfarism theorem. The proof of this theorem may be found in the Appendix.

**Theorem 3.** *For a scale-dependent social social welfare functional  $F: \mathcal{D} \rightarrow \mathcal{R}$  that satisfies Scale-Dependent Unrestricted Domain, the following conditions are equivalent:*

- (i) *Scale-Dependent Pareto Indifference and Scale-Dependent Independence of Irrelevant Alternatives,*
- (ii) *Scale-Dependent Neutrality, and*
- (iii) *for each  $S \in \mathcal{S}$ , there exists a social welfare ordering  $R_S^*$  of  $\prod_{i \in N} \mathcal{U}_i$  that is homologous to  $F|_{\mathcal{D}_S}$ .*

None of the axioms in Theorem 3 place any restrictions on how the social preferences for profiles with different scales are related. That is why the social welfare orderings in this theorem are scale dependent. Similarly, in Theorem 2, there are no cross-profile restrictions on the social preferences. Consequently, the social welfare orderings are profile-dependent in that theorem. The scale-dependent axioms in Theorem 3 do place cross-profile restrictions on the social preferences but only for fixed scales, and that is why the social welfare orderings are scale-dependent but not profile dependent.

It is straightforward to define scale-dependent versions of the various other axioms that have been considered for social welfare functionals and to determine their implications for the structure of a social welfare ordering, so we shall only do so informally. We illustrate what is involved with two Pareto and one anonymity axioms. *Scale-Dependent Weak Pareto* regards one alternative to be socially preferred to another if everybody is better off with the first alternative than with the second using a fixed profile of scales.

*Scale-Dependent Strong Pareto* regards one alternative to be weakly socially preferred to another if everybody is weakly better off with the first alternative than with the second using a fixed profile of scales and, furthermore, there is strict social preference if, in addition, at least one person is strictly better off. These Pareto principles imply that each social welfare ordering  $R_S^*$  is weakly monotonic and strictly monotonic, respectively. With *Scale-Dependent Anonymity*, the social preference is invariant to a permutation of the individual utility functions and measurement scales. If everybody uses the same scale in  $S$ , this axiom implies that  $R_S^*$  is symmetric.

Blackorby, Bossert, and Donaldson (2005a) have proposed a generalization of a social welfare functional in which non-welfare information may be used in addition to the individual utility functions in order to determine the social preference. They model non-welfare information in an abstract way that can encompass a wide variety of interpretations. Moreover, their framework is flexible enough to distinguish between social and individual non-welfare information.<sup>12</sup> Examples of non-welfare information that they consider include the presence or absence of democratic institutions and an individual’s length of life or propensity to work hard. Formally, a profile of scales can be interpreted as being non-welfare information in their sense.<sup>13</sup> As a consequence, a scale-dependent social welfare functional is a particular instantiation of their social aggregation procedure.

Blackorby, Bossert, and Donaldson (2005a) define analogues of the axioms considered here, but reformulated so as to apply to their formal framework. While it is in principle possible using their framework to take account of non-welfare information when forming a social preference, they show that their axioms imply that, in fact, non-welfare information is ignored, and so the social aggregation procedure is welfarist in the traditional sense; that is, non-welfare information is irrelevant. Their independence condition only applies when both the utilities and non-welfare information for the alternatives being

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<sup>12</sup>For an alternative approach to augmenting a social welfare functional with non-welfare information, see Roberts (2010).

<sup>13</sup>At a formal level, this is true. However, utility scales are different in kind from the examples of non-welfare information considered by Blackorby, Bossert, and Donaldson (2005a) in that scales specify information about well-beings more fully, rather than supplement well-being information with further considerations. Analogously, having said that the length of something is  $l$ , to add that this measurement is in meters is not to provide further “non-length” information, but rather to more fully specify how  $l$  should be interpreted.

compared are the same, echoing our own proposal that Independence ought to apply only when utilities are measured using the same scales. However, their Pareto indifference axiom applies when the utilities are the same even if the non-welfare information is not held fixed, which is too strong when the non-welfare information is about measurement scales. If their Pareto principle is instead formulated so that it only applies to comparisons in which the non-welfare information is held fixed, a more limited form of welfarism is characterized, one that is conditional on the non-welfare information, in precisely the same way that our formulation of welfarism is conditional on the scales being used to measure utilities. Thus, the more limited scope of our Pareto axiom is what prevents our scale-dependent account of welfarism from collapsing to the traditional form of welfarism.

## 5. A Scale-Dependent Welfarist Example

The numerical example used in Section 4 shows why it is important to augment Sen’s social welfare functionals with information about the scales being used to measure utility. However, it does not provide a fully specified scale-dependent social welfare functional. In this section, we present an example of a such a functional that is welfarist in our sense, but not in the traditional sense. This example formalizes distributional objectives that are quite plausible, but which cannot be captured by the conventional formulation of welfarism, thereby illustrating the added flexibility provided by our approach.

We suppose that everybody uses the same scale and that there is a scale-dependent utility level that is minimally acceptable socially. We take this fact into account when determining a social preference. We give priority to equity considerations when there are some individuals who have not achieved this threshold, but not if the threshold has been met by everybody. Specifically, our hybrid social aggregation procedure is utilitarian for comparisons in which nobody fails to meet the minimally acceptable level and it employs the maximin utility criterion otherwise. This functional is welfarist in the scale-dependent sense, but not in the traditional one, because which number represents the minimally acceptable utility depends on the scale used to measure utilities.

Formally, for all  $S \in \mathcal{S}$ , we assume that  $S_i = S_j$  for all  $i, j \in N$ . Let  $\mu_S$  be the *minimally acceptable utility* as expressed in this common scale. For

any profile of utility functions  $U \in \mathfrak{U}^S$  and any alternative  $x \in X$ , let

$$\mu U(x) = \min\{U_1(x), \dots, U_n(x)\}$$

be the *minimum utility* achieved in the profile  $U$  with the alternative  $x$ . The *hybrid* scale-dependent social welfare functional  $F_H: \mathcal{T} \rightarrow \mathcal{R}$  is defined by setting, for each profile  $T = (U, S) \in \mathcal{T}$  and each pair of alternatives  $x, y \in X$ ,

$$xR_T y \leftrightarrow \begin{cases} \Sigma_i U_i(x) \geq \Sigma_i U_i(y), & [\mu U(x) \geq \mu_S] \wedge [\mu U(y) \geq \mu_S] & (4a) \\ \mu U(x) \geq \mu U(y), & [\mu U(x) < \mu_S] \vee [\mu U(y) < \mu_S]. & (4b) \end{cases}$$

First, we show that  $F_H$  is a social welfare functional. That is, for any profile  $T$  in its domain, the corresponding social preference  $R_T$  is an ordering of  $X$ . Reflexivity and completeness follow directly from the corresponding properties of the relation  $\geq$  on  $\mathbb{R}$ . To show transitivity, let  $xR_T y$  and  $yR_T z$ . There are two cases to consider. In the first case,  $\mu U(y) < \mu_S$ . In this case, (4b) applies to both of the comparisons with  $y$ , and so  $\mu U(x) \geq \mu U(y) \geq \mu U(z)$ . Hence, we also have  $\mu U(z) < \mu_S$ , and therefore (4b) also applies to the comparison of  $x$  and  $z$ . Because  $\mu U(x) \geq \mu U(z)$ , it follows that indeed  $xR_T z$ . In the second case,  $\mu U(y) \geq \mu_S$ . Note that  $xR_T y$  and  $\mu U(y) \geq \mu_S$  jointly imply that  $\mu U(x) \geq \mu_S$  as well.<sup>14</sup> If it is also the case that  $\mu U(z) \geq \mu_S$ , then (4a) applies to the comparisons of both  $x$  and  $z$  with  $y$ , and so  $\Sigma_i U_i(x) \geq \Sigma_i U_i(y) \geq \Sigma_i U_i(z)$ . Because (4a) also applies to the comparison of  $x$  and  $z$  and  $\Sigma_i U_i(x) \geq \Sigma_i U_i(z)$ , it then follows that  $xR_T z$ . If, instead,  $\mu U(z) < \mu_S$ , then (4b) applies to the comparison of  $x$  and  $z$ , and we again have  $xR_T z$  because  $\mu U(x) \geq \mu_S > \mu U(z)$ , which completes the demonstration that  $R_T$  is transitive.

By construction,  $F_H$  satisfies Scale-Dependent Unrestricted Domain. We now show that it also satisfies the two scale-dependent welfarism axioms. Consider any profile  $T = (U, S) \in \mathcal{T}$  for which  $U(x) = U(y)$ . Because both  $\Sigma_i U_i(x) = \Sigma_i U_i(y)$  and  $\mu U(x) = \mu U(y)$ , we have  $xI_T y$  regardless of which of (4a) or (4b) applies. Hence,  $F_H$  satisfies Scale-Dependent Pareto Indifference. Now consider any two profiles  $T = (U, S), T' = (U', S) \in \mathcal{T}$  for which  $U(x) = U'(x)$  and  $U(y) = U'(y)$ . Then,  $\mu U(x)$  and  $\mu U(y)$  both meet

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<sup>14</sup>In general, no social improvement lets anyone slip below the minimum when previously everybody had achieved this threshold. In other words, a social improvement preserves acceptability.

the threshold  $\mu_S$  if and only if both  $\mu U'(x)$  and  $\mu U'(y)$  do as well. Thus, for both profiles, the social ranking of  $x$  and  $y$  is determined by the same case in the definition of  $F_H$ . That is, both rankings are determined by (4a) or they are both determined by (4b). Because both of these cases only consider the two alternatives being compared and the individual utilities achieved with them, it follows that  $R_T$  and  $R_{T'}$  coincide on  $\{x, y\}$ . Hence,  $F_H$  also satisfies Scale-Dependent Independence of Irrelevant Alternatives.

We have thus shown that  $F_H$  is a scale-dependent social welfare functional that satisfies the scale-dependent unrestricted domain, Pareto, and independence axioms used in Theorem 3. This does not establish that  $F_H$  is not welfarist in the traditional sense because we have not shown that there is not an alternative way of describing this functional in scale-independent terms. We now show that there is not. For this purpose, it is sufficient to show that there exist two profiles  $T = (U, S)$ ,  $T' = (U, S') \in \mathcal{T}$  for which  $F(T) \neq F(T')$ , for then there can be no social welfare ordering  $R^*$  that is homologous to  $F_H$ .

For simplicity, let there be just two individuals, 1 and 2. Suppose that with the profiles of scales  $S$  and  $S'$ , the common set of permissible grades is  $\mathbb{R}_+$  for both profiles, but their interpretations differ. Specifically, with the interpretation used in  $S'$ , the unit in which utility is measured is half of the one used in  $S$ . For  $S$ , let  $\mu_S = 25$ . It is then natural to set  $\mu_{S'}$  equal to 50, reflecting the fact that the unit of measurement has been halved.<sup>15</sup> For the alternatives  $x$  and  $y$ , suppose that the utility values are  $U_1(x) = 40$ ,  $U_1(y) = 50$ ,  $U_2(x) = 70$ , and  $U_2(y) = 50$ . Interpreting these numbers using the common scale in  $S$ , both individuals achieve the minimally acceptable utility of 25 with both  $x$  and  $y$ . That is,  $\mu U(x) \geq \mu_S$  and  $\mu U(y) \geq \mu_S$ . Thus, (4a) in the definition of  $F_H$  applies. Because  $\sum_i U_i(x) = 40 + 70 > 50 + 50 = \sum_i U_i(y)$ , we have  $x P_T y$ . On the other hand, with  $S'$ , individual 1's utility of 40 falls below the minimally acceptable utility of 50. Hence,  $\mu U(x) < \mu_{S'}$  and therefore (4b) in the definition of  $F_H$  applies. Because  $\mu U(y) = 50 > 40 = \mu U(x)$ , it then follows that  $y P_{T'} x$ . Hence,  $F(T) \neq F(T')$ , as was to be shown.

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<sup>15</sup>As an application of this formalism, suppose that utility is given by life expectancy at birth and that in  $S$  this life expectancy is measured in years, whereas in  $S'$  it is measured in half years.

## 6. Information Invariance

In Sen’s approach to determining social preferences, information invariance conditions play an important role in the axiomatic characterizations of particular classes of social welfare functionals (or of specific social welfare functionals) and their corresponding social welfare orderings (see, e.g., Bossert and Weymark, 2004). As we have explained, the combination of welfarism in its traditional formulation with an information invariance condition cannot properly distinguish real changes in utilities from mere changes in scale. In this section, we describe how information invariance conditions are modeled in our framework and how they constrain how social welfare orderings for different scales are related. We also discuss the implications of using our approach for characterizing classes of social welfare orderings.

We begin by reviewing how information invariance for a social welfare functional is typically modeled using invariance transforms.<sup>16</sup> An *invariance transform* is an  $n$ -tuple of functions  $\phi = (\phi_1, \dots, \phi_n)$ , where  $\phi_i: \mathbb{R} \rightarrow \mathbb{R}$  for all  $i \in N$ . An invariance transform is applied to a profile of utility functions  $U$  in order to obtain an informationally-equivalent profile. The transform  $\phi$  operates component-wise by means of function composition, mapping  $U$  into  $\phi \circ U = (\phi_1 \circ U_1, \dots, \phi_n \circ U_n)$ , where  $\circ$  denotes the function composition operator. Different assumptions concerning the measurability and comparability of utility are captured by specifying a class  $\Phi$  of invariance transforms and requiring a social preference to be invariant to the application of a transform from this class to any profile of utility functions in the domain of the social welfare functional  $F$ .<sup>17</sup>

**Information Invariance for  $\Phi$ .** For all  $U, U' \in \mathcal{D}$ , if  $U' = \phi \circ U$  for some  $\phi \in \Phi$ , then  $R_U = R_{U'}$ .

For example,  $\Phi$  could be the class of all  $n$ -tuples of increasing affine transforms  $\phi$  for which the unit scaling term is the same for each of the  $\phi_i$ .<sup>18</sup> Transforms in this class preserve comparisons of utility differences both

<sup>16</sup>For more details, see Bossert and Weymark (2004). This is not the only way that information invariance is modeled, but it is the most common. See Bossert and Weymark (2004, Section 5) for an alternative approach.

<sup>17</sup>In order for  $\Phi$  to partition the domain of the functional into cells of informationally-equivalent profiles,  $(\Psi, \circ)$  must be an algebraic group. See Roberts (1980).

<sup>18</sup>Formally,  $\phi \in \Phi$  if and only if for all  $i \in N$ ,  $\phi_i(t) = \alpha_i + \beta t$  for all  $t \in \mathbb{R}$ , where  $\alpha_i$  and  $\beta$  are scalars with  $\beta > 0$ .

intrapersonally and interpersonally, and so utility is *cardinally measurable and unit comparable*. As a second example,  $\Phi$  could be the class of all  $n$ -tuples of increasing transforms, in which case utility is *ordinally measurable and noncomparable*. In this case, social preferences only depend on the individual preferences, as in Arrow (1951). The main classes of invariance transforms that have been considered may be found in Bossert and Weymark (2004).

When Theorem 1 applies (i.e., when the social welfare functional is welfarist in the traditional sense), there is an equivalent invariance condition to Information Invariance for  $\Phi$  for the social welfare ordering (SWO)  $R^*$ .

**SWO Information Invariance for  $\Phi$ .** For all  $u, v, u', v' \in \Omega_{\mathcal{D}}$ , if there exists a  $\phi \in \Phi$  such that  $u' = \phi(u)$  and  $v' = \phi(v)$ , then  $uR^*v \leftrightarrow u'R^*v'$ .

An important implication of this condition is that any invariance transform  $\phi$  maps an indifference contour of the social welfare ordering  $R^*$  into another indifference contour of this relation (see Dixit, 1980; Blackorby, Donaldson, and Weymark, 1984). It is this property that provides much of the power of an information invariance condition, but it is also what accounts for the failure to distinguish between real and scale changes in utilities when the social aggregation procedure is welfarist in the traditional sense. For example, in the case in which utilities are ordinally noncomparable, one can always find an admissible transform  $\phi$  that maps any utility vector  $u$  back to itself and maps any utility vector  $v$  that is socially indifferent to it into any other vector  $v'$  that has  $v'_i \geq u_i \leftrightarrow v_i \geq u_i$  for all  $i \in N$ . Because  $R^*$  is transitive, this mapping only preserves social indifference if the social welfare ordering  $R^*$  is dictatorial or anti-dictatorial (i.e., there is some individual for whom any decrease in his utility is regarded as being a social improvement).<sup>19</sup> If  $R^*$  is weakly Paretian, we are left with an Arrovian dictator (Blackorby, Donaldson, and Weymark, 1984). If, however, utilities are distinguished by the scales in which they are measured, the pre- and post-transform indifference contours need not be the same, which opens up further possibilities.

Consider any two profiles of scales  $S, S' \in \mathcal{S}$ , where  $S_i = (\mathcal{U}_i, \mathcal{I}_i)$  and  $S'_i = (\mathcal{U}'_i, \mathcal{I}'_i)$  for all  $i \in N$ . Let  $\phi$  be any invariance transform. The profile  $S'$  is a  $\phi$ -transform of  $S$  if for all  $i \in N$ ,  $\mathcal{U}'_i = \phi_i(\mathcal{U}_i)$  and the interpretation of  $\phi_i(u_i)$  in the scale  $S'_i$  is the same as the interpretation of  $u_i$  in the scale

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<sup>19</sup>This inference implicitly assumes that  $R^*$  is continuous. If it is not, then the same conclusion follows by considering how the upper and lower contours sets of  $R^*$  are transformed by the admissible transforms.

$S_i$  for all  $u_i \in \mathcal{U}_i$ . Informally, individual  $i$  uses the transform  $\phi_i$  to translate each utility value measured using his original scale into a new value measured using the transformed scale with the transformed utilities retaining their pre-transform meanings.

The two information invariance conditions considered above can be reformulated using our framework in order to take account of the dependence on the scales used to measure utilities.

**Scale-Dependent Information Invariance for  $\Phi$ .** For all  $T = (U, S)$ ,  $T' = (U', S) \in \mathcal{D}$ , if there exists a  $\phi \in \Phi$  such that (i)  $U' = \phi \circ U$  and (ii)  $S'$  is a  $\phi$ -transform of  $S$ , then  $R_T = R_{T'}$ .

**Scale-Dependent SWO Information Invariance for  $\Phi$ .** For all  $u, v \in \Omega_S$  and  $u', v' \in \Omega_{S'}$ , if there exists a  $\phi \in \Phi$  such that (i)  $u' = \phi(u)$  and  $v' = \phi(v)$  and (ii)  $S'$  is a  $\phi$ -transform of  $S$ , then  $uR_S^*v \leftrightarrow u'R_{S'}^*v'$ .

Thus, for a scale-dependent social welfare functional, invariance of the social preference for the alternatives in  $X$  is conditional on the invariance transform being applied to both the profile of utility functions and to the profile of scales, not just to the profile of utility functions. For a social welfare ordering, the invariance condition requires the social ranking of a pair of utility vectors obtained with the pre-transform social welfare ordering  $R_S^*$  to coincide with the social ranking of the corresponding transformed utility vectors obtained with the post-transform social welfare ordering  $R_{S'}^*$ . This condition implies that an indifference contour (resp. upper contour set, lower contour set) of  $R_S^*$  is mapped by an invariance transform  $\phi \in \Phi$  into an indifference contour (resp. upper contour set, lower contour set) of  $R_{S'}^*$ , where  $S'$  is the profile of scales obtained by applying  $\phi$  to  $S$ . Moreover, nothing more can be inferred from this condition by itself or in combination with any of the standard axioms.

In contrast, with the standard formulation of welfarism, the pre- and post transformed social welfare orderings must be identical. As we have noted, it is for this reason that information invariance conditions play such an important role in characterizing classes of social welfare orderings that exhibit considerable structure when the traditional framework is employed. For example, d'Aspremont and Gevers (1977) show that the utilitarian sum-of-utilities ordering is the only social welfare ordering that is anonymous and strongly Paretian when utilities are cardinally measurable and unit comparable. In our framework, these two axioms imply that a social welfare ordering

$R_S$  for the scale  $S$  is symmetric and strictly monotonic when everybody uses the same measurement scale. Adding the information invariance condition merely links the social welfare orderings for different profiles of scales in the way described above.

In general, with scale-dependent welfarism, there is complete freedom to choose any social welfare ordering that satisfies all but the informational invariance condition for one profile of scales  $S$ . Once this is done, the information invariance condition completely determines the social welfare ordering for any other profile of scales. In particular, in the case of ordinally measurable and noncomparable utilities, any social welfare ordering that is both weakly monotonic and sensitive to the utilities of at least two individuals can be used for the profile  $S$ . This claim may appear to contradict Arrow's Theorem, but it does not. Just as with the use of grades by Balinski and Laraki (2007, 2010), the interpretation of the utilities, as provided by the scales in which they are measured, conveys useful information over and above what is provided by the individual preferences, and this is what allows us to escape from the nihilism of Arrow's Theorem when utilities are ordinally measurable and noncomparable.

## 7. Conclusion

We have introduced a generalization of the concept of a social welfare functional that explicitly takes account of the measurement scales that are used. This generalization makes it possible to distinguish between a real change in individual well-beings from one that is merely representational due to the use of different scales to measure them. Using this framework, we have described a scale-dependent form of welfarism and shown how it can be characterized using scale-dependent versions of the standard welfarism axioms. The implications of our approach for characterizing classes of social welfare orderings has also been considered. Allowing social preferences to depend on the utility scales being used limits the role that information invariance conditions play, thereby substantially enlarging the range of social aggregation procedures that are normatively appealing compared to what is possible with the traditional formulation of welfarism.

## Appendix

*Proof of Theorem 3.* Suppose that  $F$  satisfies Scale-Dependent Unrestricted Domain. Consider any  $S \in \mathcal{S}$ . This profile of scales is fixed throughout this proof.

(i)  $\Rightarrow$  (ii). Suppose that  $F$  satisfies Scale-Dependent Pareto Indifference and Scale-Dependent Independence of Irrelevant Alternatives. Consider any  $w, x, y, z \in X$  and any  $T^1 = (U^1, S), T^2 = (U^2, S) \in \mathcal{D}_S$  such that  $U^1(w) = U^2(y)$  and  $U^1(x) = U^2(z)$ . Let  $U^1(x) = U^2(z) = u$  and  $U^1(y) = U^2(w) = \bar{u}$ . By Scale-Dependent Unrestricted Domain, there exists an alternative  $t \in X$  and profiles  $T^3 = (U^3, S), T^4 = (U^4, S), T^5 = (U^5, S) \in \mathcal{D}_S$  such that (a)  $U^3(x) = U^3(t) = u$  and  $U^3(y) = \bar{u}$ , (b)  $U^4(z) = U^4(t) = u$  and  $U^4(w) = \bar{u}$ , and (c)  $U^5(t) = u$  and  $U^5(y) = U^5(w) = \bar{u}$ .<sup>20</sup> By Scale-Dependent Independence of Irrelevant Alternatives,  $xR_{T^1}y \leftrightarrow xR_{T^3}y$ . Scale-Dependent Pareto Indifference and the transitivity of  $R_{T^3}$  imply that  $xR_{T^3}y \leftrightarrow tR_{T^3}y$ . Similarly,  $tR_{T^3}y \leftrightarrow tR_{T^5}y \leftrightarrow tR_{T^5}w$ . Applying the same argument once again,  $tR_{T^5}w \leftrightarrow tR_{T^4}w \leftrightarrow zR_{T^4}w$ . Scale-Dependent Independence of Irrelevant Alternatives then implies that  $zR_{T^4}w \leftrightarrow zR_{T^2}w$ . We have thus shown that  $xR_{T^1}y \leftrightarrow zR_{T^2}w$ . Hence,  $F$  satisfies Scale-Dependent Strong Neutrality.

(ii)  $\Rightarrow$  (i). Suppose that  $F$  satisfies Scale-Dependent Strong Neutrality. By setting  $w = y$  and  $x = z$  in the definition of this neutrality axiom, Scale-Dependent Independence of Irrelevant Alternatives follows. By setting  $U = U'$  and  $x = y = z$ , Scale-Dependent Strong Neutrality implies that  $wR_Tx \leftrightarrow xR_Ux$  when  $U(w) = U(x)$ . Reflexivity of  $R_T$  then implies that  $wI_Tx$  and, hence, that Pareto indifference is satisfied.

(ii)  $\Rightarrow$  (iii). Suppose that  $F$  satisfies Scale-Dependent Strong Neutrality. For any  $u, \bar{u} \in \Pi_{i \in N} \mathcal{U}_i$ , by Scale-Dependent Unrestricted Domain, there exists a profile  $T = (U, S) \in \mathcal{D}_S$  and a pair of alternatives  $x, y \in X$  such that  $U(x) = u$  and  $U(y) = \bar{u}$ . Define  $R_S^*$  on  $\{u, \bar{u}\}$  by setting  $uR_S^*\bar{u} \leftrightarrow xR_Ty$  and  $\bar{u}R_S^*u \leftrightarrow yR_Tx$ . By Scale-Dependent Strong Neutrality, this ranking of  $u$  and  $\bar{u}$  does not depend on the profile in  $\mathcal{D}_S$  or on the alternatives in  $X$  used to obtain  $u$  and  $\bar{u}$ . To show that  $R_S^*$  is transitive, consider any any  $u, \bar{u}, \hat{u} \in \Pi_{i \in N} \mathcal{U}_i$  with  $uR_S^*\bar{u}$  and  $\bar{u}R_S^*\hat{u}$ . By Scale-Dependent Unrestricted Domain, there exists a profile  $T = (U, S) \in \mathcal{D}_S$  and alternatives  $x, y, z \in X$  such that  $U(x) = u, U(y) = \bar{u}$ , and  $U(z) = \hat{u}$ . Because  $U(x)R_S^*U(y)$  and  $U(y)R_S^*U(z)$ , by the construction of  $R_S^*$  it follows that  $xR_Ty$  and  $yR_Tz$ . Transitivity of  $R_T$

<sup>20</sup>Except for the case in which  $w = x$  and  $y = z$ ,  $t$  can be chosen from the set  $\{w, x, y, z\}$ .

then implies that  $xR_Tz$ . Hence,  $U(y)R_S^*U(z)$  or, equivalently,  $uR_S^*\hat{u}$ . Thus,  $R_S^*$  is transitive.

(iii)  $\Rightarrow$  (ii). This implication follows immediately from the definitions.<sup>21</sup>

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<sup>21</sup>Our proof follows the same basic proof strategy that is used by Bossert and Weymark (2004) to prove Theorem 1. Their proof in turn draws on the proof of Theorem 1 in d'Aspremont (1985).

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