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Majority Rule and Selfishly Optimal Nonlinear Income Tax Schedules with Discrete Skill Levels

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Keywords: nonlinear income taxation; political economy of taxation; optimal bunching; redistributive taxation; voting over tax schedules.

JEL classification numbers: D72, D82, H21.

1 Introduction

An alternative that obtains at least as many votes as any other alternative on the agenda in a pairwise majority contest is a Condorcet winner. As Condorcet (1785) himself has shown with his famous paradox of majority voting, without restrictions on the voter's preferences or on the set of alternatives, there may not be any Condorcet winner. However, for a one-dimensional set of alternatives, Black's Median Voter Theorem (Black, 1948) shows that a most-preferred alternative of a median voter is a Condorcet winner if preferences are single-peaked. With a single-peaked preference, each voter has a set of most-preferred alternatives, with preference declining monotonically in either direction from this preference peak.[1]

Röell (2012) shows that Black's Median Voter Theorem applies to voting over nonlinear income tax schedules that satisfy the constraints of a finite type version of the Mirrlees (1971) optimal income tax problem, as in Guesnerie and Seade (1982), when voting takes place only over the tax schedules that are selfishly optimal for some individual and preferences are quasilinear. In this article, we provide an alternative way of establishing Röell's median voter result that offers a different perspective on her findings, drawing on insights obtained by Brett and Weymark (2017b) in their analysis of a version of this problem with a continuum of types. In order to characterize a selfishly optimal schedule, we determine how to optimally bunch different types of individuals. Our approach to characterizing optimal bunching can be applied to other asymmetric information problems with a finite number of types that feature self-selection constraints.

In the Mirrlees (1971) optimal income tax problem, there are two commodities, consumption and labor supply, with individuals differing in their labor productivity, which is private information. Individuals with the same productivity are of the same type and behave identically. A tax schedule is feasible if the resulting allocation of consumptions and incomes (i) satisfies the economy's production feasibility constraint (or, equivalently, the government budget constraint) and (ii) the incentive-compatibility constraint that each type optimally chooses its consumption and labor supply (or, equivalently, its income) given the tax schedule. If no further restrictions are placed on the tax schedules being voted on, then no Condorcet winner exists.

In the literature on voting over nonlinear income tax schedules, voting in fact takes place over feasible allocations. This is equivalent to voting over feasible tax schedules because, by the Taxation Principle of Hammond (1979) and Guesnerie (1995), choosing a tax schedule subject to the incentive-compatibility constraint is equivalent to choosing the allocation of consumption and income directly subject to standard self-selection constraints. We follow this practice here by identifying an income tax schedule with its associated allocation.

A type's selfishly optimal allocation is the one that this type would choose from among the feasible allocations if it were a dictator. Röell (2012) restricts voting to these allocations. In effect, each type gets to propose one allocation that will subsequently be voted on along with the proposals of the other types. The selfishly optimal allocations can be indexed by the types' labor productivities, and so are one-dimensional in this parameter. Röell shows that when

[1] A preference can alternatively be described as being single-peaked if on any triple of alternatives there is one of them that is not ranked last by anyone who is not indifferent among all three (Arrow, 1951).

the agenda is restricted in this way and utility is quasilinear in consumption, then each type's preferences are single-peaked on the agenda and, hence, the preferred allocation of the type with the median labor productivity is a Condorcet winner.[2]

Röell (2012) does not completely characterize a type's proposal. Instead, she identifies some of its qualitative properties and then uses these properties to show that the types' preferences are single-peaked. Here, we show that it is possible to provide a simple characterization of the selfishly optimal allocations in the quasilinear case. Röell's single-peakedness result easily follows from this characterization. Our alternative approach to analyzing Röell's problem thus provides further insight into her median voter theorem. So as to take advantage of a theorem established by Brett and Weymark (2017a), we assume that preferences are quasilinear in labor, but otherwise our model is the same as that of Röell (2012).[3]

Brett and Weymark (2017a) show that a type's selfishly optimal allocation can be determined in two steps when preferences are quasilinear in labor. A type first chooses the consumptions for each type (including its own type) by solving a reduced-form problem in which an additively separable function of the consumptions is maximized subject to the consumptions being nonnegative and nondecreasing in type. Recursion formulae are then used to determine the optimal incomes. An implication of their results is that a selfishly optimal allocation is completely identified by the schedule showing the optimal consumption as a function of type.

Brett and Weymark (2017a) do not solve their reduced-form problem. We do that here. In order to solve this problem, we identify which types should be bunched together. The methodology used to do this generalizes an approach introduced by Weymark (1986a) for identifying optimal bunching patterns in an optimal nonlinear income tax problem with a weighted utilitarian objective when the number of types is finite so as to allow for a non-uniform type distribution. As we have noted, this way of characterizing optimal bunching is applicable to other kinds of finite-type asymmetric information problems.

We show that a type's proposed consumption schedule consists of up to three regions. In the first and third regions, the consumptions track the maximax and maximin schedules, respectively. The middle region consists of the proposer's type and all of those types (if any) that are bunched with it (i.e., have the same consumption). One or the other of the first and third regions may not be present. We also show that marginal tax wedges (i.e., implicit marginal tax rates) are negative in the first region and positive in the third.

[2] There is an extensive literature on majority voting over income tax schedules, some of which restricts the voting to be over selfishly optimal schedules. For a brief introduction to this literature, see Brett and Weymark (2017b).

[3] It is straightforward to reformulate our analysis in terms of quasilinear-in-consumption preferences. See Section 6.

When there is a continuum of types, Brett and Weymark (2017b) show that a type's proposed consumption schedule exhibits the same features as is found here for a finite number of types. In their problem, the monotonicity constraints on consumptions are the second-order conditions in the corresponding reduced-form problem of the proposer. If these constraints are ignored, in the resulting relaxed problem, the proposed consumptions coincide with the maximax schedule for lower-skilled types and with the maximin schedule for the higher-skilled types, with a downward discontinuity at the proposer's own type, violating the monotonicity constraints. In order to satisfy these constraints, it is necessary to "iron" (i.e., smooth) the schedule by bunching the proposer with nearby types. The standard way of ironing a non-monotone schedule is to use the control-theoretic techniques described by Guesnerie and Laffont (1984).[4] Brett and Weymark (2017b) demonstrate that the bunching region containing the proposer's type can be identified more simply using calculus. Our characterization of the three regions for a selfishly optimal consumption schedule also makes use of a relaxed problem in which the monotonicity constraints on consumptions are ignored. However, because the type space is discrete, we cannot use calculus to determine which types should be in the second region. It is for this reason that we need to generalize the procedure for identifying optimal bunching regions in Weymark (1986a).

The plan for the rest of this article is as follows. In Section 2, we introduce the model. A proposer's reduced-form problem is described in Section 3. The solution to this problem is characterized in Section 4. Voting over the selfishly optimal allocations is analyzed in Section 5. In Section 6, we offer some concluding remarks.

2 The Model

There are two commodities, consumption and labor. The consumption good is produced using using a constant-returns-to-scale technology with labor as its single input. Markets are competitive. The price of the consumption good is normalized to equal 1.

There are N individuals, each of whom is one of $n \geq 2$ types, with n_i of them being of type i. Type i is characterized by its marginal product of labor w_i (its skill level), which is also type i's wage rate. Types are ordered so that

$$0 < w_1 < w_2 < \cdots < w_n. \tag{1}$$

Everybody knows what the n skill levels are and how many individuals there are of each type, but nobody knows anyone's type except for its own.

The consumption and labor supply of an individual of type i are c_i and l_i, respectively. All types have the same quasilinear-in-labor utility function $u\colon \mathbb{R}^2_+ \to \mathbb{R}$ given by

[4] Arrow (1968) was the first to show how to iron a non-monotone schedule in his analysis of optimal capital policy with irreversible investment.

$$u(l_i, c_i) = v(c_i) - l_i. \tag{2}$$

It is assumed that the function v is strictly increasing, strictly concave, and twice continuously differentiable with $v(0) = 0$, $v'(0) = \infty$, and $\lim_{r\to\infty} v'(r) = 0$.

Type i's (pretax) income is

$$y_i = w_i l_i, \quad i = 1, \dots, n, \tag{3}$$

which is also its labor supply in efficiency units. Because the price of the consumption good is equal to 1, c_i is also type i's after-tax income. An individual knows its own labor supply, but not that of anybody else. The *allocation* $\mathbf{a} = (\mathbf{y}, \mathbf{c}) \in \mathbb{R}^{2n}_{+}$, which consists of an *income vector* $\mathbf{y} = (y_1, \dots, y_n)$ and a *consumption vector* $\mathbf{c} = (c_1, \dots, c_n)$, is publicly observable. Type i's *commodity bundle* is $(y_i, c_i) \in \mathbb{R}^2_{+}$.

Expressed in terms of publicly observable variables, type i has a type-specific utility function $U^i\colon \mathbb{R}^2_{+} \to \mathbb{R}$ given by

$$U^i(y_i, c_i) = v(c_i) - \frac{y_i}{w_i}, \quad i = 1, \dots, n, \tag{4}$$

which is obtained by substituting (3) into (2). This type's marginal rate of substitution at (y_i, c_i) is

$$\mathrm{MRS}^i(y_i, c_i) = \frac{1}{w_i v'(c_i)}, \quad i = 1, \dots, n, \tag{5}$$

which is independent of income. The Mirrlees (1971) *single-crossing property*, which requires that indifference curves of two different types cross at most once, is satisfied because the marginal rate of substitution for any commodity bundle is decreasing in the skill level.

The income tax (subsidy, if negative) paid by an individual of type i is

$$t_i = y_i - c_i, \quad i = 1, \dots, n. \tag{6}$$

Taxation is only used for redistributive purposes. Hence, the *government budget constraint* is

$$\sum_{i=1}^{n} n_i t_i \geq 0. \tag{7}$$

This constraint is satisfied if and only if

$$\sum_{i=1}^{n} n_i c_i \leq \sum_{i=1}^{n} n_i y_i, \tag{8}$$

which is the *production feasibility constraint*. Informally, aggregate consumption cannot exceed aggregate labor supply in efficiency units. The latter is the amount of the consumption good that is produced. Constraint (8) binds if and only if (7) does. That is, production efficiency is equivalent to the government balancing its budget.

An income tax schedule specifies the tax paid as a function of income. Choosing an income tax schedule is equivalent to choosing a budget set consisting of the income-consumption pairs (y, c) for which c is affordable given the after-tax income left after paying the tax due when the income is y. An allocation $\mathbf{a}$ is *incentive compatible* if there exists a tax schedule such that each type's commodity bundle is utility maximal for it in the corresponding budget set. By the Taxation Principle of Hammond (1979) and Guesnerie (1995), an allocation $\mathbf{a}$ is incentive compatible if and only if it satisfies the following *self-selection constraints*,

$$U^i(y_i, c_i) \geq U^i(y_j, c_j), \quad \forall i, j = 1, \ldots, n. \tag{9}$$

In view of this equivalence, henceforth, we suppose that an allocation is chosen directly subject to the self-selection constraints (9) rather than indirectly by specifying a common income tax schedule and then having individuals optimize. By the single-crossing property of the preferences, to verify that (9) holds, it is sufficient to show that all of the adjacent downward and upward self-selection constraints are satisfied.

An allocation $\mathbf{a}$ is *feasible* if it satisfies the production feasibility constraint (8) and the self-selection constraints (9). Equivalently, $\mathbf{a}$ is feasible if it satisfies the government budget constraint (7) in addition to the self-selection constraints.

Two types are *bunched* in an allocation if they have the same commodity bundle. The following implications of the self-selection constraints when the single-crossing property is satisfied are well known. (i) Any allocation that satisfies the self-selection constraints must have both consumption and income nondecreasing in the type parameter. (ii) Two types are not bunched if and only if their consumptions and incomes both differ. (iii) A set of types who are bunched together has the form $\{i, i+1, \ldots, j-1, j\}$ for some $i < j$.[5] (iv) Utility is nondecreasing in type. It is strictly increasing in type for types that have positive income.

Type i's *marginal tax wedge* at the commodity bundle (y_i, c_i) is

$$\tau^i(y_i, c_i) = 1 - \mathrm{MRS}^i(y_i, c_i) = 1 - \frac{1}{w_i v'(c_i)}, \quad i = 1, \ldots, n, \tag{10}$$

which is independent of income. If the income tax schedule is differentiable at y_i, this wedge is the marginal tax rate.

3 A Proposer's Reduced-Form Problem

Each type proposes a *selfishly optimal income tax schedule* that generates a feasible allocation when individuals optimally choose from the corresponding

[5] We refer to a set of types of this form as being an *interval of types*.

budget set. As we have seen in the preceding section, this is equivalent to proposing a feasible allocation that is utility maximal for it—a *selfishly optimal allocation*. Formally, a proposer of type k solves the following problem.

Proposer k's Problem. Choose an allocation $\mathbf{a}^k$ to maximize type k utility (4) subject to the allocation satisfying the production feasibility constraint (8) and the self-selection constraints (9).

In Brett and Weymark (2017a), we show that this problem can be solved in two steps. First, the optimal consumptions are determined and then these consumptions are used to determine the optimal incomes. We summarize their findings in this section.[6]

A type k proposer would like to redistribute resources towards itself from all of the other types, but is limited in doing so by the self-selection constraints. As a consequence, all of the adjacent downward (resp. upwards) self-selection constraints for higher-skilled (resp. lower-skilled) types bind in its selfishly optimal allocation. Because utility is nondecreasing in type when the self-selection constraints are satisfied, in effect, a type k proposer behaves as if it is employing a maximax social welfare function for lower-skilled types and a maximin social welfare function for higher-skilled types. An allocation that satisfies this pattern of binding self-selection constraints necessarily satisfies the other self-selection constraints if consumption is nondecreasing in type.

It is also optimal for the production feasibility constraint to bind. Because the utility function is quasilinear in labor, for any consumption vector $\mathbf{c}^k \in \mathbb{R}^n_+$, there is a unique income vector $\mathbf{y}^k(\mathbf{c}^k)$ such that $(\mathbf{y}^k(\mathbf{c}^k), \mathbf{c}^k)$ exhibits the pattern of binding production feasibility and self-selection constraints described above. It is given by

$$\begin{aligned} y^k_k(\mathbf{c}^k) = \frac{1}{N}\Bigg[\sum_{i=1}^{n} n_i c^k_i - \sum_{i=1}^{k-1} n_i \left(\sum_{j=i}^{k-1} w_j[v(c^k_j) - v(c^k_{j+1})]\right) \\ - \sum_{i=k+1}^{n} n_i \left(\sum_{j=k+1}^{i} w_j[v(c^k_j) - v(c^k_{j-1})]\right)\Bigg], \end{aligned} \tag{11}$$

$$y^k_i(\mathbf{c}^k) = y^k_k(\mathbf{c}^k) + \sum_{j=i}^{k-1} w_j[v(c^k_j) - v(c^k_{j+1})], \quad i = 1, \ldots, k-1, \tag{12}$$

and

[6] The methodology used by Brett and Weymark (2017a) to derive a type's reduced-form problem is based on a similar methodology to the one used by Weymark (1986b) to reduce the dimensionality of an optimal nonlinear income tax problem with a weighted utilitarian objective when the number of types is finite. Using a different approach, Lollivier and Rochet (1983) had previously shown that such a dimensionality reduction is possible when there is a continuum of types.

$$y_i^k(\mathbf{c}^k) = y_k^k(\mathbf{c}^k) + \sum_{j=k+1}^{i} w_j[v(c_j^k) - v(c_{j-1}^k)], \quad i = k+1, \ldots, n. \tag{13}$$

Once the proposer's own income (11) is known, the incomes for the other types can be determined using the recursion formulae in (12) and (13). The allocation $(\mathbf{y}^k(\mathbf{c}^k), \mathbf{c}^k)$ is feasible if

$$0 \leq c_1^k \leq c_2^k \leq \cdots \leq c_n^k. \tag{14}$$

When the income vector is given by $\mathbf{y}^k(\mathbf{c}^k)$, it is possible to rewrite type k's utility function solely in terms of the consumption vector using the function $W^k\colon \mathbb{R}_+^n \to \mathbb{R}$ defined by

$$W^k(\mathbf{c}^k) = U^k(\mathbf{y}^k(\mathbf{c}^k), \mathbf{c}^k) = \frac{1}{Nw_k}\left[\sum_{i=1}^{n} n_i\beta_i^k v(c_i^k) - \sum_{i=1}^{n} n_i c_i^k\right], \tag{15}$$

where

$$\beta_i^k = w_i + \left[\sum_{j=1}^{i-1} \frac{n_j}{n_i}\right](w_i - w_{i-1}), \quad i = 1, \ldots, k-1, \tag{16}$$

$$\beta_i^k = w_i + \left[\sum_{j=i+1}^{n} \frac{n_j}{n_i}\right](w_i - w_{i+1}), \quad i = k+1, \ldots, n, \tag{17}$$

and

$$\beta_k^k = w_k + \left[\sum_{j=1}^{k-1} \frac{n_j}{n_k}\right](w_k - w_{k-1}) + \left[\sum_{j=k+1}^{n} \frac{n_j}{n_k}\right](w_k - w_{k+1}). \tag{18}$$

In these expressions, w_0 and w_{n+1} can be chosen arbitrarily. Let $\beta^k = (\beta_1^k, \ldots, \beta_n^k)$.

The parameter β_i^k is the *virtual wage* of type i. It is obtained by adjusting this type's wage w_i so as to take account of the informational externalities that it generates. For types below k, this adjustment is positive, except for the lowest skilled for whom $\beta_1^k = w_1$. For types above k, this adjustment is negative except for the highest skilled for whom $\beta_n^k = w_n$. The adjustment to type k's wage could be of either sign.

Proposer k's optimal consumption vector solves the following problem.

Proposer k's Reduced-Form Problem. Choose a consumption vector $\mathbf{c}^k$ to maximize the function $W(\mathbf{c}^k)$ in (15) subject to the nonnegativity and monotonicity constraints on consumption in (14).

Proposer's k's Reduced-Form Problem has a number of features that facilitate its analysis. Its objective function is additively separable. This function is also strictly concave if all of the virtual wages rates are positive. Moreover, the constraint set is defined by a system of linear inequalities.

In summary, we have the following: (i) If $\mathbf{a}^{k*} = (\mathbf{y}^{k*}, \mathbf{c}^{k*})$ solves Proposer k's Problem, then $\mathbf{c}^{k*}$ solves Proposer k's Reduced-Form Problem. (ii) If $\mathbf{c}^{k*}$ solves Proposer k's Reduced-Form Problem, then $\mathbf{a}^{k*} = (\mathbf{y}^{k}(\mathbf{c}^{k*}), \mathbf{c}^{k*})$ solves Proposer k's Problem.

4 The Selfishly Optimal Proposals

We have seen in the previous section that type k's optimal allocation $\mathbf{a}^{k*}$ is uniquely determined by the consumption vector $\mathbf{c}^{k*}$ that solves its Reduced-Form Problem. As a consequence, voting over selfishly optimal allocations is equivalent to voting over the the n consumption vectors $\mathbf{c}^{k*}$, $k = 1, \ldots, n$. In this section, we determine what these selfishly optimal consumptions are by solving each type's Reduced-Form Problem.

In deriving a reduced-form problem for a weighted utilitarian social welfare function, Weymark (1986b) assumes that there is only one individual with each skill level. If we had made the same assumption about the distribution of types, Proposer k's Reduced-Form Problem would be essentially the same as its reduced-form problem except for how the virtual wages are defined.[7] The optimal consumption vector for Weymark's problem is identified in Weymark (1986a). Because we do not assume that there is only one individual of each type, we cannot appeal to this solution here. Instead, we must modify his analysis by allowing for a non-uniform type distribution. As we shall see, the optimal consumptions for each proposer depend on the virtual wage rates that appear in its Reduced-Form Problem. Different types of proposers use different virtual wages, which result in them having different optimal consumption vectors.

4.1 The Relaxed Problem

Let $\hat{\mathbf{c}}^k = (\hat{c}_1^k, \ldots, \hat{c}_n^k)$ be the solution to *the relaxed version of Proposer's k's Reduced-Form Problem* in which the monotonicity constraints on consumption are ignored. Because of the additive separability of the objective function in (15), $\hat{c}_i^k$ is determined by solving the following optimization problem:

$$\max \ \beta_i^k v(c_i^k) - c_i^k \quad \text{subject to} \quad c_i^k \geq 0. \tag{19}$$

[7] We say "essentially" because there are some minor differences. Weymark (1986b) allows the marginal disutility of labor γ and the price p of the consumption good to differ from 1, which results in the analogue to the second term in brackets in (15) being multiplied by γp. Here, this product is 1. His objective function does not include an analogue to the scaling factor $1/Nw_k$. This is of no consequence as only the ordinal properties of the objective function matter.

This problem has a unique solution given by

$$\hat{c}_i^k = \begin{cases} {v'}^{-1}(1/\beta_i^k) & \text{if } \beta_i^k > 0, \\ 0 & \text{if } \beta_i^k \leq 0, \end{cases} \quad i = 1, \ldots, n. \tag{20}$$

The function v is strictly concave, so it follows from (20) that a necessary condition for it to be optimal to have no bunching at the solution to Proposer's k's Reduced-Form Problem is

$$\beta_1^k < \cdots < \beta_n^k \quad \text{with} \quad \beta_2^k > 0 \tag{21}$$

and a necessary condition for it to be optimal to have no bunching with all consumptions being positive is

$$0 < \beta_1^k < \cdots < \beta_n^k. \tag{22}$$

When (21) is satisfied, $\hat{\mathbf{c}}^k$ is the unique solution to the relaxed problem described above. Because this solution also satisfies the monotoniciity constraints in (14) that have been ignored, $\hat{\mathbf{c}}^k$ in fact solves Proposer's k's Reduced-Form Problem. Hence, (21) is also a sufficient condition for it to be optimal to have no bunching. Similarly, (22) is also a sufficient condition for it to be optimal to have no bunching with all consumptions being positive. We have therefore established the following theorem.

Theorem 1. *If* $\mathbf{c}^{k*}$ *solves Proposer k's Reduced-Form Problem, then* (i) $\mathbf{c}^{k*}$ *exhibits no bunching if and only* (21) *holds and* (ii) $\mathbf{c}^{k*}$ *exhibits no bunching and all consumptions are positive if and only if* (22) *holds. If either* (21) *or* (22) *is satisfied, then* $\mathbf{c}^{k*} = \hat{\mathbf{c}}^k$*, where* $\hat{\mathbf{c}}^k$ *is defined in* (20).

4.2 Sufficient Conditions for Bunching

The inequalities in (21) and (22) need not be satisfied, so in order to determine the solution to Proposer's k's Reduced-Form Problem, it is necessary to identify the optimal pattern of bunching and to identify which types are to have zero consumption. In order to address the issue of bunching and of the positivity of a type's consumption in a unified way, we introduce an additional type, type 0, whose consumption c_0 is identically 0. Thus, constraint (14) can be rewritten as

$$c_{i-1} \leq c_i, \quad i = 1, \ldots, n. \tag{23}$$

We abuse notation somewhat by continuing to write consumption vectors as having n components, with it being understood that we are implicitly adding type 0's consumption when needed. We shall show that when account is taken of the monotonicity constraints, the optimal consumptions are given by a modified version of (20) in which the virtual wages β_i^k are adjusted to account for the optimal pattern of bunching.

We begin the task of providing a complete characterization of the solution to Proposer's k's Reduced-Form Problem by establishing two lemmas that provide sufficient conditions for an interval of types to be optimally bunched together or to have zero consumption.

The objective function in Proposer's k's Reduced-Form Problem is ordinally equivalent to the function $\overline{W}^k \colon \mathbb{R}^n_+ \to \mathbb{R}$ defined by

$$\overline{W}^k(\mathbf{c}^k) = \sum_{i=1}^{n} n_i \beta_i^k v(c_i^k) - \sum_{i=1}^{n} n_i c_i^k. \tag{24}$$

We use this form of the objective function in the rest of this section.

In (24), the benefit from marginally increasing c_i^k is $n_i \beta_i^k v'(c_i^k)$, whereas the marginal cost is n_i. As a consequence, if $\beta_i^k \leq 0$ and $c_i^k > c_{i-1}^k$, the proposer's utility can be increased by decreasing c_i^k until the constraint $c_i^k \geq c_{i-1}^k$ binds. Similarly, if this constraint binds, a marginal increase in the common value of c_{i-1}^k and c_i^k has a marginal benefit of $[n_{i-1}\beta_{i-1}^k + n_i \beta_i^k] v'(c_i^k)$ and a marginal cost of $n_{i-1} + n_i$. Hence, if $n_{i-1}\beta_{i-1}^k + n_i \beta_i^k \leq 0$, it is optimal to have $c_{i-2}^k = c_{i-1}^k = c_i^k$. This is the case even if $\beta_{i-1}^k > 0$. In general, if there is an interval of types $\{j, \ldots, m\}$ for which, starting with type m and working down the skill distribution, the cumulative sums of the product of a type's virtual wage and the number of individuals with this type is nonpositive, then all of these types are optimally allocated the same consumption as type $j-1$.

Lemma 1. *If $\mathbf{c}^{k*}$ is a solution to Proposer k's Reduced-Form Problem and if there exists an interval of types $\{j, \ldots, m\}$ with $1 \leq j \leq m$ such that*

$$\sum_{i=h}^{m} n_i \beta_i^k \leq 0 \quad \textit{for all } h \in \{j, \ldots, m\}, \tag{25}$$

then

$$c_i^{k*} = c_{j-1}^{k*} \quad \textit{for all } i \in \{j, \ldots, m\}. \tag{26}$$

If $j = 1$ in Lemma 1, then all of the types in the interval have zero consumption. Note that this lemma applies if $j = m = 1$.

By considering transfers of consumption between types, we are able to identify a second sufficient condition for an interval of types to be optimally bunched.

Lemma 2. *If $\mathbf{c}^{k*}$ is a solution to Proposer k's Reduced-Form Problem and if there exists an interval of types $\{j, \ldots, m\}$ with $1 \leq j < m$ such that*

$$\frac{\sum_{i=j}^{h} n_i \beta_i^k}{\sum_{i=j}^{h} n_i} \geq \frac{\sum_{i=j}^{m} n_i \beta_i^k}{\sum_{i=j}^{m} n_i} > 0 \quad \textit{for all } h \in \{j, \ldots, m\}, \tag{27}$$

then

$$c_i^{k*} = c_j^{k*} \quad \textit{for all } i \in \{j, \ldots, m\}. \tag{28}$$

Note that the inequality for $h = m$ in (27) is vacuous. The second fraction in (27) is the population-share weighted average of the virtual wages of all types in the interval of types being considered. The first fraction is the population-share weighted average of the virtual wages of all types in the subinterval from j to h. The sufficient condition requires the weighted average of the virtual wages for the whole interval of types not to exceed that of any of the subintervals that include type j, with all of these averages being positive.

To gain some intuition for Lemma 2, we consider the special case in which $j = m - 1$. If $c_m^k > c_{m-1}^k$, it is possible to increase the consumption of each person of type $m - 1$ by a small amount $\delta > 0$ by taking $[n_{m-1}/n_m]\delta$ units of consumption from each person of type m. Using (24) and some simple algebra, such a change increases the proposer's utility if and only if

$$\beta_{m-1}^k v'(c_{m-1}^k) > \beta_m^k v'(c_m^k). \tag{29}$$

Because $v'(c_{m-1}^k) > v'(c_m^k)$, (29) is satisfied if $\beta_{m-1}^k \geq \beta_m^k$. The latter inequality holds if and only if

$$\beta_{m-1}^k = \frac{n_{m-1}\beta_{m-1}^k}{n_{m-1}} \geq \frac{n_{m-1}\beta_{m-1}^k + n_m\beta_m^k}{n_{m-1} + n_m}, \tag{30}$$

which is (27) for $h = j = m - 1$. In other words, β_{m-1}^k is greater than or equal to the population-share weighted average of the virtual wages of these two types.

4.3 The Optimal Solution

Our characterization of the optimal solution to Proposer k's Reduced-Form Problem involves constructing a particular convex function that can be used to identify the optimal pattern of binding monotonicity and nonnegativity constraints on consumption. Let $\sigma^k = (\sigma_1^k, \ldots, \sigma_n^k)$ be defined by setting

$$\sigma_i^k = \sum_{h=1}^{i} n_h \beta_h^k, \quad i = 1, \ldots, n. \tag{31}$$

Informally, σ_i^k is the sum of the virtual wages of all of the individuals who have a skill that does not exceed w_i. Let $\sigma_0^k = 0$. We construct a function whose graph is obtained by plotting the points (n_i, σ_i^k), $i = 0, \ldots, n$, and then connecting consecutive points by a line segment. Formally, the function $f_{\sigma^k}\colon [0, N] \to \mathbb{R}$ is given by

$$f_{\sigma^k}(r) = \sigma^k_{i-1} + \frac{\left[r - \sum_{j=1}^{i-1} n_j\right]}{n_i}(\sigma^k_i - \sigma^k_{i-1}) \quad \text{for all } r \in \left[\sum_{j=1}^{i-1} n_j, \sum_{j=1}^{i} n_j\right], \quad i = 1, \ldots, n. \tag{32}$$

Note that the ith virtual wage is

$$\beta^k_i = \frac{f_{\sigma^k}\left(\sum_{j=1}^{i} n_j\right) - f_{\sigma^k}\left(\sum_{j=1}^{i-1} n_j\right)}{n_i}, \quad i = 1, \ldots, n. \tag{33}$$

Let $\bar{f}_{\sigma^k}\colon [0, N] \to \mathbb{R}$ be the maximal convex function that lies nowhere above f. Formally, $\bar{f}_{\sigma^k}$ is defined by setting

$$\bar{f}_{\sigma^k}(r) = \min\left\{\zeta f_{\sigma^k}(r_1) + (1-\zeta) f_{\sigma^k}(r_2) \mid \zeta \in [0,1],\ r_1, r_2, \in [0, N], \text{ and } \zeta r_1 + (1-\zeta) r_2 = r\right\}. \tag{34}$$

In (32), the virtual wages are used to define f_{σ^k}. Applying the inverse of this procedure to $\bar{f}_{\sigma^k}$, we obtain the *adjusted virtual wage* for each type i,

$$\bar{\beta}^k_i = \frac{\bar{f}_{\sigma^k}\left(\sum_{j=1}^{i} n_j\right) - \bar{f}_{\sigma^k}\left(\sum_{j=1}^{i-1} n_j\right)}{n_i}, \quad i = 1, \ldots, n. \tag{35}$$

Let $\bar{\beta}^k = (\bar{\beta}^k_1, \ldots, \bar{\beta}^k_n)$. Because $\bar{f}_{\sigma^k}$ is a convex function, the $\bar{\beta}^k_i$ are nondecreasing in type. Note that if $\bar{\beta}^k_i \leq 0$, then the most skilled type with this adjusted virtual wage must have a nonpositive virtual wage. The function $\bar{f}_{\sigma^k}$ and the vector $\bar{\beta}^k$ are discrete counterparts of analogous constructions used by Myerson (1981) in his characterization of an optimal auction.

We illustrate these constructions in Example 1.

Example 1. There are four types, with one person of type 2 and two people of each of the other three types. Let $\beta^k = (1, -4, 2.25, 0.25)$. For this β^k, $\sigma^k = (2, -2, 2.5, 3)$. The graphs of f_{σ^k} and $\bar{f}_{\sigma^k}$ are shown in Figure 1. The slopes of the first and second segments of $\bar{f}_{\sigma^k}$ are $-2/3$ and $5/4$, respectively, so $\bar{\beta}^k = (-2/3, -2/3, 5/4, 5/4)$. By Lemma 1 with $j = 1$ and $m = 2$, types 1 and 2 should both have zero consumption. By Lemma 2 with $j = 3$ and $m = 4$, types 3 and 4 should be bunched together.

Using the function $\bar{f}_{\sigma^k}$ together with Lemmas 1 and 2, we obtain the explicit solution to Proposer k's Reduced-Form Problem shown in Theorem 2. This theorem generalizes Theorem 3 in Weymark (1986a) by allowing for a non-uniform distribution of types.

Theorem 2. *The unique solution* $\mathbf{c}^{k*}$ *to Proposer* k*'s Reduced-Form Problem is*

$$c^{k*}_i = \begin{cases} {v'}^{-1}(1/\bar{\beta}^k_i) & \text{if } \bar{\beta}^k_i > 0, \\ 0 & \text{if } \bar{\beta}^k_i \leq 0, \end{cases} \quad i = 1, \ldots, n. \tag{36}$$

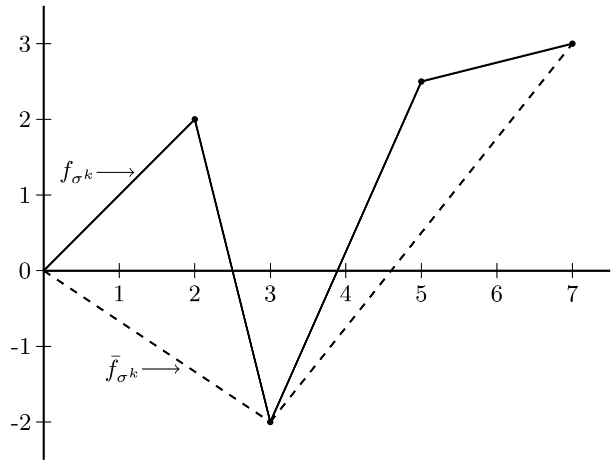

Fig. 1. Computing the Adjusted Virtual Wages

Types are bunched together if they share a common adjusted virtual wage, with their common consumption equaling zero if and only if this adjusted virtual wage is nonpositive. An implication of Theorem 2 is that the optimal pattern of bunching for Proposer k only depends on its identity, the set of skill levels, and the distribution of types, as these are the variables that are used to compute $\bar{\beta}^k$. The function v used to measure the utility of consumption helps determine the magnitudes of the optimal consumptions, but not who should be bunched together.

The only difference between the formula in (20) used to compute the consumptions $\hat{c}^k$ that solve the relaxed version of Proposer k's Reduced-Form Problem in which the monotonicity constraints on consumption are ignored and the formula in (36) used to compute c^{k*} is that the former uses the virtual wages, whereas the latter uses the adjusted virtual wages. By Theorem 1, it is optimal to have no bunching if and only (21) holds. When this is the case, f_{σ^k} is convex and, hence, f_{σ^k} coincides with $\bar{f}_{\sigma^k}$.

By Theorem 2 and (35), it follows that

$$\bar{\beta}_i^k = \frac{\sum_{h=j}^m n_h \beta_h^k}{\sum_{h=j}^m n_h}, \tag{37}$$

where $\{j, \ldots, m\}$ is the interval of types that share this value of the adjusted virtual wage. That is, $\bar{\beta}_i^k$ is the population-share weighted average of the virtual wages of these types. If $\bar{\beta}_i^k > 0$, these are all of the types that are bunched with type i. If $\bar{\beta}_i^k \leq 0$, there may be more than one value of the adjusted virtual wage which is nonpositive, so this interval of types may not be the complete set of types that are bunched with type i. Rather, type i is bunched with all types whose adjusted virtual wages are nonpositive, and

Proposer	$k=1$		$k=2$		$k=3$		$k=4$		$k=5$	
	β_i^k	$\bar{\beta}_i^k$	β_i^k	$\bar{\beta}_i^k$	β_i^k	$\bar{\beta}_i^k$	β_i^k	$\bar{\beta}_i^k$	β_i^k	$\bar{\beta}_i^k$
$(w_1, n_1) = (5, 4)$	1.25	1.25	5	5	5	5	5	5	5	5
$(w_2, w_2) = (6, 6)$	4.5	4.5	5.17	5.17	6.67	6.67	6.67	6.67	6.67	6.67
$(w_3, n_3) = (7, 5)$	6.2	6.2	6.2	6.2	8.2	7.88	9	9	9	9
$(w_4, n_4) = (8, 3)$	7.33	7.33	7.33	7.33	7.33	7.88	12.33	11.75	13	13
$(w_5, n_5) = (10, 1)$	10	10	10	10	10	10	10	11.75	46	46

Table 1. Virtual wages and adjusted virtual wages in Example 2

they all have zero consumption. The population-share weighted average of the virtual wages of all of these types is nonpositive.

The general form of the solution to a proposer's reduced-form problem does not depend on its identity. However, the optimal consumptions are a function of the adjusted virtual wages, and the values of these parameters depend on the proposer's type. In particular, the adjusted virtual wages depend on the number of individuals of each type and their skill levels, with the way that this information is aggregated to obtain the adjusted virtual wages being type specific.

In Example 2, we determine the selfishly optimal consumption schedules for each of the types in a five-type example and discuss how these schedules are related.

Example 2. There are five types with $(w_1, w_2, w_3, w_4, w_5) = (5, 6, 7, 8, 10)$ and $(n_1, n_2, n_3, n_4, n_5) = (4, 6, 5, 3, 1)$. The utility function is $v(c) = \ln c$. For this utility function, ${v'}^{-1}(x) = 1/x$, so $c_i^{k*} = \bar{\beta}_i^k$. Thus, a proposer's optimal consumption schedule coincides with the adjusted virtual wages. Table 1 provides the virtual wages and adjusted virtual wages (consumption) for each possible type of proposer. The virtual wages are computed using the formulas in (16)–(18). Reading down the columns of the table, we see that the virtual wages are monotonic when the proposers are of types 1 (maximin), 2, or 5 (maximax). For the other proposer types, the virtual wage increases up to the proposer's type and then decreases going to the next type. This necessitates some bunching of types near the proposer. For the parameter values in this example, the type just above the proposer is bunched with the proposer. Reading across the rows, we see that any type's virtual wage coincides with the maximax (respectively, maximin) virtual wage whenever that type is less (respectively, greater) than the proposer's type.

4.4 Necessary Conditions for an Optimum

Further insight into the optimal bunching pattern may be obtained by considering the necessary conditions for a solution to Proposer's k's Reduced-Form Problem. The Lagrangian for this problem is

$$\sum_{i=1}^{n} \left[n_i \beta_i^k v(c_i^k) - n_i c_i^k \right] + \sum_{i=1}^{n} \mu_i \left[c_i^k - c_{i-1}^k \right], \tag{38}$$

where $\mu_i \geq 0$ for all $i = 1, \ldots n$. Letting $\mu_{n+1} = 0$, differentiating (38), the first-order necessary conditions for an optimum can be written as

$$n_i \beta_i^k v'(c_i^k) + \mu_i - \mu_{i+1} - n_i \begin{cases} = 0 & \text{if } c_i^{k*} > 0, \\ \geq 0 & \text{if } c_i^{k*} = 0, \end{cases} \quad i = 1, \ldots n. \tag{39}$$

It follows from (39) that if types $j, \ldots, m$ are optimally bunched at c_i^{k*} with $c_{j-1}^{k^*} < c_i^{k*} < c_{m+1}^{k*}$, then

$$v'(c_i^{k*}) \left[\frac{\sum_{h=j}^{m} n_h \beta_h^k}{\sum_{h=j}^{m} n_h} \right] \begin{cases} = 1 & \text{if } c_i^{k*} > 0, \\ \geq 1 & \text{if } c_i^{k*} = 0, \end{cases} \quad i = 1, \ldots, n, \tag{40}$$

where use is made of the fact that $\mu_j = \mu_{m+1} = 0$ because the corresponding monotonicity constraints do not bind.[8] The term is square brackets in (40) is $\bar{\beta}_i^k$. Because $v'(0) = \infty$, (40) implies that a necessary condition for $\mathbf{c}^{k*}$ to solve Proposer k's Reduced-Form Problem is that it satisfy (36) in Theorem 2.

If $j = m$ in (40) (and, hence, $i = j$), this bunching interval consists of a single type. It follows from (36) and the concavity of v that the optimal consumptions for each proposer must be of the form of a collection of bunching intervals with the property that the population-share weighted average of the virtual wages in each bunching interval is increasing as one goes up the type distribution.

5 Voting Over the Proposals

The self-selection constraints imply that the utilities obtained with any type's proposal are nondecreasing in type. Hence, type 1 proposes the maximin consumptions and type n proposes the maximax consumptions. Consider any $k \neq 1, n$. From (16)–(18),

$$\beta^k = (\beta_1^n, \ldots, \beta_{k-1}^n, \beta_k^k, \beta_{k+1}^1, \ldots, \beta_n^1). \tag{41}$$

Thus, for types below itself, a proposer of type k uses the virtual wages of the highest type, whereas for higher types, it uses the virtual wages of the lowest type. In the relaxed problem in which the monotonicity constraints on consumption are ignored, (20) implies that type k proposes the maximax consumptions for lower types and the maximin consumptions for higher types. It follows from (16) and (17) that

[8] If $j = 1$, the inequality $c_{j-1}^{k^*} < c_i^{k*}$ does not apply. Similarly, if $m = n$, the inequality $c_i^{k*} < c_{m+1}^{k*}$ does not apply.

$$\beta_n^1 = w_n \quad \text{and} \quad \beta_i^1 < w_i \text{ for all } i \neq n, \tag{42}$$

and

$$\beta_1^n = w_1 \quad \text{and} \quad \beta_i^n > w_i \text{ for all } i \neq 1. \tag{43}$$

As a consequence, for the relaxed problems of types 1 and n, (20) also implies that the maximax consumption schedule lies above the maximin schedule. Hence, in order to satisfy the monotonicity constraints, it may be necessary for the proposer's type to be part of a bunching interval. The schedules in Example 2 have these features.

These observations suggest that a proposer's selfishly optimal consumption schedule potentially has three regions. Starting with the lowest skilled, there is first an interval of types that are allocated the maximax consumptions. Next, there is an interval of types that includes the proposer who are bunched together. Finally, there is an interval of types that are allocated the maximin consumptions. The middle region could consist of just the proposer's type and either the first or third region may not be present. In this section, we show that this is the general structure of the selfishly optimal proposals, and use this structure to show that each type has a single-peaked preference on the set of these proposals and that there is a Condorcet winner. We also investigate the signs of the marginal tax wedges in the winning type's proposal.

5.1 The Shape of the Selfishly Optimal Consumption Schedules

In order to focus on the essentials of the argument, we make the simplifying assumption that there is no bunching in the maximin and maximax consumption schedules. By Lemma 1, this is equivalent to (21) holding for $k = 1, n$. That is, the virtual wages for these two types of proposer must be increasing in type with at most one negative value.[9] Our conclusions about the shape of the selfishly optimal consumption schedule and the single-peakedness of each type's preferences over the proposals hold more generally, but to show that this is the case requires first ironing the maximin and maximax schedules and then using the virtual wages for these ironed schedules.

Example 3 describes a situation in which our simplifying assumption is satisfied.

Example 3. There are an equal number of individuals of each type ($n_i = n_j$ for all i, j) and the skills of adjacent types are equally spaced ($w_i - w_{i-1} = \Delta > 0$, $i = 2, \ldots, n$). It follows from (17) and (18) for $k = 1$ that

$$\beta_i^1 = w_1 - [n - (2i - 1)]\Delta, \quad i = 1, \ldots, n. \tag{44}$$

Because $\Delta > 0$, the virtual wages in (44) are increasing in type and, therefore, there is no bunching in the maximin schedule. It follows from (16) and (17) for $k = n$ that

[9] It follows from (43) that the virtual wages for proposer n are all positive because the skill levels are also all positive.

$$\beta_i^n = w_1 + [2(i-1)]\Delta, \quad i = 1, \ldots, n. \tag{45}$$

Because $w_1 > 0$ and $\Delta > 0$, the virtual wages in (45) are positive and increasing in type and, therefore, there is no bunching and each type has positive consumption in the maximin schedule.

Provided that the maximin and maximax consumption schedules are monotonically increasing, for any proposer of type $k \neq 1, n$, there can be at most one bunching interval. Moreover, if there is a bunching interval, then it must contain the proposer's type.

Lemma 3. *If there is no bunching in the selfishly optimal consumption schedules for types 1 and n, then any bunching interval in the selfishly optimal consumption schedule for any type $k \neq 1, n$ includes type k.*

With a continuum of types, Brett and Weymark (2017b) show that the proposer must be bunched with types both below and above itself. This is the case because there are types with skill levels arbitrarily close to that of the proposer, with the consequence that there must be a downward discontinuity in the relaxed solution at the proposer's type when transitioning from the maximax to the maximin consumptions schedules that must be ironed. With discrete skill levels, there must be bunching with the proposer if there are skill levels sufficiently close to that of the proposer. However, if there is a wide gap between the proposer's skill and those of the adjacent types, no ironing may be needed.

Because $\bar{\beta}_i^k = \beta_i^k$ if type i is not bunched, it follows immediately from Lemma 3, Theorem 2, and (20) that each proposer's selfishly optimal consumption schedule has the three-region structure described above.

Theorem 3. *If $\mathbf{c}^{k*}$ solves Proposer k's Reduced-Form Problem for $k \neq 1, n$ and there is no bunching in the selfishly optimal consumption schedules for types 1 and n, then there exist types j and m with $j \leq k \leq m$ such that* (i) *$c_i^{k*} = c_i^{n*}$ for $i = 1, \ldots, j-1$,* (ii) *$c_i^{k*} = c_k^{k*}$ for $i = j, \ldots, m$, and* (iii) *$c_i^{k*} = c_i^{1*}$ for $i = j+1, \ldots, n$.*

5.2 The Median Voter Theorem

Type i, $i = 1, \ldots, n$, has a *single-peaked preference* on the set of selfishly optimal allocations if

$$U^i(y_i^{k*}(\mathbf{c}^{k*}), c_i^{k*}) \geq U^i(y_i^{(k-1)*}(\mathbf{c}^{(k-1)*}), c_i^{(k-1)*}), \quad k = 1, \ldots i, \tag{46}$$

and

$$U^i(y_i^{k*}(\mathbf{c}^{k*}), c_i^{k*}) \geq U^i(y_i^{(k+1)*}(\mathbf{c}^{(k+1)*}), c_i^{(k+1)*}), \quad k = i, \ldots n. \tag{47}$$

As in Arrow (1951), this definition of a single-peaked preference does not require there to be a unique most-preferred alternative.

Theorem 4 establishes that each type's preferences are single-peaked.

Theorem 4. *If there is no bunching in the selfishly optimal consumption schedules for types 1 and n, then each type has single-peaked preferences on the set of selfishly optimal allocations.*

Some intuition for this result can be obtained by considering how the virtual wages for different proposers are related. An implication of (16)–(18) is that a type k proposer's own virtual wage lies between the virtual wages assigned to it by types 1 and n. Hence, in the relaxed problems for types 1, k, and n, the consumption type k would like for itself lies between what it would obtain with the maximin and maximax schedules. It also follows from (16)–(18) that as the proposer's type increases from k to $k+1$, only the virtual wages of these two types change, and they both increase. In their relaxed problems, when type k is replaced by type $k+1$ as the proposer, (i) the consumption of type k jumps up from what it would propose for itself to the maximax consumption and (ii) the consumption of type $k+1$ jumps up to what it would propose for itself from the maximin consumption. As a consequence, the bunching interval cannot start or finish at lower-skilled types as the proposer's type increases. It is this (weakly) rightward shift in the bunching interval that accounts for the single-peakedness of the preferences.

A direct consequence of Theorem 4 and Black's Median Voter Theorem (Black, 1948) is that the proposal of a type with a median skill level is weakly preferred by a majority to any of the other proposals.[10]

Theorem 5. *If there is no bunching in the selfishly optimal consumption schedules for types 1 and n, then the selfishly optimal allocation for a median skill type is a Condorcet winner when pairwise majority voting is restricted to the allocations that are selfishly optimal for some skill type.*

5.3 Marginal Tax Wedges

Because $\bar{\beta}_i^k = \beta_i^k$ if type i is not bunched, using the definition of a marginal tax wedge in (10) and (41)–(43), Theorems 2 and 3 imply that the marginal tax wedges of types in the first (respectively, third) region of the Condorcet-winning consumption schedule are negative (respectively, positive) except for the lowest (respectively, highest) type, who is undistorted.

Theorem 6. *If type k's selfishly optimal allocation* $\mathbf{a}^{k*}$ *is the Condorcet winner and there is no bunching in the selfishly optimal consumption schedules for types 1 and n, then in* $\mathbf{a}^{k*}$, (i) *for any type* $i < k$ *with* $i > 1$ *that is not*

[10] With majority rule, if preferences are single-peaked, nobody can obtain a preferred outcome by reporting a single-peaked preference different from the true one. This does not imply that someone cannot manipulate the outcome by reporting a preference that is not single-peaked. Such a false report could be detected and ruled to be inadmissible. Applied to the tax voting problem, if a proposed income tax schedule or corresponding allocation is not selfishly optimal for any type, then this is publicly observable and could be disallowed.

bunched with type k, the marginal tax wedge is negative, (ii) *for any type $i > k$ with $i < n$ that is not bunched with type k, the marginal tax wedge is positive, and* (iii) *for types 1 and n, the marginal tax wedges are zero.*

For the types considered in Theorem 6, the signs of these marginal tax wedges are the familiar ones for maximax and maximin social welfare functions. On the maximax part of the consumption schedule, marginal income subsidies are used to relax adjacent upward self-selection constraints, except for the lowest type who does not face such a constraint. Analogously, on the maximin part of the consumption schedule, marginal income taxes are used to relax adjacent downward self-selection constraints, except for the highest type who does not face such a constraint. For types bunched with the proposer, these considerations work against each other, so, in general, it is not possible to sign their marginal tax wedges.

6 Concluding Remarks

Each type of proposer wants to redistribute resources towards itself. The incentive constraints ensure that higher-skilled types are no worse off than the proposer. However, those who are lower skilled are not protected in this way. Röell (2012) suggests subjecting proposers to a further constraint that guarantees each type a minimum level of utility. Using a version of our model with a continuum of types, Brett and Weymark (2016) show that the addition of this constraint results in each proposer maximizing a weighted average of its type's utility and the utility of the least skilled. We conjecture that the same conclusion obtains with the finite-type model considered here.

For a weighted utilitarian objective, Simula (2010) derives a reduced-form problem for utility that is quasilinear in consumption that is an analogue of the one in Weymark (1986b) for utility that is quasilinear in labor supply. In Simula's reduced-form problem, it is the incomes that are chosen. The arguments used in Brett and Weymark (2017a) can be adapted to determine a reduced-form problem for this kind of quasilinearity when the objective is to maximize a type's utility. The analysis provided here can then be used to show that preferences are single-peaked over the selfishly optimal income schedules proposed by each type.

The assumption that utility is quasilinear is used in a fundamental way. We conjecture that by drawing on the analysis of Chambers (1989), our proof that preferences over the proposals are single-peaked generalizes to utility that is additively separable. However, in this case, it would no longer be possible to express the optimal incomes explicitly as a function of the optimal consumptions. Without such a restriction on the functional form of the utility function, we do not believe that single-peakedness generally obtains. Nevertheless, as shown by Bohn and Stuart (2013) when there is a continuum of types, there is a Condorcet winner when voting is over the selfishly optimal tax schedules even if utility is not restricted to be quasilinear or additively separable. They establish their result by considering, for each proposal, the function that specifies the utility obtained with it for each type, and showing that the graphs for any pair of these functions cross only once.[11]

Appendix

Proof of Lemma 1. Suppose by way of contradiction that (26) does not hold. Let h be the smallest type in $\{j,\dots,m\}$ such that $c_h^{k*} > c_{h-1}^{k*}$. Because $\mu_h = 0$, summing (39) over types $h,\dots,m$ yields

$$v'(c_i^{k*})\left[\sum_{i=h}^{m} n_i\beta_i^k\right] = \sum_{i=h}^{m} n_i + \mu_{m+1}. \tag{A.1}$$

The left-hand side of (A.1) is nonpositive by (25), whereas the right-hand side is positive, a contradiction. □

Proof of Lemma 2. If $j = m$, (27) is vacuous and the lemma is trivially true.

Now, suppose that $j < m$. By way of contradiction, suppose that there is some type in $\{j,\dots,m\}$ with $c_i^{k*} > c_j^{k*}$. Let l be the lowest such type. Then, $\mu_l = 0$.

Suppose, first, that type l is bunched with type m (l need not be distinct from m). By (39),

$$v'(c_j^{k*})\sum_{i=j}^{l-1} n_i\beta_i^k \geq \sum_{i=j}^{l-1} n_i - [\mu_j + \mu_{l-1}], \tag{A.2}$$

which implies that

$$v'(c_j^{k*})\left[\frac{\sum_{i=j}^{l-1} n_i\beta_i^k}{\sum_{i=j}^{l-1} n_i}\right] \geq 1 - \frac{[\mu_j + \mu_{l-1}]}{\sum_{i=j}^{l-1} n_i}. \tag{A.3}$$

By assumption, the fraction on the left-hand side of (A.3) is positive. Because $v'(0) = \infty$, if $c_j^{k*} = 0$, this inequality is violated. Hence, $c_j^{k*} > 0$ and both (A.2) and (A.3) hold with equality. Also by (39)

[11] This kind of single-crossing property is related to the single-crossing property considered by Gans and Smart (1996), which is satisfied if (i) voters are linearly ordered and (ii) if a pair of voters agree on how to rank a pair of alternatives, then all voters in between them concur with this ranking. A median voter according to the linear ordering of the voters is decisive in a pairwise majority vote. Gans and Smart apply this median voter theorem to voting over nonlinear income tax schedules. Bierbrauer and Boyer (2017) prove a median voter theorem for tax reforms from a status quo situation in which the change in the tax liability is a monotonic function of income. They also make use of a single-crossing property.

$$v'(c_l^{k*})\left[\frac{\sum_{i=l}^{m} n_i\beta_i^k}{\sum_{i=l}^{m} n_i}\right] = 1 + \frac{[\mu_{l+1} + \mu_{m+1}]}{\sum_{i=l}^{m} n_i}. \tag{A.4}$$

It follows from the equality version of (A.3), (A.8), and the concavity of v that

$$v'(c_j^{k*})\left[\frac{\sum_{i=j}^{l-1} n_i\beta_i^k}{\sum_{i=j}^{l-1} n_i}\right] < v'(c_l^{k*})\left[\frac{\sum_{i=l}^{m} n_i\beta_i^k}{\sum_{i=l}^{m} n_i}\right] < v'(c_j^{k*})\left[\frac{\sum_{i=l}^{m} n_i\beta_i^k}{\sum_{i=l}^{m} n_i}\right]. \tag{A.5}$$

Thus,

$$\frac{\sum_{i=j}^{l-1} n_i\beta_i^k}{\sum_{i=j}^{l-1} n_i} < \frac{\sum_{i=l}^{m} n_i\beta_i^k}{\sum_{i=l}^{m} n_i}, \tag{A.6}$$

which contradicts (27).

We have thus shown that there must be a maximal type $q < m$ that is bunched with l. Note that it may be the case that $q = l$. By (40),

$$v'(c_l^{k*})\left[\frac{\sum_{i=l}^{q} n_i\beta_i^k}{\sum_{i=l}^{q} n_i}\right] = 1. \tag{A.7}$$

Using (39) for types $q+1$ to m implies

$$v'(c_l^{k*})\left[\frac{\sum_{i=q+1}^{m} n_i\beta_i^k}{\sum_{i=q+1}^{m} n_i}\right] = 1 + \frac{[\mu_{q+2} + \mu_{m+1}]}{\sum_{i=q+1}^{m} n_i}. \tag{A.8}$$

The argument leading to (A.6) may be repeated to conclude that

$$\frac{\sum_{i=l}^{l-1} n_i\beta_i^k}{\sum_{i=j}^{l-1} n_i} < \frac{\sum_{i=l}^{m} n_i\beta_i^k}{\sum_{i=l}^{m} n_i}, \tag{A.9}$$

which contradicts (27).

Having shown that both possibilities for type l result in a contradiction, no such type is possible and, hence, (28) holds. □

Proof of Theorem 2. Because $\bar{f}_{\sigma^k}$ is a convex function and v' is decreasing, the consumption vector $\mathbf{c}^{k*}$ defined in (36) satisfies the constraints in Proposer k's Reduced-Form Problem. Because $\bar{\beta}^k$ is nondecreasing in type, the set of all types that share a common value of the adjusted virtual wage is an interval of types. Let $\{j, \ldots, m\}$ be such an interval. Because $\bar{f}_{\sigma^k}$ is the highest convex function that lies nowhere above f_{σ^k}, (25) holds if $\bar{\beta}_j^k \leq 0$ and (27) holds if $\bar{\beta}_j^k > 0$. Hence, by Lemmas 1 and 2, respectively, all types in $\{j, \ldots, m\}$ are bunched together. Furthermore, because the types for which the adjusted virtual wages are nonpositive are the first l types for some $l \in \{0, 1, \ldots, n\}$, Lemma 1 also implies that c_i^{k*} is zero for any type $i \leq l$.

For any type i for which $\bar{\beta}_i^k > 0$, let $c_i^{k\circ}$ be the common value of the consumption of the types $\{j, \ldots, m\}$ that are bunched with type i. The optimal value of $c_i^{k\circ}$ is the solution to

$$\max \sum_{h=j}^{m} n_h \bar{\beta}_i^k v(c_i^{k\circ}) - \sum_{h=j}^{m} n_h c_i^{k\circ} \quad \text{subject to} \quad c_i^{k\circ} \geq 0. \tag{A.10}$$

Because $\bar{\beta}_i^k > 0$, v is strictly concave, and $v'(0) = \infty$, this problem has a unique solution and it is positive. This solution is given by the first equality in (36). □

Proof of Lemma 3. Suppose, by way of contradiction, that there is a complete bunching interval $\{j, \ldots, m\}$ with $j < m$ that does not include type k. We first consider the case in which $m < k$. Because $\mu_j = \mu_{m+1} = 0$, by (39) and (41),

$$n_j \beta_j^n v'(c_j^{k*}) \geq n_j + \mu_{j+1} \quad \text{and} \quad n_m \beta_m^n v'(c_j^{k*}) \geq n_m - \mu_m \tag{A.11}$$

or, equivalently,

$$\beta_j^n \geq \frac{1}{v'(c_j^{k*})} \left[1 + \frac{\mu_{j+1}}{n_j}\right] \quad \text{and} \quad \beta_m^n \geq \frac{1}{v'(c_j^{k*})} \left[1 - \frac{\mu_m}{n_m}\right]. \tag{A.12}$$

The inequalities in (A.11) and (A.12) bind if $c_j^{k*} > 0$. In this case, (A.12) implies that $\beta_j^n \geq \beta_m^n$. If $c_j^{k*} = 0$, then $j = 1$ and (A.12) implies that $\beta_j^n \geq 0$. Because there are no types above m with zero consumption, by Theorem 2 and the way that the adjusted virtual wages are determined, it must be the case that $\beta_m^n \leq 0$. Thus, in the case as well, we have $\beta_j^n \geq \beta_m^n$. However, because type n's optimal consumption schedule does not exhibit bunching, by Theorem 1, $\beta_j^n < \beta_m^n$, so we have a contradiction.

A similar argument can be used to show that no complete bunching interval of the form $\{j, \ldots, m\}$ with $j < m$ and $j > k$ exists. In this case, c_j^{k*} is necessarily positive, so the analogues to (A.11) and (A.12) for β_j^1 and β_m^1 hold with equality. □

Proof of Theorem 4. We first suppose that $i < k$. We show that the utility that type i receives with type k's selfishly optimal allocation is greater than or equal to the utility that type i receives with type $(k+1)$'s selfishly optimal allocation. Formally,

$$w_i v(c_i^k) - y_i^k \geq w_i v(c_i^{k+1}) - y_i^{k+1}.^{12} \tag{A.13}$$

Substituting (12) into the left-hand side of (A.13) yields

[12] For notational simplicity, we omit the superscript * and the dependence of the optimal incomes on the consumptions in this proof. That is, all allocations in this proof are assumed to be optimal for the relevant proposer.

$$w_i v(c_i^k) - y_i^k = w_i v(c_i^k) - \left[y_k^k + \sum_{j=i}^{k-1} w_j [v(c_j^k) - v(c_{j+1}^k)] \right]. \tag{A.14}$$

Rearranging terms in (A.14) gives

$$\begin{aligned} w_i v(c_i^k) - y_i^k = w_i v(c_i^k) - y_k^k - w_i v(c_i^k) \\ + \sum_{j=i}^{k-2} [w_j - w_{j+1}] v(c_{j+1}^k) + w_{k-1} v(c_k^k). \end{aligned} \tag{A.15}$$

Cancelling the terms in $w_i v(c_i^k)$ and adding and subtracting $w_k v(c_k^k)$ on the right-hand side of (A.15) yields

$$w_i v(c_i^k) - y_i^k = w_k v(c_k^k) - y_k^k + \sum_{j=i}^{k-1} [w_j - w_{j+1}] v(c_{j+1}^k). \tag{A.16}$$

Substituting (12) into the right-hand side of (A.13) yields

$$w_i v(c_i^{k+1}) - y_i^{k+1} = w_i v(c_i^{k+1}) - \left[y_{k+1}^{k+1} + \sum_{j=i}^{k} w_j [v(c_j^{k+1}) - v(c_{j+1}^{k+1})] \right]. \tag{A.17}$$

Rearranging terms on the right-hand side of (A.17) gives

$$w_i v(c_i^{k+1}) - y_i^{k+1} = w_k v(c_{k+1}^{k+1}) - y_{k+1}^{k+1} + \sum_{j=i}^{k-1} [w_j - w_{j+1}] v(c_{j+1}^{k+1}). \tag{A.18}$$

Subtracting (A.18) from (A.16), we obtain

$$\begin{aligned} \left[w_i v(c_i^k) - y_i^k \right] - \left[w_i v(c_i^{k+1}) - y_i^{k+1} \right] = \left[w_k v(c_k^k) - y_k^k \right] \\ - \left[w_k v(c_{k+1}^{k+1}) - y_{k+1}^{k+1} \right] + \sum_{j=i}^{k-1} [w_j - w_{j+1}] [v(c_{j+1}^k) - v(c_{j+1}^{k+1})]. \end{aligned} \tag{A.19}$$

The first term on the right-hand side of (A.19) is the maximal utility of type k when the proposer is of this type. Because type k's adjacent upward self-selection constraint binds at type $(k+1)$'s selfishly optimal bundle, the second term is the utility that type k receives when the proposer is of type $k+1$. Because someone of type k can do no better than with its own proposal, the difference between the first and second terms is nonnegative. The final term is also nonnegative because any selfishly optimal consumption schedule is nondecreasing in type. Hence, the entire right-hand side of (A.19) is nonnegative, which establishes (A.13).

If $i > k$, we can use a similar argument to show that the utility that type i receives with type k's selfishly optimal allocation is greater than or equal to the utility that i receives with type $(k-1)$'s selfishly optimal allocation. This argument uses (13) instead of (12). □

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